Lecture 21: Properties of the diamond norm

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Recall that last time we defined two norms on transformations. If $\Phi \in \mathrm{T}(\mathcal{F}, \mathcal{G})$ is a transformation, the *trace norm* of Φ is defined as

$$\|\Phi\|_{\mathrm{tr}} = \max\left\{\|\Phi(X)\|_{\mathrm{tr}} : X \in \mathrm{L}(\mathcal{F}),\ \|X\|_{\mathrm{tr}} = 1\right\}$$

and the *diamond norm* of Φ is defined as

$$\|\Phi\|_{\diamond} = \left\|\Phi \otimes I_{\mathrm{L}(\mathcal{F})}\right\|_{\mathrm{tr}} = \max\left\{\left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{F})})(X)\right\|_{\mathrm{tr}} : X \in \mathrm{L}(\mathcal{F} \otimes \mathcal{F}),\ \|X\|_{\mathrm{tr}} = 1\right\}.$$

We proved two lemmas in the previous lecture: Lemmas 20.3 and 20.4. Now let us see what those lemmas say about the diamond norm.

Theorem 21.1. *Let $\Phi \in \mathrm{T}(\mathcal{F}, \mathcal{G})$, and let $\mathcal{H}$ be a space of arbitrary dimension. Then*

$$\left\|\Phi \otimes I_{\mathrm{L}(\mathcal{H})}\right\|_{\mathrm{tr}} \leq \|\Phi\|_{\diamond},$$

with equality if (but not necessarily only if) $\dim(\mathcal{H}) \geq \dim(\mathcal{F})$.

Proof. Consider any operator $X \in \mathrm{L}(\mathcal{F} \otimes \mathcal{H})$ for which $\|X\|_{\mathrm{tr}} = 1$. Let

$$X = \sum_{i=1}^{r} s_i u_i v_i^*$$

be a singular value decomposition of X. Because $\|X\|_{\mathrm{tr}} = 1$, we have that $(s_1, \ldots, s_r)$ represents a probability distribution. By the triangle inequality we have

$$\left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{H})})(X)\right\|_{\mathrm{tr}} \leq \sum_{i=1}^{r} s_i \left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{H})})(u_i v_i^*)\right\|_{\mathrm{tr}}.$$

By Lemma 20.4 there exist unit vectors $x_1, \ldots, x_r, y_1, \ldots, y_r \in \mathcal{F} \otimes \mathcal{F}$ such that

$$\left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{H})})(u_i v_i^*)\right\|_{\mathrm{tr}} = \left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{F})})(x_i y_i^*)\right\|_{\mathrm{tr}}$$

for each $i = 1, \ldots, r$. Because $\|x_i y_i^*\|_{\mathrm{tr}} = 1$ we have $\left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{F})})(x_i y_i^*)\right\|_{\mathrm{tr}} \leq \|\Phi\|_{\diamond}$ for each i, and thus

$$\left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{H})})(X)\right\|_{\mathrm{tr}} \leq \|\Phi\|_{\diamond} \sum_{i=1}^{r} s_i = \|\Phi\|_{\diamond}.$$

Now let us prove that equality holds whenever $\dim(\mathcal{H}) \geq \dim(\mathcal{F})$. Let $U \in \mathrm{U}(\mathcal{F}, \mathcal{H})$. For any $X \in \mathrm{L}(\mathcal{F} \otimes \mathcal{F})$ we have

$$\begin{aligned}\left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{H})})\right\|_{\mathrm{tr}} &\geq \left\|(I \otimes U^*)\,(\Phi \otimes I_{\mathrm{L}(\mathcal{H})})((I \otimes U) X (I \otimes U^*))\,(I \otimes U)\right\|_{\mathrm{tr}} \\ &= \left\|(\Phi \otimes I_{\mathrm{L}(\mathcal{F})})(X)\right\|_{\mathrm{tr}}.\end{aligned}$$

Maximizing over all $X \in \mathrm{L}(\mathcal{F} \otimes \mathcal{F})$ with $\|X\|_{\mathrm{tr}} = 1$ establishes the claim. □

Next let us consider the notion of distance between admissible transformations that we defined in the previous lecture:

$$\begin{aligned}&\operatorname{dist}(\Phi_0,\Phi_1)\\&\quad=\sup\left\{\left\|(\Phi_0\otimes I_{\mathrm{L}(\mathcal{H})})(\rho)-(\Phi_1\otimes I_{\mathrm{L}(\mathcal{H})})(\rho)\right\|_{\mathrm{tr}}\,:\,\rho\in\mathrm{D}\left(\mathcal{F}\otimes\mathcal{H}\right),\ \mathcal{H}=\mathbb{C}(n),\ n\in\mathbb{N}\right\}.\end{aligned}$$

The following theorem relates this quantity to the diamond norm.

Theorem 21.2. *Let* $\Phi_0,\Phi_1\in\mathrm{T}\left(\mathcal{F},\mathcal{G}\right)$ *be admissible. Then*

$$\operatorname{dist}(\Phi_0,\Phi_1)=\left\|\Phi_0-\Phi_1\right\|_{\diamond}.$$

Moreover, the value of the supremum in the definition of $\operatorname{dist}(\Phi_0,\Phi_1)$ *is achieved when* $\dim(\mathcal{H})=\dim(\mathcal{F})$.

Proof. Choose any space $\mathcal{H}$ and any $\rho\in\mathrm{D}\left(\mathcal{F}\otimes\mathcal{H}\right)$. Then by the previous theorem, along with the fact that $\left\|\rho\right\|_{\mathrm{tr}}=1$, we have

$$\begin{aligned}\left\|(\Phi_0\otimes I_{\mathrm{L}(\mathcal{H})})(\rho)-(\Phi_1\otimes I_{\mathrm{L}(\mathcal{H})})(\rho)\right\|_{\mathrm{tr}}&=\left\|\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{H})}\right)(\rho)\right\|_{\mathrm{tr}}\\&\leq\left\|(\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{H})}\right\|_{\mathrm{tr}}\\&\leq\left\|\Phi_0-\Phi_1\right\|_{\diamond}.\end{aligned}$$

Thus, we have $\operatorname{dist}(\Phi_0,\Phi_1)\leq\left\|\Phi_0-\Phi_1\right\|_{\diamond}$.

Now we need to prove the opposite inequality. For any $X\in\mathrm{L}\left(\mathcal{F}\otimes\mathcal{F}\right)$ we have

$$\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{F})}\right)(X^*)=\left(\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{F})}\right)(X)\right)^*,$$

and so by Lemma 20.3 we have, under the assumption $\left\|X\right\|_{\mathrm{tr}}=1$, that there exists a unit vector $u\in\mathcal{F}\otimes\mathcal{F}\otimes\mathcal{Q}$ such that

$$\left\|\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{Q})}\right)(uu^*)\right\|_{\mathrm{tr}}\geq\left\|\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{F})}\right)(X)\right\|_{\mathrm{tr}}.$$

By Lemma 20.4 there exists a unit vector $x\in\mathcal{F}\otimes\mathcal{F}$ such that

$$\left\|\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{F})}\right)(xx^*)\right\|_{\mathrm{tr}}=\left\|\left((\Phi_0-\Phi_1)\otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{Q})}\right)(uu^*)\right\|_{\mathrm{tr}}$$

Taking $\mathcal{H}=\mathcal{F}$ and $\rho=xx^*$ establishes the required result. □

Multiplicativity of the diamond norm

The diamond norm satisfies several nice properties. One of these properties is that it is multiplicative with respect to tensor products.

Theorem 21.3. *Let* $\Phi\in\mathrm{T}\left(\mathcal{F},\mathcal{G}\right)$ *and* $\Psi\in\mathrm{T}\left(\mathcal{H},\mathcal{K}\right)$ *be arbitrary transformations. Then*

$$\left\|\Phi\otimes\Psi\right\|_{\diamond}=\left\|\Phi\right\|_{\diamond}\left\|\Psi\right\|_{\diamond}.$$

Proof. By definition we have

$$\|\Phi \otimes \Psi\|_{\diamond} = \left\|\Phi \otimes \Psi \otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{H})}\right\|_{\mathrm{tr}}.$$

Consider any $X \in \mathrm{L}(\mathcal{F} \otimes \mathcal{H} \otimes \mathcal{F} \otimes \mathcal{H})$ with $\|X\|_{\mathrm{tr}} = 1$. Let

$$Y = \left(I_{\mathrm{L}(\mathcal{F})} \otimes \Psi \otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{H})}\right)(X).$$

Then $\|Y\|_{\mathrm{tr}} \leq \|\Psi\|_{\diamond}$. Moreover,

$$\left\|\left(\Phi \otimes I_{\mathrm{L}(\mathcal{K})} \otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{H})}\right)(Y)\right\|_{\mathrm{tr}} \leq \|\Phi\|_{\diamond}\|Y\|_{\mathrm{tr}}.$$

Thus,

$$\begin{aligned}\left\|\left(\Phi \otimes \Psi \otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{H})}\right)(X)\right\|_{\mathrm{tr}} &= \left\|\left(\Phi \otimes I_{\mathrm{L}(\mathcal{K})} \otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{H})}\right)\left(I_{\mathrm{L}(\mathcal{F})} \otimes \Psi \otimes I_{\mathrm{L}(\mathcal{F}\otimes\mathcal{H})}\right)(X)\right\|_{\mathrm{tr}} \\ &\leq \|\Phi\|_{\diamond}\|\Psi\|_{\diamond}.\end{aligned}$$

As this holds for all choices of $X \in \mathrm{L}(\mathcal{F} \otimes \mathcal{H} \otimes \mathcal{F} \otimes \mathcal{H})$, we have

$$\|\Phi \otimes \Psi\|_{\diamond} \leq \|\Phi\|_{\diamond}\|\Psi\|_{\diamond}.$$

For the reverse inequality, choose operators $X_1 \in \mathrm{L}(\mathcal{F} \otimes \mathcal{F})$ and $X_2 \in \mathrm{L}(\mathcal{H} \otimes \mathcal{H})$ such that $\|X_1\|_{\mathrm{tr}} = \|X_2\|_{\mathrm{tr}} = 1$, $\|\Phi\|_{\diamond} = \|(\Phi \otimes I_{\mathcal{F}})(X_1)\|_{\mathrm{tr}}$, and $\|\Psi\|_{\diamond} = \|(\Psi \otimes I_{\mathcal{H}})(X_2)\|_{\mathrm{tr}}$. Then $\|X_1 \otimes X_2\|_{\mathrm{tr}} = 1$, and so

$$\begin{aligned}\|\Phi \otimes \Psi\|_{\diamond} &= \left\|\Phi \otimes \Psi \otimes I_{\mathrm{L}(\mathcal{F})} \otimes I_{\mathrm{L}(\mathcal{H})}\right\|_{\mathrm{tr}} \\ &\geq \left\|\left(\Phi \otimes \Psi \otimes I_{\mathrm{L}(\mathcal{F})} \otimes I_{\mathrm{L}(\mathcal{H})}\right)(X_1 \otimes X_2)\right\|_{\mathrm{tr}} \\ &= \left\|\left(\Phi \otimes I_{\mathrm{L}(\mathcal{F})}\right)(X_1) \otimes \left(\Psi \otimes I_{\mathrm{L}(\mathcal{H})}\right)(X_2)\right\|_{\mathrm{tr}} \\ &= \left\|\left(\Phi \otimes I_{\mathrm{L}(\mathcal{F})}\right)(X_1)\right\|_{\mathrm{tr}}\left\|\left(\Psi \otimes I_{\mathrm{L}(\mathcal{H})}\right)(X_2)\right\|_{\mathrm{tr}} \\ &= \|\Phi\|_{\diamond}\|\Psi\|_{\diamond}\end{aligned}$$

as required. □