Final Exam Practice Problems

Here are a few random practice problems that you may find helpful in preparing for the final exam for CS365. (Of course these problems do not represent all possible topics that you might be asked about on the final—it is just a sample of questions.)

Assume $\Sigma = \{0, 1\}$ for all of the questions.

1. Suppose that $A \subseteq \Sigma^*$ is a regular language and $y \in \Sigma^*$ is any string. Prove that the language

$$B = \{x \in \Sigma^* \,:\, xy \in A\}$$

is regular.

2. Prove that the language BAL $\subseteq \{\,(\,,\,)\,\}^*$ containing all strings of properly balanced parentheses is in L (i.e., is decidable by a DTM running in logarithmic space).

3. Prove that the language

$$\text{CON} = \{\langle G\rangle \,:\, G \text{ is a } \underline{\text{connected}} \text{ undirected graph}\}$$

is in NL. (In fact this language is known to be in L, but this is significantly harder to prove than its membership in NL.)

4. Suppose that $A, B \subseteq \Sigma^*$ are languages in BPP. Prove that $A \cup B$ and $A \cap B$ are also in BPP.

5. Determine whether the following statements are true or false, and defend your answer (as usual).

   (a) Suppose that $A \subseteq \Sigma^*$ is any nontrivial language (meaning that $A \notin \{\varnothing, \Sigma^*\}$), and define

   $$B = \{\langle x, y\rangle \,:\, x \in A \text{ and } y \notin A\}.$$

   Then it holds that $A \leq_m^p B$ and $\overline{A} \leq_m^p B$.

   (b) It holds that PSPACE $\not\subseteq$ DTIME(n^{100}).

   (c) For every language $A \subseteq \Sigma^*$ it holds that $A^* \leq_T A$.

6. For any function $f : \Sigma^* \to \Sigma^*$, define a language

$$A_f = \left\{x \in \Sigma^* \,:\, \exists y \in \Sigma^{|x|} : f(\langle x, y\rangle) = y\right\}.$$

   (a) Prove that $A_f \in$ NP for every polynomial-time computable function f.

   (b) Prove that there exists a particular choice of a polynomial-time computable function f such that A_f is NP-complete.

7. Define a language

$$\text{SLOWER} = \{\langle M_1, M_2 \rangle \,:\, M_1 \text{ and } M_2 \text{ are DTMS, and } M_1 \text{ is always slower than } M_2\}\,.$$

When we say that M_1 is always slower than M_2, we means that if M_1 halts on any given input string x within t steps, then M_2 must halt on x within $t' < t$ steps.

Prove that E $\equiv_m$ SLOWER (i.e., E $\leq_m$ SLOWER and SLOWER $\leq_m$ E), where

$$\text{E} = \{\langle M \rangle \,:\, M \text{ is a DTM with } L(M) = \varnothing\}\,.$$