The spectrum of α-resolvable designs with block size four

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Abstract

An α-resolvable BIBD is a BIBD with the property that the blocks can be partitioned into disjoint classes such that every class contains each point of the design exactly α times. In this paper, we show that the necessary conditions for the existence of α-resolvable designs with block size four are sufficient, with the exception of $(\alpha, v, \lambda) = (2, 10, 2)$.

1 Introduction

An α-resolvable BIBD is a BIBD with the property that the blocks can be partitioned into disjoint classes such that every class contains each point of the design exactly α times. Furino [7] shows that the necessary conditions for the existence of α-resolvable $(v, 4, \lambda)$ BIBDs are sufficient for $v \not\equiv 2, 6 \bmod 12$ with 110 possible exceptions. Furino and Mullin [10] improve this result, and Vasiga [17] completes the missing congruence classes of $v \equiv 2, 6 \mod 12$ and reduces the number of possible exceptions to 79.

In this paper, we show that the necessary conditions for the existence of α-resolvable designs with block size four are sufficient, with the exception of $(\alpha, v, \lambda) = (2, 10, 2)$. We demonstrate this by eliminating the possible exceptions, and by provide shorter and neater proofs that use more recent techniques.

We begin with the required definitions, and then prove the main theorem of this paper by examining each congruence class.

2 Primary Definitions

A (v, k, λ) *balanced incomplete block design (BIBD)* is a pair $(\mathcal{V}, \mathcal{B})$, where

1. $\mathcal{V}$ is a set of v points
2. $\mathcal{B}$ is a collection of k-subsets, called *blocks* of $\mathcal{V}$
3. $k < v$
4. each pair of elements from $\mathcal{V}$ occur together in exactly λ blocks.

Each point in a BIBD occurs $r = \frac{\lambda(v-1)}{k-1}$ times and the numbers of blocks in a BIBD is $b = \frac{vr}{k}$.

A BIBD is called α*-resolvable* if its blocks are partitioned into classes (called α*-resolution classes*) such that each point of the design occurs in precisely α blocks in each class.

In this paper, we investigate the existence of α-resolvable $(v, 4, \lambda)$ BIBDs. Based on the relations defined above, we can derive the following two relations

$$bk = rv \tag{1}$$
$$\lambda(v-1) = r(k-1) \tag{2}$$

which must hold for any BIBD. Further, for an α-resolvable (v, k, λ) BIBD, the conditions,

$$k \mid \alpha v \tag{3}$$
$$\alpha \mid r \tag{4}$$

must hold.

From these conditions, we can derive minimum values for α and λ. Call these *minimum values* α_0 and λ_0. We then call an α_0-resolvable (v, k, λ_0) BIBD a *minimum design.* Using these minimum values, Furino [7] derives the following lemma, which is straightforward to prove using Conditions 1-4.

Lemma 2.1 *If an α_1-resolvable (v, k, λ_1) BIBD exists, then $\alpha_0|\alpha_1$ and $\lambda_0|\lambda_1$.*

Additionally, by repeating a design and combining resolution classes, it is clear that

Lemma 2.2 *If an α-resolvable (v, k, λ) BIBD exists, then a $t\alpha$-resolvable $(v, k, n\lambda)$ BIBD exists, where n is a positive integer and t divides the number of resolution classes, which is $\frac{\lambda(v-1)}{\alpha(k-1)}$.*

The above two lemmas imply the following theorem regarding minimal designs.

Theorem 2.3 *If an α_0-resolvable (v, k, λ_0) BIBD exists, and α_1 and λ_1 satisfy Conditions 1-4, then an α_1-resolvable (v, k, λ_1) BIBD exists.*

Thus, in order to show that the necessary conditions for α-resolvable designs are sufficient, we simply need to prove the existence of minimal designs: that is, for a given v, demonstrate the existence of an α_0-resolvable (v, k, λ_0) BIBD.

For $k = 3$, it was shown in [13] that these necessary conditions are also sufficient except for $v = 6$ and $\lambda = 4t + 2$. For $k = 4$, we have the minimum values listed in Table 2.

v mod 12	λ	α
0	1	3
1	4	1
2	2	6
3	4	6
4	1	1
5	4	3
6	2	6
7	4	2
8	1	3
9	4	3
10	2	2
11	4	6

Table 1: Minimum values for an α-resolvable (v, k, λ)

Additionally, the following results have been proved:

Theorem 2.4 *[10] In a λ-resolvable $(v, 4, \lambda)$ BIBD, the condition $v \equiv 1 \pmod 3$ must hold.*

1. *Let $\lambda \equiv 0 \pmod 4$. There exists a λ-resolvable $(v, 4, \lambda)$ BIBD if and only if $v \equiv 1 \pmod 3$.*

2. *Let $\lambda \equiv 2 \pmod 4, \lambda \geq 6$. There exists a λ-resolvable $(v, 4, \lambda)$ BIBD if and only if $v \equiv 4 \pmod 6$ with the possible exception of $v \in \{10, 118, 262\}$. Further if $\lambda = 2$, there exists a λ-resolvable $(v, 4, \lambda)$ BIBD if and only if $v \equiv 4 \pmod 6$ with the exception of a $v = 10$ and possible exception of $v \in \{46, 118, 262, 550, 658\}$.*

3 Auxiliary Definitions

In order to solve the existence problem, we need some auxiliary designs. Let K be a set of positive integers. A *group divisible design* (K, λ)-GDD is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ where

1. $\mathcal{X}$ is a finite set of *points* of cardinality v,

2. $\mathcal{G}$ is a set of subsets of $\mathcal{X}$, called *groups*, which partition $\mathcal{X}$,

3. $\mathcal{A}$ is a collection of subsets of $\mathcal{X}$ with sizes from K ($2 < k < v$ for each $k \in K$), called *blocks*, such that every pair of points from distinct groups occurs in exactly λ blocks, and

4. no pair of points belonging to a group occurs in any block.

We use the usual exponential notation for the types of GDDs. Thus a GDD of *type* $1^i 2^j \dots$ has i groups of size 1, j groups of size 2, and so on.

A *transversal design*, $TD(k, g)$ is a $GDD(kg, g, k)$. A *resolvable transversal design*, denoted $RTD(k, g)$ is a $TD(k, g)$ whose blocks can be partitioned into resolution classes.

A *partial α-resolution class* is a collection of blocks in which each point of $\mathcal{X}$ occurs in precisely α or zero blocks. The set of points not occurring in a partial parallel class is called the *complement* of the class.

An $(\alpha; k, \lambda)-$frame is a GDD $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ such that

1. $\mathcal{A}$ is a collection of α-partial resolution classes of the set $\mathcal{X}$

2. All blocks are of cardinality k

3. The complement of each partial α-resolution class A of $\mathcal{A}$ is a group G of $\mathcal{G}$;

As with GDDs, we use exponential notation for the types of frames. Thus a frame of *type* $1^i 2^j \dots$ in which there are i groups of size 1, j groups of size 2, and so on.

The final structure that we will require are *difference families*. Let G be an additive Abelian group, and let $B = \{b_1, b_2, \cdots, b_k\}$ be a set of k distinct elements of G. Define $B + s = \{b_1 + s, b_2 + s, ..., b_k + s\}$ where $s \in G$. Define the *development* of B as dev $B = \{B + s : s \in G\}$. If dev B is a (v, k, λ) design, then we call B a (v, k, λ) *difference set* (where B is the *base block*). If $|\text{dev } B| < |G|$, then we call B a *short orbit* under G.

Extending this idea to a collection of base blocks, we let $\mathcal{F} = (B_1, B_2, ..., B_t)$ be a list of subsets of G. Let dev $\mathcal{F} = \sum_{i=1}^{t}$ dev B_i. We call $\mathcal{F}$ a (v, k, λ) *difference family* (denoted as DF(v, k, λ)) if dev $\mathcal{F}$ is a (v, k, λ) BIBD. Formally, we have the following lemma proved in Furino [7]:

Lemma 3.1 *If $\mathcal{F}$ is a (v, k, λ) difference family and* dev $\mathcal{F}$ *contains no short orbits, then* dev $\mathcal{F}$ *is a k-resolvable (v, k, λ)* BIBD.

4 Recursive Constructions

The following construction for $(\alpha; k, \lambda)$-frames is a straightforward generalization of that of Construction 3.1 in [16].

Theorem 4.1 *Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD of index λ_1, and let w be a mapping $w : \mathcal{X} \to \mathbb{Z}^+ \cup \{0\}$. For each $A \in \mathcal{A}$, suppose that there is an $(\alpha; k, \lambda_2)$-frame of type $\{w(x) : x \in A\}$. Then there is an $(\alpha\lambda_1; k, \lambda_1\lambda_2)$-frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

The following theorems are straightforward generalizations of resolvable design constructions in [9].

Theorem 4.2 *If an $(\alpha; k, \lambda)$-frame of type h^t exists and an α-resolvable $(h+1, k, \lambda)$ BIBD exists, then an α-resolvable $(ht+1, k, \lambda)$ BIBD exists.*

Theorem 4.3 *If there exists an $(\alpha; k, \lambda)$-frame of type T and a $TD(k+1, n)$, then there exists an $(\alpha; k, \lambda)$-frame of type nT (where nT signifies that each group in T inflates in size by a factor of n).*

We need several results on frames.

Theorem 4.4 *[15] There exists a $(1; 4, 1)$-frame of type 12^n for all $n \geq 5$ and $n \notin \{8, 12\}$. Also, there exists a $(1; 4, 1)$-frame of type 24^6.*

Theorem 4.5 *[8] There exists a $(1; 4, 3)$-frame of type 4^n for all $n \geq 5$.*

5 $v \equiv 0 \pmod 4$

When $v \equiv 4 \pmod{12}$, it was shown by Hanani, Ray-Chaudhuri and Wilson [12] that there exists a 1-resolvable $(v, 4, 1)$ BIBD. Hence, the basic necessary condition is also sufficient. When $v \equiv 0, 8 \pmod{12}$, it has been shown by Baker [3] that there exists a 1-resolvable $(v, 4, 3)$ BIBD. Hence, the basic necessary condition is also sufficient.

As a corollary to this result, we will require the following simple result.

Corollary 5.1 *There exists a 2-resolvable $(v, 4, 6)$ BIBD where $v \in \{8, 12\}$.*

Proof: Take the results of Baker [3], and duplicate each block within each resolution class. This doubles α and λ, and gives us the desired designs. $\Box$

6 $v \equiv 1 \pmod 4$

When $v \equiv 1 \pmod 4$, and $k | \alpha v$, we must have α as a multiple of 4.

Lemma 6.1 *There exists a 4-resolvable $(12t+1, 4, 1)$ BIBD for $t = 1, 2, 3, 4, 8$.*

Proof: It has been shown [1] that there exists a difference family $(12t+1, 4, 1)$ BIBD over $\mathbb{Z}_{12t+1}$. This gives a 4-resolvable $(12t+1, 4, 1)$ BIBD for $t = 1, 2, 3, 4, 8$. $\Box$

Lemma 6.2 *There exists a 4-resolvable $(12t+1, 4, 1)$ BIBD for $t \geq 1$.*

Proof: When $t \geq 5$ and $t \neq 8, 12$, take a $(1; 4, 1)$-frame of type 12^t to obtain a $(4; 4, 1)$-frame of type 12^t by grouping the frame parallel classes. Add a new point and apply Theorem 4.2 to get a 4-resolvable $(12t+1, 4, 1)$ BIBD for $t \geq 5$ and $t \neq 8, 12$ since a 4-resolvable $(13, 4, 1)$ BIBD exists. When $t = 1, 2, 3, 4, 8$, it is dealt with in Lemma 6.1. When $t = 12$, take a $(1; 4, 1)$-frame of type 24^6 to obtain a $(4; 4, 1)$-frame of type 24^6. Add a new point and apply Theorem 4.2 to get a 4-resolvable $(12t+1, 4, 1)$ BIBD for $t = 12$ since a 4-resolvable $(25, 4, 1)$ BIBD exists. The required frames are taken from Theorem 4.4. $\square$

For $v \equiv 5, 9 \pmod{12}$ the existence of a $(v, 4, \lambda)$ BIBD implies that λ is a multiple of 3.

Lemma 6.3 *There exists a 4-resolvable* $(4t+1, 4, 3)$ *BIBD when* $t \geq 1$.

Proof: When $t = 1, 2, 3, 4$, there exists a DF$(4t+1, 4, 3)$. Hence there exists a 4-resolvable $(4t+1, 4, 3)$ BIBD when $t = 1, 2, 3, 4$ by Lemma 3.1. When $t \geq 5$, take a $(1; 4, 3)$-frame of type 4^t and group the partial parallel classes to obtain $(4; 4, 3)$-frame of type 4^t. Add a new point and apply Theorem 4.2 with a 4-resolvable $(5, 4, 3)$ BIBD to obtain a 4-resolvable $(4t+1, 4, 3)$ BIBD for all $t \geq 5$. $\square$

7 $v \equiv 3 \pmod 4$

If $v \equiv 7 \pmod{12}$, then λ is a multiple of 2. If $v \equiv 3, 11 \pmod{12}$, then λ is a multiple of 6.

We first deal with the case $v \equiv 7 \pmod{12}$.

Lemma 7.1 *There exists a 4-resolvable* $(12t+7, 4, 2)$ *BIBD when* $t = 0, 1$.

Proof: There exists a DF$(12t+7, 4, 2)$ when $t = 0, 1$ [1]. Apply Lemma 3.1. $\square$

Lemma 7.2 *There exists a 4-resolvable* $(12t+7, 4, 2)$ *BIBD when* $t \geq 2$.

Proof: We first construct a $(4; 4, 2)$-frame of type 6^{2t+1} when $t \geq 2$. If $t \geq 2$, it is known that there exists a TD$(5, 2t+1)$ (or a (5,1)-GDD of type $(2t+1)^5$) [2]. Let the point set of the TD be $\mathbb{Z}_{2t+1} \times \{0, 1, 2, 3, 4\}$ and suppose there are blocks of the form $\{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4)\}$ for $i \in \mathbb{Z}_{2t+1}$. Let the groups be $\mathbb{Z}_{2t+1} \times \{i\}$ for $i \in \{0, 1, 2, 3, 4\}$. We construct a $(4; 4, 2)$-frame of type 6^{2t+1} on $\mathbb{Z}_6 \times \mathbb{Z}_{2t+1}$. The groups of the frame are $\mathbb{Z}_6 \times \{j\}$ for $j \in \mathbb{Z}_{2t+1}$. For every block of the form $\{(i, 0), (a, 1), (b, 2), (c, 3), (d, 4)\}$ where $i \neq a$, we construct a block of the form $\{(j, a), (j, b), (j+1, c), (j+4, d)\}$ for $j \in \mathbb{Z}_6$ and the first component is computed mod 6. We first check that it yields a $(4, 2)$-GDD of type 6^{2t+1}. For any pair of the form $\{(j, x), (j, y)\}$ $(x \neq y)$, they appear together in two blocks constructed from the blocks containing the pair $\{(x, 1), (y, 2)\}$ and $\{(y, 1), (x, 2)\}$. For any pair of the form $\{(j, x), (j+1, y)\}$ $(x \neq y)$, they appear together in two blocks constructed from the blocks containing the pair $\{(1, x), (3, y)\}$ and $\{(2, y), (3, x)\}$. It is a easy exercise to check for the remaining cases. To show that this is a frame, we note that if $\mathcal{A}_i$ is the set of blocks in the TD containing $(i, 0)$, then the set of blocks constructed from $\mathcal{A}_i$ induces a partial parallel class missing $\mathbb{Z}_6 \times \{i\}$. Hence, there exists a $(4; 4, 2)$-resolvable frame of type 6^{2t+1} for all $t \geq 2$. Add a new point to the frame and fill in the holes with a 4-resolvable $(7,4,2)$ BIBD to yield a 4-resolvable $(12t+7,4,2)$ BIBD. □

We next deal with the case when $v \equiv 11 \pmod{12}$.

Lemma 7.3 *There exists a 4-resolvable $(v,4,6)$ BIBD when $v \equiv 11 \pmod{12}$.*

Proof: It has been shown in [6] that there exists a DF$(v,4,6)$ when $v \equiv 11 \pmod{12}$. The result follows from Lemma 3.1. □

We finally deal with the case when $v \equiv 3 \pmod{12}$.

Lemma 7.4 *There exists a 4-resolvable $(v,4,6)$ BIBD when $v = 15, 27, 39$.*

Proof: If $v = 15$, there exists a DF$(15,4,6)$ over $\mathbb{Z}_{15}$ with base blocks $\{1,2,4,8\}$, $\{1,2,4,8\}, \{1,2,4,8\}, \{7,11,13,14\}, \{7,11,13,14\}, \{0,5,10,12\}$ and $\{0,5,9,10\}$ [11]. If $v = 27$, there exists a DF$(27,4,6)$ over $\mathbb{Z}_{27}$ since 27 is a prime power [5]. If $v = 39$, there exists a DF$(39,4,6)$ over $\mathbb{Z}_{13} \times \mathbb{Z}_3$ with base blocks $\{(0,0),(1,0),(4,0),(8,0)\}, \{(0,0),(1,0),(4,0),(8,0)\}, \{(0,0),(1,1),(3,1),(4,1)\}$, $\{(0,1),(1,0),(3,0),(4,0)\}, \{(t,0),(12t,0),(t,1),(12t,1)\}$ for $t = 1,3,4$, $\{(0,0),(t,0),(2t,1),(3t,2)\}$ for $t = 1,2,3,4,5,6,7,8,9,10,11,12$. Hence there exists a 4-resolvable $(v,4,6)$ BIBD when $v = 15, 27, 39$ by applying Lemma 3.1. □

Lemma 7.5 *There exists a $(4;4,6)$-frame of type 2^n for $n = 5,6$.*

Proof: A $(1;4,3)$-frame of type 2^5 is constructed in [8]. Take a $(5,2)$-GDD of type 2^6 [5] and give weight 1 and apply Wilson's Fundamental Construction with a $(1;4,3)$-frame of type 1^5. □

Lemma 7.6 *[8] If $q \equiv 1 \pmod{4}$, and q is a prime power, then there exists a $(1;4,3)$-frame of type 2^q.*

Lemma 7.7 *There exists a $(2;4,3)$-frame of type 2^q when $q \equiv 3 \pmod{4}$ and q is a prime power.*

Proof: Let α denote a generator of $GF(q)$; that is $\langle\alpha\rangle = GF(q)^*$. Define the point set of the design by $GF(q) \times \mathbb{Z}_2$. Define the groups of the design by $G_\beta = \{(\beta,0),\ (\beta,1)\}$ for each $\beta \in GF(q)$, and associate with G_β the partial 2-resolution class $\mathcal{B}_\beta$

$$\mathcal{B}_\beta = \{(\beta+\alpha^j, i), (\beta+\alpha^{\frac{q-1}{2}+j}, i), (\beta+\alpha^{1+j}, i), (\beta+\alpha^{\frac{q+1}{2}+j}, i+1)\}$$
$$j = 1,2,\ldots,\tfrac{q-1}{2},\ \ i = 0,1$$

Developing these blocks for all $\beta \in GF(q)$ yields the desired frame. □

Lemma 7.8 *If $v \equiv 3 \pmod{12}$, then there exists a 4-resolvable $(v,4,6)$ BIBD for all v.*

Proof: When $v \equiv 1 \pmod 2$ and $v \not\equiv 3 \pmod{12}$, it has been shown that the basic necessary condition for the existence of 4-resolvable $(v, 4, \lambda)$ BIBD is also sufficient. By taking multiple copies, we can obtain a 4-resolvable $(v, 4, 6)$ BIBD for all $v \equiv 1, 5, 7, 9, 11 \pmod{12}$. When $v = 51$, take a TD$(5, 5)$ and give weight 2 and add an infinite point. When $v = 63$, take a RTD$(7, 9)$, fill in each group by a 4-resolvable $(9, 4, 6)$ BIBD and each block by a 4-resolvable $(7, 4, 6)$ BIBD. If there exists a TD$(7, n)$, we truncate in two groups and give weight two to obtain a $(4; 4, 6)$-frame of type $(2n)^5(2a)^1(2b)^1$ where $a, b \leq n$. We add an infinite point to this frame to obtain a 4-resolvable $(10n + 2a + 2b + 1, 4, 6)$ BIBD if 4-resolvable $(2x + 1, 4, 6)$ BIBDs exists for $x = a, b, n$. We note that if $b = 0$, we just require a TD$(6, n)$ instead. Explicit values for $v \leq 171, n, a, b$ are given below:

1. $v = 75$, $n = 7$, $a = 2$, $b = 0$.
2. $v = 87$, $n = 7$, $a = 2$, $b = 6$.
3. $v = 99$, $n = 9$, $a = 4$, $b = 0$.
4. $v = 111$, $n = 9$, $a = 4$, $b = 6$.
5. $v = 123$, $n = 11$, $a = 6$, $b = 0$.
6. $v = 135$, $n = 13$, $a = 2$, $b = 0$.
7. $v = 147$, $n = 13$, $a = 8$, $b = 0$.
8. $v = 159$, $n = 13$, $a = 8$, $b = 6$.
9. $v = 171$, $n = 13$, $a = 8$, $b = 12$.

We note that there for every three consecutive integers $\{n-2, n-1, n\}$, there must exist a TD$(7, k)$ where $k \in \{n-2, n-1, n\}$. Additionally, note that if $v \geq 183$, we may write $v = 10n + 2n + 3$ where $n \geq 15$. Based on these two observations, $v - 12n \leq 27$ and thus appropriate a, b values can be found. Therefore, there exists a 4-resolvable $(v, 4, 6)$ BIBD for all $v \equiv 3 \pmod{12}$ and $v \geq 183$. □

8 $v \equiv 2 \pmod 4$

In this section, we deal with the existence of 2-resolvable $(v, 4, \lambda)$ BIBD when $v \equiv 2 \pmod 4$.

We first consider the case when $v \equiv 10 \pmod{12}$, which has minimum values $\alpha = 2$ and $\lambda = 2$.

In [10], the following result is proved.

Lemma 8.1 *There does not exist a 2-resolvable* $(10, 4, 2)$ *BIBD.*

We now examine other values for α and λ when $v = 10$.

To begin, consider different values of λ.

Lemma 8.2 *There exists a* 2*-resolvable* $(10, 4, 6)$ *BIBD.*

Proof: Take the point set to be $\mathbb{Z}_9 \cup \{\infty\}$. Develop the following blocks (mod 9), which are a 2-resolution class:

$$\{0, 1, 2, 3\}$$
$$\{0, 2, 5, 6\}$$
$$\{1, 4, 5, 7\}$$
$$\{3, 7, 8, \infty\}$$
$$\{4, 6, 8, \infty\}$$

to get the desired design. $\square$

Lemma 8.3 *[6] There exists a 2-resolvable* $(10, 4, 4)$ *BIBD.*

We then can prove the following lemma to combine the previous two lemmas.

Lemma 8.4 *There exists a 2-resolvable* $(10, 4, m)$ *for all* $m \geq 4, m \equiv 0 \pmod 2$*.*

Proof: Combine copies of the 2-resolvable $(10, 4, 4)$ and 2-resolvable $(10, 4, 6)$ to get the appropriate value for m. $\square$

In order to cover the remaining minimal values for $v = 10$, Furino [6] has demonstrated the following:

Lemma 8.5 *There exists a* 6*-resolvable* $(10, 4, 2)$ *BIBD.*

Thus, for $v = 10$, we have demonstrated that the minimal designs exist, and thus the basic necessary conditions are sufficient, except in the case when $(\alpha, v, \lambda) = (2, 10, 2)$.

In view of the work of Furino and Mullin [10], we only need to deal with the remaining values in Theorem 2.4.

Lemma 8.6 *There exists a 2-resolvable* $(3v+1, 4, 2)$ *BIBD when* $v = 15, 39, 87, 183, 219$*.*

Proof: We shall prove this theorem using the frame construction of Theorem 4.2 to $(2; 4, 2)$-frames of type 3^v.

When $v = 15$, take the point set to be $\mathbb{Z}_{45}$ and take the groups to be $\{0, 15, 30\} + i$, $i = 0, \ldots, 14$. Let A denote the following blocks:

$$\begin{array}{ccc} \{1,4,9,17\} & \{1,5,14,21\} & \{2,3,5,25\} \\ \{3,9,27,44\} & \{4,10,21,28\} & \{7,8,41,43\} \\ & \{7,12,26,38\} & \end{array}$$

Then A, $A{+}15$ and $A{+}30$ form a 2-resolution class with the group $\{0, 15, 30\}$. Thus, developing A for each respective group yields the desired frame.

When $v = 39, 87, 219$, take a $(5, 2)$-GDD of type $1^{v-7}7^1$, which is shown to exist in [14]. Give weight 3 using an ingredient $(1; 4, 1)$-frame of type 3^5 and fill in the hole with a $(2;4,2)$-frame of type 3^7 to obtain a $(2;4,2)$-frame of type 3^v. The required frames exist by [10]. Add a new point to obtain a 2-resolvable $(3v+1,4,2)$ BIBD when $v = 39, 87, 219$ (Theorem 4.2).

Finally, for the case of $v = 183$, take a TD$(6,17)$ and truncate in one group to obtain a $\{5,6\}$-GDD of type $17^5 6^1$. Give weight 2 using a $(5,2)$-GDD of type 2^5 and 2^6 [5] to obtain a $(5,2)$-GDD of type $34^5 12^1$. Give weight 3 using a $(1;4,1)$-frame of type 3^5 to obtain a $(2;4,2)$-frame of type $102^5 36^1$. Add three infinite points and fill in the group with $(2;4,2)$-frame of type 3^{35} and 3^{13} to obtain a $(2;4,2)$-frame of type 3^{183}. The required frames exist by [10]. The result follows from Theorem 4.2.

□

We construct the smallest case for $v \equiv 2, 6 \pmod{12}$, which is $v = 6$.

Lemma 8.7 *A 2-resolvable $(6,4,6)$ BIBD exists.*

Proof: Take the point set to be $\mathbb{Z}_5 \cup \{\infty\}$. Then develop the following blocks (mod 5):

$$\{0,1,2,3\}$$
$$\{0,3,4,\infty\}$$
$$\{1,2,4,\infty\}$$

Since each point occurs exactly twice in these starter blocks, and since each difference occurs exactly six times, the developed blocks form the desired 5 2-resolution classes of the 2-resolvable $(6,4,6)$ BIBD. □

Continuing with direct constructions in the class $v \equiv 2$ or $6 \pmod{12}$, we now present the next largest values of v, which are $v = 14$ and $v = 18$.

Lemma 8.8 *A 2-resolvable $(14,4,6)$ BIBD exists.*

Proof: Take the point set to be $\mathbb{Z}_{13} \cup \{\infty\}$. Then develop the blocks (mod 13) given by:

$$\begin{array}{ccc} \{0,1,2,3\} & \{0,1,3,7\} & \{2,6,8,11\} \\ \{4,6,10,11\} & \{4,7,9,12\} & \{5,8,12,\infty\} \\ & \{5,9,10,\infty\} & \end{array}$$

which gives the desired 2-resolvable $(14,4,6)$ BIBD, since each difference occurs 6 times and since each element appears twice in the starter blocks. □

Lemma 8.9 *A 2-resolvable $(18,4,6)$ BIBD exists.*

Proof: Take the point set to be $\mathbb{Z}_{17} \cup \{\infty\}$. Then develop the blocks (mod 17) given by:

$$\begin{array}{ccc} \{0,1,2,6\} & \{0,1,3,7\} & \{2,5,10,15\} \\ \{3,5,9,12\} & \{4,8,10,13\} & \{4,11,12,15\} \\ \{6,7,14,\infty\} & \{8,13,14,16\} & \{9,11,16,\infty\} \end{array}$$

which gives the desired 2-resolvable $(18,4,6)$ BIBD, since these blocks form the 2-resolution class, and each difference occurs exactly six times. □

Next, we prove a lemma that will provide both a 2-resolvable $(26,4,6)$ BIBD and a more general result that will be used later in this section.

Lemma 8.10 *There exists a 2-resolvable* $(20t+6,4,6)$ *BIBD for all* $t>0$

Proof: We know there exists a (near resolvable design) NRB$(4t+1,4)$ by [9], which is equivalent to a $(1;4,3)$ frame of type 1^{4t+1}. Also, since 5 is a prime power, there exists an $RTD(5,5)$, and hence there exists an $RTD(4,5)$ (see [2]). Applying Theorem 4.3, we can construct a $(1;4,3)$ frame of type 5^{4t+1}. By Lemma 8.7, we know that a 2-resolvable $(6,4,6)$ BIBD exists. Hence, we may apply Theorem 4.2 to derive a 2-resolvable $(20t+6,4,6)$ BIBD. □

This lemma easily leads to the following corollary.

Corollary 8.11 *There exists a 2-resolvable* $(26,4,6)$ *BIBD.*

We now present the final two direct constructions of this section.

Lemma 8.12 *A 2-resolvable* $(30,4,6)$ *BIBD exists.*

Proof: Take the point set to be $\mathbb{Z}_{24}\cup\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\}$. Then develop the following blocks (mod 24) :

$$\begin{array}{ccccc}
\{0,1,2,\infty_1\} & \{0,5,7,14\} & \{1,2,20,23\} & \{3,4,5,\infty_1\} & \{3,4,12,22\} \\
\{6,11,17,\infty_2\} & \{6,17,19,\infty_2\} & \{7,12,22,\infty_3\} & \{8,11,18,\infty_3\} & \{8,15,21,\infty_4\} \\
\{9,15,19,\infty_4\} & \{9,16,19,\infty_5\} & \{10,13,20,\infty_5\} & \{10,16,21,\infty_6\} & \{12,14,23,\infty_6\}
\end{array}$$

along with the groups $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\}$ and $\{0,4,8,12,16,20\}$ developed by rotating through 0, 1, 2, 3 (mod 24) to form a 2-resolvable GDD of type 6^5 with index 6. Since each point occurs exactly twice in the blocks, these blocks form a total of 24 2-resolution classes. Expanding out the groups using the 2-resolvable $(6,4,6)$ BIBD displayed in Lemma 8.7 on the respective group points, forms the final five 2-resolution classes on all the points, since the 2-resolvable $(6,4,6)$ BIBD has a total of five 2-resolution classes. Also, since the starter blocks have each difference six times, except for the differences $\{4,8,12\}$, which the groups have exactly six times, we have $\lambda=6$ for the collection of blocks. In total, these blocks form the desired 2-resolvable $(30,4,6)$ BIBD. □

Lemma 8.13 *A 2-resolvable* $(38,4,6)$ *BIBD exists.*

Proof: Define the point set by $\mathbb{Z}_{30}\cup\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6,\infty_7,\infty_8\}$. Develop the difference system (mod 30) given by:

$$\begin{array}{ccc}
\{0,2,8,11\} & \{1,3,15,19\} & \{1,9,10,17\} \\
\{0,13,26,\infty_1\} & \{2,4,13,\infty_1\} & \{3,4,22,\infty_2\} \\
\{5,11,18,\infty_2\} & \{5,16,28,\infty_3\} & \{6,8,19,\infty_3\} \\
\{6,12,20,\infty_4\} & \{7,14,20,\infty_4\} & \{7,14,16,\infty_5\} \\
\{9,18,25,\infty_5\} & \{10,21,24,\infty_6\} & \{12,21,29,\infty_6\} \\
\{15,23,27,\infty_7\} & \{17,23,29,\infty_7\} & \{22,25,26,\infty_8\} \\
 & \{24,27,28,\infty_8\} &
\end{array}$$

Notice that each difference occurs in these blocks exactly six times, except for the differences $\{0,5,10\}$ which do not occur at all; the differences $\{1,3\}$ which occur four times; and the differences $\{2,4\}$ occur five times. Consider the block $P=\{0,1,3,4\}$ which contains the missing $\{1,2,3,4\}$ differences. Now, we take this block P as two partial 2-resolution classes; that is, take the two collections of blocks generated by $P_0=\{P+i \pmod{30} | i \equiv 0 \pmod{2}\}$ and $P_1=\{P+i \pmod{30} | i \equiv 1 \pmod{2}\}$, which have each element of $\mathbb{Z}_{30}$ exactly twice. Then, taking the groups to be $\{0,5,10,15,20,25\}$ developed through 0, 1, 2, 3, 4 (mod 30), along with $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6,\infty_7,\infty_8\}$ forms a GDD of type $6^5 8^1$ with index 6. By Lemma 8.7, there exists a 2-resolvable $(6,4,6)$ BIBD, which we will use on the groups of size 6. On the group of size 8, we use a 2-resolvable $(8,4,6)$ BIBD (which exists by Corollary 5.1). Pairing up these resolution classes formed by the groups leaves two 2-resolution classes on the infinity points. We can then pair up these remaining two 2-resolution classes with P_0 and P_1, to yield a total of 37 2-resolution classes, 30 from the blocks of the GDD, 5 from the groups, and 2 from the 2-parallel classes. Thus, this collection of blocks is a 2-resolvable $(38,4,6)$ BIBD. $\Box$

The following three lemmas focus on incomplete resolvable BIBDs, which are used in this paper as ingredients for recursive constructions. Recall that an incomplete resolvable BIBD is a triple $(\mathcal{V},\mathcal{W},\mathcal{B})$ with $\mathcal{W}\subset\mathcal{V}$ and $|B|=k$ for all blocks $B\in\mathcal{B}$, such that

- every pair of points of $\mathcal{V}\backslash\mathcal{W}$ occurs in λ blocks
- no block contains two elements from $\mathcal{W}$
- the blocks can be grouped into resolution classes or partial resolution classes missing $\mathcal{W}$

We call $\mathcal{W}$ the *hole* of the design. In most cases, we are only concerned with $|\mathcal{W}|$, which is referred to as the *size of the hole*. Thus, an incomplete resolvable BIBD is referred to as an α-resolvable $(|\mathcal{V}|,k,\lambda)$ BIBD with a hole of size $|\mathcal{W}|$.

Thus, we present three α-resolvable designs with a hole of size 2.

Lemma 8.14 *There exists a 2-resolvable $(10,4,6)$ BIBD with a hole of size two.*

Proof: Take the point set to be $\mathbb{Z}_8\cup\{\infty_1,\infty_2\}$. Take the starter blocks to be

$$\begin{array}{ccc} \{0,2,5,\infty_1\} & \{0,1,2,3\} & \{3,6,7,\infty_2\} \\ \{1,4,5,\infty_1\} & & \{4,6,7,\infty_2\} \end{array}$$

and rotate each block (mod 8). Additionally, rotate the block $\{0,2,4,6\}$ through a half-cycle; that is, rotate through $0,1,2,3 \pmod{8}$. Then, we have a total of eight 2-resolution classes (from the starter blocks) and one partial 2-resolution classes on all points not in the hole, namely on $\mathbb{Z}_8$. This yields each pair of points exactly six times, except for the pair $\{\infty_1,\infty_2\}$ which never occurs, and is the hole of size two. $\Box$

Lemma 8.15 *There exists a 2-resolvable* $(12, 4, 6)$ *BIBD with a hole of size two.*

Proof: Take the point set to be $\mathbb{Z}_5 \times \{0, 1\} \cup \{\infty_1, \infty_2\}$. Take the blocks to be the following starter blocks, and develop (mod 5),

$$\begin{array}{cc}
\{0_0, 0_1, 2_1, 4_1\} & \{0_0, 1_1, 3_1, 4_0\} \\
\{0_0, 0_1, 1_0, 1_1\} & \{0_0, 1_0, 2_0, 2_1\} \\
\{1_0, 2_0, 3_0, \infty_1\} & \{0_1, 1_0, 3_0, \infty_2\} \\
\{2_0, 3_1, 4_0, \infty_1\} & \{2_0, 4_0, 4_1, \infty_2\} \\
\{1_1, 3_0, 4_1, \infty_2\} & \{0_1, 1_1, 3_0, \infty_1\} \\
\{2_1, 3_1, 4_0, \infty_2\} & \{2_1, 3_1, 4_1, \infty_1\}
\end{array}$$

Additionally, rotate the block $\{0_0, 2_0, 0_1, 3_1\}$ through (mod 5). These blocks in total have each pair exactly six times, except for the $\{\infty_1, \infty_2\}$ pair, since the starter blocks have all differences exactly six times, except those differences found in the special block. It is worth noting that the starter blocks form ten 2-resolution classes, and the special block forms a single partial 2-resolution class on all points not in the hole. Thus, we have the desired design. $\square$

Lemma 8.16 *There exists a 2-resolvable* $(14, 4, 6)$ *BIBD with a hole of size two.*

Proof: Take the point set to be $\mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$. Take the blocks to be the following starter blocks, and develop (mod 12),

$$\begin{array}{ccc}
\{0, 1, 4, 9\} & \{2, 7, 8, 10\} & \{3, 5, 7, 8\} \\
\{0, 1, 2, \infty_1\} & \{3, 5, 9, \infty_1\} & \{4, 6, 11, \infty_2\} \\
 & \{6, 10, 11, \infty_2\} &
\end{array}$$

Additionally, include the blocks $\{0, 3, 6, 9\}$, $\{1, 4, 7, 10\}$, and $\{2, 5, 8, 11\}$ repeated twice. The developed starter blocks contain every pair, except the $\{\infty_1, \infty_2\}$ pair, exactly six times, except for those pairs that are contained in the special blocks. Now, the developed starter blocks create twelve 2-resolution classes, and the special blocks create a single partial 2-resolution class, which yields the desired design. $\square$

As final ingredients for the recursive constructions, we present several frames that will be used later in this section.

Lemma 8.17 *There exist* $(2; 4, 6)$*-frames of type* 2^5, 2^{13} *and* 2^{17}.

Proof: Since $5, 13$ and 17 are all prime powers, and all are congruent to 1 (mod 4), we can obtain a $(1; 4, 3)$-frame of type 2^5, 2^{13} and 2^{17} from Lemma 7.6. Duplicating each block within each resolution class doubles both λ and α, yielding the desired frames. $\square$

Lemma 8.18 *There exists a* $(2; 4, 6)$*-frame of type* 2^6.

Proof: Take a $(5,2)$-GDD of type 2^6 [11]. Replace each block of size five by a $(1;4,3)$-frame of type 1^5. This gives a $(2;4,6)$-frame of type 2^6. □

Lemma 8.19 *There exists a $(2;4,6)$-frame of type 2^7 and 2^{11}.*

Proof: Since 7 and 11 are both prime powers and both are congruent to 3 (mod 4), we can obtain a $(2;4,3)$-frame of type $2^7, 2^{11}$ from Lemma 7.7. Duplicating each resolution class, which doubles both λ and the number of resolution classes, yields the desired frames.□

We now have sufficient ingredients to complete the remaining designs of this section. We present a very useful lemma that allows us to derive α-resolvable designs from α-resolvable frames.

Lemma 8.20 *If there exists a $(2;4,6)$-frame of type $8^x 10^y 12^z (2v-8x-10y-12z)^1$, $x,y,z \geq 0$ and, if there exists a 2-resolvable $(2v-8x-10y-12z+2,4,6)$ BIBD, then there exists a 2-resolvable $(2v+2,4,6)$ BIBD.*

Proof: Define $a = v-4x-5y-6z$ and let F be a $(2;4,6)$-frame of type $8^x 10^y 12^z (2v-8x-10y-12z)^1$. To the group of size $2a = 2v-8x-10y-12z$, append two points $\{\infty_1, \infty_2\}$. Replace this appended group with the 2-resolvable $(2a+2,4,6)$ *BIBD*, which has a total of $(2a+1)$ 2-resolution classes. For the remaining groups, replace each group G with a 2-resolvable $(|G|+2,4,6)$ *BIBD* with a hole of size 2, where $|G| \in \{8,10,12\}$, on the point set $G \cup \{\infty_1, \infty_2\}$ where the hole is $\{\infty_1, \infty_2\}$. These holey designs exist by Lemmas 8.14, 8.15, and 8.16. Since each group G was associated with $\rho(G) = \frac{\lambda|G|}{\alpha(k-1)} = |G|$ partial 2-resolution classes on the point set outside of G, and since there are $|G|$ 2-resolution classes in the 2-resolvable $(|G|+2,4,6)$ with a hole of size 2, we can pair up each partial 2-resolution class associated with G on the point set outside of G with a 2-resolution class on the point set $G \cup \{\infty_1, \infty_2\}$. This leaves exactly one partial 2-resolution class on the point set G, namely the 2-resolution class that misses the hole (i.e. contains no infinite points), for each group G.

Now, we deal with the group of size $2a+2$. Before appending the two infinity points to the group of size $2a$, this group was associated with $2a$ partial 2-resolution classes on the $2v-2a$ points not in the group. Thus, we may pair up $(2a)$ 2-resolution classes of the 2-resolvable $(2a+2,4,6)$ BIBD with the $(2a)$ partial 2-resolution classes, leaving one 2-resolution class as yet unpaired. This remaining 2-resolution class on $2a+2$ points along with the remaining unpaired 2-resolution classes from the holey designs can be combined to form an additional 2-resolution class. In summation, we have $2a+8x+10y+12z+1 = 2v+1$ total 2-resolution classes in the collection of blocks, which is the desired number of 2-resolution classes for the 2-resolvable $(2v+2,4,6)$ BIBD.

All that is left to show is that $\lambda = 6$. We consider the following four cases:

- If $\{x,y\}$ are contained in a block of F, then they occur together exactly 6 times in the blocks, and do not occur together in the groups. Thus, after all groups are replaced, they still occur six times together in the blocks.

- If $\{x,y\}$ are contained in a group, G, of F, they do not occur in any other block or group of F. Replacing the group by a 2-resolvable $(|G|+2,4,6)$ BIBD, which may or may not have a hole of size 2, pairs up x and y exactly 6 times.

- If $x \in \{\infty_1,\infty_2\}$ and $y \notin \{\infty_1,\infty_2\}$, then $\{x,y\}$ are not paired together in any group or block of F. Since y is in exactly one group, G_y, the points x and y can only be paired together in the 2-resolvable $(|G_y|+2,4,6)$ BIBD, and hence, $\{x,y\}$ are paired together exactly 6 times.

- If $\{x,y\} = \{\infty_1,\infty_2\}$, then this pair must occur once in the group of size $2a+2$. Since this group is replaced with a 2-resolvable $(2a+2,4,6)$ BIBD, the pair $\{x,y\}$ occur together exactly 6 times.

Thus, this collection of blocks is the desired 2-resolvable $(2v+2,4,6)$ BIBD. □

We now use this lemma as an intermediate step to derive α-resolvable designs from PBDs.

Lemma 8.21 *If there exists a $PBD(v,\{5,6,7\},1)$, then there exists a 2-resolvable $(2v,4,6)$ BIBD.*

Proof: Let $P = PBD(v,\{5,6,7\},1)$. Delete a point X and create a $\{5,6,7\}$-GDD of type $4^x5^y6^z$ by taking the blocks of P that contain X as the groups and the blocks of P not containing X as the blocks. This forms a $\{5,6,7\}$-GDD of type $4^x5^y6^z$ on $v-1$ points

We may then apply the GDD construction (Theorem 4.1) with weight 2, since there exists $(2;4,6)$-frames of type 2^5, 2^6 and 2^7 by Lemmas 8.17, 8.18, and 8.19 respectively.

Hence, we get a $(2;4,6)$-frame of type $8^x10^y12^y$. Arbitrarily choose a group G as the "distinct group" of Lemma 8.20. By Lemmas 8.2, 8.8, and Corollary 5.1,we know there exists a 2-resolvable $(|G|+2,4,6)$ BIBD, where $|G| \in \{8,10,12\}$. Thus, we may apply Lemma 8.20 with F and the 2-resolvable $(|G|+2,4,6)$ BIBD to derive the desired 2-resolvable $(2v,4,6)$ BIBD. □

This lemma, along with previously proven lemmas, allow us to deduce the following result:

Theorem 8.22 *There exists a 2-resolvable $(v,4,6)$ BIBD for all $v \equiv 2,6 \pmod{12}$, $v \notin \{54,138,198,218\}$.*

Proof: The cases where $v \leq 38$ are covered by Lemmas 8.7, 8.8, 8.9, 8.11, 8.12, and 8.13. Lemma 8.21 along with results in [4] provide existence for all $v \notin \{54,66,138,186,198,218\}$. However, Lemma 8.10 proves existence for $v \in \{66,186\}$, leaving the remaining cases as stated. □

All that is left is to show existence of the remaining four cases. They are handled in the following four lemmas.

Lemma 8.23 *There exists a 2-resolvable $(54,4,6)$ BIBD.*

Proof: Since 5 is a prime number, there exists $TD(6;5)$. Delete 4 points from one group to get a $\{5,6\}$-GDD of type 1^15^5 by taking as groups the singleton group and any group that goes through a specific deleted point X. We may then inflate using Wilson's construction of Theorem 4.1, assigning every point weight 2. Since there exists $(2;4,6)$-frame of type 2^5 and 2^6 by Lemma 8.17 and 8.18, we indeed derive a $(2;4,6)$-frame of type 10^52^1. Since the block $\{1,2,3,4\}$ repeated 6 times yields a 2-resolvable $(4,4,6)$ BIBD, we apply Lemma 8.20 to form the desired 2-resolvable $(54,4,6)$ BIBD. □

Lemma 8.24 *There exists a 2-resolvable* $(138,4,6)$ *BIBD.*

Proof: We know there exists a $TD(6;13)$. Delete 10 points from one group. Pick a deleted point x, and take as a collection of groups of a GDD to be the group of size 3 and any block containing x. The remaining blocks and groups of the TD form the blocks of this $\{5,6,13\}$-GDD of type 3^15^{13}. Since there exists a $(2;4,6)$-frame of type 2^5, 2^6 and 2^{13} by Lemma 8.17 and 8.18, we can apply Theorem 4.1 with weight 2 for each point, which yields a $(2;4,6)$-frame of type $10^{13}6^1$.

Since there exists a 2-resolvable $(8,4,6)$ BIBD by Corollary 5.1, we can use this constructed frame and the 2-resolvable $(8,4,6)$ BIBD in Lemma 8.20 to yield a 2-resolvable $(138,4,6)$ BIBD. □

Lemma 8.25 *There exists a 2-resolvable* $(198,4,6)$ *BIBD.*

Proof: There exists a $TD(6;17)$ since 17 is prime. Delete 4 points from any group to form a $\{5,6,17\}$-GDD of type $5^{17}13^1$. Since there exists a $(2;4,6)$-frame of type 2^w for $w \in \{5,6,17\}$ by Lemmas 8.17 and 8.18, we can use Theorem 4.1 with weight 2 to form a $(2;4,6)$-frame of type $10^{17}26^1$. Since there exists a resolvable $(28,4,1)$ BIBD, we can replicate each resolution class 6 times, implying $\lambda' = 6$, and combine pairs of resolution classes to yield $\alpha' = 2$. Since the resolvable $(28,4,1)$ had 9 resolution classes, replicating by six and combining pairs yields the desired 27 2-resolution classes. This design and frame can be applied with Lemma 8.20 to yield the desired 2-resolvable $(198,4,6)$.□

Lemma 8.26 *There exists a 2-resolvable* $(218,4,6)$ *BIBD.*

Proof: There exists a $TD(7;17)$. Delete 11 points from one group to form a $\{6,7,17\}$-GDD of type 6^{18}. We know there exists a $(2;4,6)$-frame of type 2^w for $w \in \{6,7,17\}$ by Lemma 8.18, 8.19, and 8.17. Thus, we may apply Theorem 4.1 with weight 2 to form a $(2;4,6)$-frame of type 12^{18}. By Lemma 8.8, we know there exists a 2-resolvable $(14,4,6)$ BIBD. Using the frame and 2-resolvable $(14,4,6)$ BIBD, we may apply Lemma 8.20 to form the desired 2-resolvable $(218,4,6)$ BIBD. □

These results can be summarized into the following theorem:

Theorem 8.27 *If* $v \equiv 2,6 \pmod{12}$ *then there exists a 2-resolvable* $(v,4,6)$ *BIBD.*

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