Euclidean Bounded-Degree Spanning Tree Ratios

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ABSTRACT

Let τ_K be the worst-case (supremum) ratio of the weight of the minimum degree-K spanning tree to the weight of the minimum spanning tree, over all finite point sets in the Euclidean plane. It is known that $\tau_2 = 2$ and $\tau_5 = 1$. In STOC'94, Khuller, Raghavachari, and Young established the following inequalities: $1.103 < \tau_3 \leq 1.5$ and $1.035 < \tau_4 \leq 1.25$. We present the first improved upper bounds: $\tau_3 < 1.402$ and $\tau_4 < 1.143$. As a result, we obtain better approximation algorithms for Euclidean minimum bounded-degree spanning trees.

Let $\tau_K^{(d)}$ be the analogous ratio in d-dimensional space. Khuller *et al.* showed that $\tau_3^{(d)} < 1.667$ for any d. We observe that $\tau_3^{(d)} < 1.633$.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*computations on discrete structures, geometrical problems and computations*; G.2.2 [**Discrete Mathematics**]: Graph Theory—*graph algorithms, trees*

General Terms

Algorithms, Theory

Keywords

Minimum spanning trees, discrete geometry, approximation

1. INTRODUCTION

The starting point of this work is the following well-known observation [5, 19]: for finite point sets in any metric space, we can construct a spanning path (or cycle) of at most twice the weight of the minimum spanning tree (MST), by doubling the MST edges, taking an Euler tour, and short-cutting repeated vertices. This strategy yields a simple factor-2 approximation algorithm for the traveling salesman path/tour problem. It was shown by Fekete *et al.* [8] that even in the geometric case of the Euclidean metric in the plane, the analysis cannot be improved upon (in other words, short-cutting doesn't help much in the worst case), as there exists point sets whose traveling salesman path weight is more than $2 - \varepsilon$ times the MST weight for any $\varepsilon > 0$. However, by using other lower bounds besides the MST weight, it is possible to obtain better approximation guarantees for the traveling salesman problem, as was demonstrated by Christofides [4] (factor 3/2) for general metrics, and Arora [1] and Mitchell [13] (factor $1 + \varepsilon$) for the Euclidean metric in fixed dimensions.

The focus of the present paper is on the following generalization of the traveling salesman path problem (which corresponds to the $K = 2$ case): given K, find a spanning tree, of minimum weight such that the maximum degree is at most K. The degree constraint is natural to consider, since high-degree nodes in networks are in many ways undesirable. The $K = 3$ case is especially appealing, since once rooted, a degree-3 tree becomes a binary tree.

For this bounded-degree spanning tree problem, Christofides' algorithm no longer gives a 3/2 approximation factor; the celebrated techniques of Arora and Mitchell do not seem to work either [2]. We thus return to the idea of constructing a solution by traversing the MST and analyzing the weight of the solution as a factor of the MST weight. The doubling strategy still applies; in fact, it is possible, using the triangle inequality alone, to get an approximation factor of $2 - (K - 2)/(K_{\max} - 2)$ [8], where $K_{\max}$ is the maximum degree of an MST, thus showing that "short-cutting" does help for $K \geq 3$. The analysis is tight for arbitrary metric spaces. For the Euclidean metric in the plane, every point set already possesses an MST of maximum degree 5 [14], so this yields factors 5/3 and 4/3 for $K = 3$ and $K = 4$ respectively.

In as early as 1984, Papadimitriou and Vazirani [15] asked whether the geometry of the Euclidean case (besides the triangle inequality) can be exploited to prove better approximation factors for bounded-degree spanning trees. Khuller, Raghavachari, and Young [10] took an in-depth look into this question and managed to achieve factors 3/2 and 5/4 for $K = 3$ and $K = 4$ respectively in the plane. Since then, no improvements have been made, despite frequent references to their work [2, 3, 7, 8, 11, 16, 17].

We report the first progress in 8 years: in the Euclidean plane, there always exists degree-3 and degree-4 spanning

*Work supported in part by NSERC.

SoCG'03, June 8–10, 2003, San Diego, California, USA.

trees with weights within factors 1.402 and 1.143 respectively of the MST weight. Such trees can be constructed in polynomial time.

Immediately, we obtain a factor-1.402 and factor-1.143 approximation algorithm for the minimum Euclidean degree-3 and degree-4 spanning tree problem in the plane. Note that Papadimitriou and Vazirani [15] have shown the NP-hardness of the minimum Euclidean degree-3 spanning tree problem, but the status of the corresponding degree-4 problem remains open. However, regardless of algorithmic implications, our result is important in that it provides new information on a universal constant (the largest ratio of the minimum degree-3/-4 spanning tree weight to the MST weight) similar to the the Steiner ratio (the smallest ratio of the minimum Steiner tree weight to the MST weight) [6] and other constants studied in discrete geometry (such as [9]).

The new algorithms are not complicated and involve some interesting, cleverer recursive tree constructions. Their analyses, though, require more cases and demand techniques more versatile than those of Khuller *et al.*'s; still, with proper planning, we get proofs that are (hopefully) not too difficult to verify. We briefly review Khuller *et al.*'s previous algorithm in the next section and explain why 3/2 and (a value close to) 5/4 are particularly difficult barriers to break. In Sections 3 and 4 we present the new recursive algorithms and analyses.

The study of these worst-case ratios in d-dimensional Euclidean space is even more vital, because the maximum degree of an MST can be much larger (a constant that depends exponentially on d [18]). In their paper, Khuller *et al.* [10] analyzed a simple algorithm and proved a remarkable 5/3 upper bound for degree-3 spanning trees in *any* number of dimensions. In Section 5, we mention how the bound can be reduced slightly to $\frac{2}{3}\sqrt{6} < 1.633$ using essentially the same algorithm.

2. KHULLER, RAGHAVACHARI, AND YOUNG'S APPROACH

To facilitate comparisons, we begin with a recursive interpretation of Khuller *et al.*'s approach [10].

2.1 Degree-3 spanning trees

We are given an MST T of an n-point set in the plane, which we may assume [14] has maximum degree at most 5. Let $w(T)$ denote the sum of the edge weights in T. Root T at a fixed non-degree-5 vertex (e.g., a leaf) so that each vertex has at most 4 children.

Khuller *et al.*'s approach can be viewed as a recursive algorithm that transforms the rooted tree T into a new degree-3 spanning tree, with the inductive hypothesis that *the root v of T should have degree 1 in the new tree.* The algorithm is simple:

> First pick a permutation $v_1, \ldots, v_k$ of the children of v in T; recursively transform the subtrees $T_1, \ldots, T_k$ rooted at $v_1, \ldots, v_k$; finally, add the edges $vv_1, v_1v_2, \ldots, v_{k-1}v_k$ to the new tree, as illustrated by the diagram below.

Showing that the new spanning tree has weight at most $1.5w(T)$ amounts to showing at every step the existence of

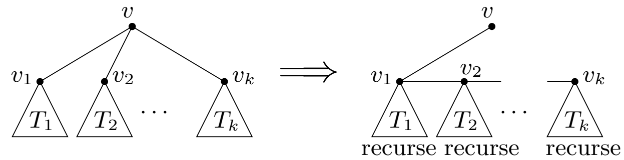


Figure 1: Khuller *et al.*'s degree-3 algorithm. (Pictures drawn graph-theoretically, not geometrically.)

a "good" permutation $v_1, \ldots, v_k$ satisfying the inequality

$$|vv_1| + |v_1v_2| + \cdots + |v_{k-1}v_k| \le 1.5 \sum_{i=1}^{k} |vv_i|,$$

or more loosely, that the ratio of the weight of some path starting at v to the weight of a star is bounded by 1.5. Theoretically, the proof can be carried out in $O(1)$ time because $k \le 4$ only (and the theory of the reals is decidable). Khuller *et al.* used a combination of nontrivial ideas in order to obtain a "presentable" proof.

2.2 Degree-4 spanning trees

Khuller *et al.*'s algorithm for producing a degree-4 spanning tree of weight at most $1.25w(T)$ is similar, except for a weakened inductive hypothesis: *the root v of T now should have degree at most 2 in the new tree.* Instead of adding a path that must start at v, we can now add any path visiting $v, v_1, \ldots, v_k$ to the new tree, as in the diagram below.

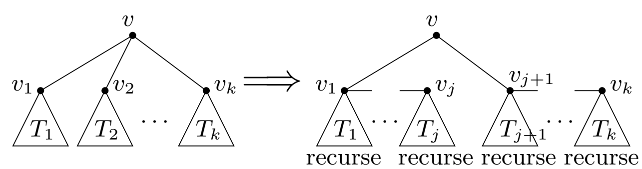


Figure 2: Khuller *et al.*'s degree-4 algorithm.

In the analysis, it is shown that the ratio of some such path to the weight of the star is bounded by 1.25, again via a careful case study.

2.3 Limits to the approach

The 1.5 bound for degree-3 spanning trees is tight if we insist that the designated root must have degree 1 in the new tree. The example in Figure 3(a) indicates why. Even if all vertices with 2 children were to magically disappear, we still have the configurations of Figure 3(d) (which requires a ratio of $\frac{1}{3}(2\sqrt{3}+1) > 1.488$) and Figure 3(b,c) (where paths starting at v require a ratio arbitrarily close to 1.5) to contend with.

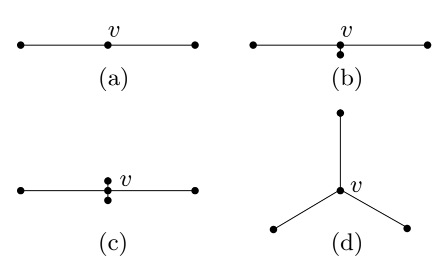


Figure 3: Bad examples.

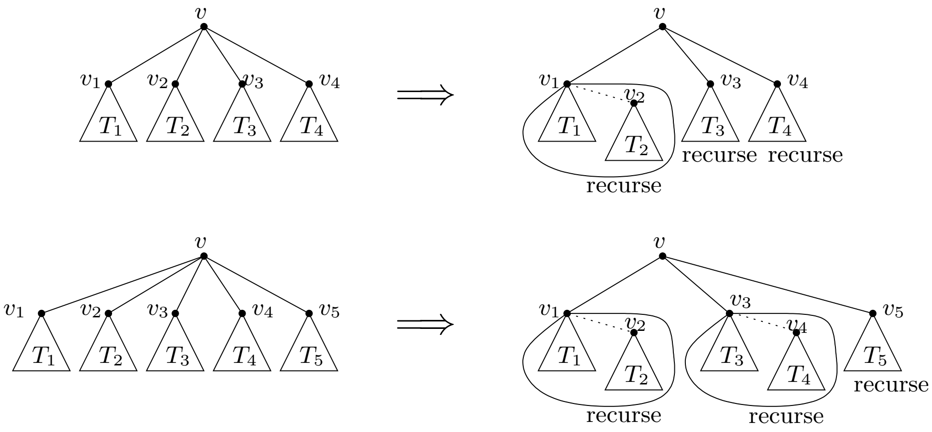


Figure 4: The new degree-4 algorithm in a nutshell.

Khuller *et al.* did not claim that their 1.25 bound is tight for degree-4 spanning trees, but even if their analysis could be refined, the improvement would be marginal, because the example in Figure 3(d) requires a ratio of $\frac{1}{3}(\sqrt{3}+2) > 1.244$, under the condition that the designated root has degree at most 2 in the new tree.

So, in order to get better results, we need to relax the inductive hypotheses. For instance, as Khuller *et al.* have noticed, in the recursion to the subtree T_k (but not the other subtrees) in Figure 1, we could transform its root to have degree 2 instead of 1. Similarly, in the recursion to the subtrees T_j and T_k (but not the others) in Figure 2, we could transform their roots to have degree 3 instead of 2. Unfortunately, such refinements do not necessarily translate to improved approximation factors in the worst case. Alternatively, we could try to exploit "upward" information (the parent) instead of just "downward" information (the children) at each vertex v; Khuller *et al.* even proposed rerooting T at different vertices. Again, it is unclear how to get general worst-case improvements this way.

3. BETTER DEGREE-4 SPANNING TREES IN THE PLANE

Let $\tau = 1.143$ in this section.

We describe our result for degree-4 spanning trees first, as it is simpler to explain. Our approach is indeed to adopt a weaker inductive hypothesis that consistently permits the root v to have degree at most 3 instead of 2 in the new tree. To do so, we have to recurse not just on subtrees of the original MST, but on subtrees "attached to" other subtrees. The key technical idea is to strengthen the weakened hypothesis by forcing special "attachment" edges to cost less, with factor 1 (instead of τ) in the analysis.

3.1 The new approach

In the sequel, let $T\nwarrow T'$ denote the (rooted) tree obtained by making the root of T' a child of T. Given $T\nwarrow T'$ where T and T' are subtrees of the original MST, with roots v and v', we describe a recursive algorithm that transforms $T\nwarrow T'$ to a new tree, such that *the root v has degree at most 3 in the new tree, and the new tree has weight at most* $|vv'| + \tau(w(T) + w(T'))$.

The algorithm works basically as in Figure 4. Pick a permutation $v_1, \ldots, v_{k+1}$ of the k children of v in T together with v'. Let $T_1, \ldots, T_{k+1}$ be their corresponding subtrees.

- **Case** $k \leq 2$**.** Just transform $T_1, \ldots, T_{k+1}$ recursively and leave the edges $vv_1, \ldots, vv_{k+1}$ in.
- **Case** $k = 3$**.** We can transform $T_1\nwarrow T_2$, T_3, and T_4 recursively and put in the edges vv_1, vv_3, vv_4. By hypothesis, the weight of the resulting tree is bounded by $|vv_1| + |v_1v_2| + |vv_3| + |vv_4| + \tau\sum_{i=1}^{4} w(T_i)$. Call $|v_1v_2| - |vv_2|$ the *excess* of the permutation $v_1, \ldots, v_4$.
- **Case** $k = 4$**.** We can transform $T_1\nwarrow T_2$, $T_3\nwarrow T_4$, and T_5 recursively and put in the edges vv_1, vv_3, vv_5. By hypothesis, the weight of the resulting tree is bounded by $|vv_1| + |v_1v_2| + |vv_3| + |v_3v_4| + |vv_5| + \tau\sum_{i=1}^{5} w(T_i)$. Call $|v_1v_2| - |vv_2| + |v_3v_4| - |vv_4|$ the *excess* of the permutation $v_1, \ldots, v_5$.

It suffices to choose a permutation with excess smaller than $(\tau - 1)\sum_{v_i \neq v'} |vv_i|$. We will prove that such a permutation always exists for both $k = 3$ and $k = 4$.

Clearly, the algorithm runs in linear time, given the MST (which can be constructed in $O(n \log n)$ time [14]).

3.2 Preliminaries for the proof

Our analysis, though somewhat lengthy (due to our desire to obtain the lowest constant), relies on very elementary tools—just an angle-sensitive version of the triangle inequality, and a useful min trick:

LEMMA 3.1. *If a triangle has sides x, y, z with $x \leq y$, and the angle opposite z is θ, then*

$$z \leq f(\theta)x + y, \text{ where } f(\theta) := \max\{2\sin(\theta/2) - 1, 0\}.$$

PROOF. When $x = y$, we have $z = 2\sin(\theta/2)x$. As y increases by δ while x is fixed, z can increase by at most δ. □

LEMMA 3.2. *If $a_1, \ldots, a_m \geq 0$, then*

$$\min\{a_1x_1, \ldots, a_mx_m\} \leq \tfrac{1}{m}\,\text{H.M.}\{a_1, \ldots, a_m\}\,(x_1 + \cdots + x_m),$$

where H.M. *denotes the Harmonic mean.*

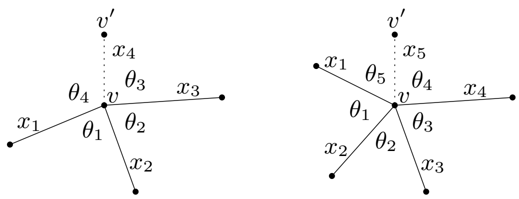
PROOF. Just take a convex combination:

$$\min\{a_1x_1, \ldots, a_mx_m\} \leq \alpha_1a_1x_1 + \cdots + \alpha_ma_mx_m,$$

with $\alpha_i = (1/a_i)/[1/a_1 + \cdots + 1/a_m]$. □

3.3 The analysis

Case $k = 3$**.** Let v_a, v_b, v_c be the children of v in T sorted by angle, with v' between v_a and v_c. Let $x_1 = |vv_a|$, $x_2 = |vv_b|$, $x_3 = |vv_c|$, $x_4 = |vv'|$, $\theta_1 = \angle v_a v v_b$, $\theta_2 = \angle v_b v v_c$, $\theta_3 = \angle v_c v v'$, and $\theta_4 = \angle v' v v_a$, as in Figure 5(a).



(a) $k = 3$ (b) $k = 4$

Figure 5: Notation for the degree-4 analysis.

We want to show that some permutation has excess less than $(\tau - 1)(x_1 + x_2 + x_3)$. By Lemma 3.1, we have permutations with excesses bounded by these numbers:

$$\begin{array}{ll} f(\theta_1)\min\{x_1, x_2\}, & f(\theta_2)\min\{x_2, x_3\}, \\ f(\theta_3)\min\{x_3, x_4\}, & f(\theta_4)\min\{x_4, x_1\}. \end{array}$$

By Lemma 3.2, the minimum excess is at most

$$\tfrac{1}{3}\,\text{H.M.}\{\, \min\{f(\theta_4), f(\theta_1)\}, \min\{f(\theta_1), f(\theta_2)\}, \min\{f(\theta_2), f(\theta_3)\}\,\}\,(x_1 + x_2 + x_3).$$

Since $\min\{\theta_4, \theta_1\} + \min\{\theta_1, \theta_2\} + \min\{\theta_2, \theta_3\} \le 270°$, it can be verified (as the H.M. is no more than the arithmetic mean and f is concave on the interval of interest) that the above coefficient is bounded by $\frac{1}{3}f(90°) = (\sqrt{2} - 1)/3 < 0.139$.

Case $k = 4$**.** Let v_a, v_b, v_c, v_d be the children of v in T sorted by angle, with v' between v_a and v_d. Let $x_1 = |vv_a|$, $x_2 = |vv_b|$, $x_3 = |vv_c|$, $x_4 = |vv_d|$, $x_5 = |vv'|$, $\theta_1 = \angle v_a v v_b$, $\theta_2 = \angle v_b v v_c$, $\theta_3 = \angle v_c v v_d$, $\theta_4 = \angle v_d v v'$, and $\theta_5 = \angle v' v v_a$, as in Figure 5(b). Because angles between two adjacent MST edges must exceed $60°$ (e.g., see [14]), we know that $\theta_1, \theta_2, \theta_3, \theta_4 + \theta_5 \ge 60°$. Furthermore, because one of the MST edge at v (the parent) is not present in T, we also know the following fact, which will be helpful later (though not necessary to get a new result):

$$\max\{\theta_1, \theta_2, \theta_3, \theta_4 + \theta_5\} \ge 120°. \qquad (1)$$

We want to show that some permutation has excess smaller than $(\tau - 1)(x_1 + x_2 + x_3 + x_4)$. By Lemma 3.1, we have permutations with excesses bounded by

$$\begin{array}{c} f(\theta_1)\min\{x_1, x_2\} + f(\theta_3)\min\{x_3, x_4\}, \\ f(\theta_2)\min\{x_2, x_3\} + f(\theta_4)\min\{x_4, x_5\}, \\ f(\theta_3)\min\{x_3, x_4\} + f(\theta_5)\min\{x_5, x_1\}, \\ f(\theta_4)\min\{x_4, x_5\} + f(\theta_1)\min\{x_1, x_2\}, \\ f(\theta_5)\min\{x_5, x_1\} + f(\theta_2)\min\{x_2, x_3\}, \\ f(\theta_3 + \theta_4)\min\{x_3, x_5\} + f(\theta_4 + \theta_5)\min\{x_4, x_1\}, \\ f(\theta_4 + \theta_5)\min\{x_4, x_1\} + f(\theta_5 + \theta_1)\min\{x_5, x_2\}. \end{array}$$

Consider three subcases (the missing one is symmetric):

- Subcase $\theta_4, \theta_5 \le 60°$. Then $f(\theta_4) = f(\theta_5) = 0$. By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{4}\,\text{H.M.}\{\, f(\theta_1),\ \min\{f(\theta_1), f(\theta_2)\}, \min\{f(\theta_2), f(\theta_3)\},\ f(\theta_3)\,\}\,(x_1 + x_2 + x_3 + x_4).$$

  Since $\theta_1 + \min\{\theta_1, \theta_2\} + \min\{\theta_2, \theta_3\} + \theta_3 \le 400°$, it can be verified that the above coefficient is bounded by $\frac{1}{4}f(100°) < 0.134$.

- Subcase $\theta_4 \ge 60°$, $\theta_5 \le 60°$. Here, $f(\theta_5) = 0$. By (1), we have the following possibilities:

  – *Subsubcase* $\theta_1 \ge 120°$ *or* $\theta_4 + \theta_5 \ge 120°$. By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{3}\,\text{H.M.}\{f(\theta_2), f(\theta_2), f(\theta_3)\}\,(x_2 + x_3 + x_4).$$

  Since $\theta_2 + \theta_3 \le 360° - 120° - 60° = 180°$, it can be confirmed numerically that the above coefficient is maximized near $\theta_2 \approx 95°$, $\theta_3 \approx 85°$ and is below 0.142.

  – *Subsubcase* $\theta_2 \ge 120°$. By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{3}\,\text{H.M.}\{\max\{f(\theta_4 + \theta_5), f(\theta_5 + \theta_1)\},\ f(\theta_3),\ f(\theta_3)\}\,[(x_1 + x_2) + x_3 + x_4].$$

  Since $\max\{\theta_4 + \theta_5, \theta_5 + \theta_1\} + \theta_3 \le 360° - 120° - 60° = 180°$, the above coefficient is again bounded by 0.142.

  – *Subsubcase* $\theta_3 \ge 120°$. By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{3}\,\text{H.M.}\{\max\{f(\theta_4), f(\theta_1)\},\ f(\theta_2),\ f(\theta_2)\}\,[(x_4 + x_1) + x_2 + x_3].$$

  Since $\max\{\theta_4, \theta_1\} + \theta_2 \le 360° - 120° - 60° = 180°$, the above coefficient is also bounded by 0.142.

- Subcase $\theta_4, \theta_5 \ge 60°$. Consider which of the angles $\theta_1, \ldots, \theta_5$ is the largest (the missing subsubcases are symmetric):

  – *Subsubcase:* θ_1 *is the largest.* By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{2}\,\text{H.M.}\{\max\{f(\theta_3), f(\theta_5)\}, \max\{f(\theta_2), f(\theta_4)\}\}\,[(x_3 + x_1) + (x_2 + x_4)].$$

  Since $\max\{\theta_3, \theta_5\} + \max\{\theta_2, \theta_4\} \le \min\{2\theta_1,\ 360° - 60° - 60° - \theta_1\} \le 160°$, it can be verified that the above coefficient is bounded by $\frac{1}{2}f(80°) < 0.143$.

  – *Subsubcase:* θ_5 *is the largest.* By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{2}\,\text{H.M.}\{\max\{f(\theta_1), f(\theta_3)\}, \max\{f(\theta_2), f(\theta_4)\}\}\,[(x_1 + x_3) + (x_2 + x_4)].$$

  By a similar argument, the coefficient is again bounded by 0.143.

  – *Subsubcase:* θ_2 *is the largest.* By Lemma 3.2, the minimum excess is at most

  $$\tfrac{1}{2}\,\text{H.M.}\{\max\{f(\theta_3), f(\theta_5)\}, \max\{f(\theta_4), f(\theta_1)\}\}\,[(x_3 + x_1) + (x_4 + x_2)].$$

  The coefficient is again bounded by 0.143.

A ratio of 1.143 has thus been established.

Remark: There might be room for improvement in the last subcase, by a more detailed case analysis, or by not bounding distances linearly with Lemma 3.1 (which is tight only when $x = y$). The room would be small though, considering that our analysis for the $k = 3$ case (with ratio > 1.138) is tight under our inductive hypothesis.

4. BETTER DEGREE-3 SPANNING TREES IN THE PLANE

Let $\tau = 1.402$ in this section.

Logically, our approach for degree-3 spanning trees should adopt a similar relaxed condition where the root v is permitted to have degree 2 in the new tree, instead of 1 as in Khuller *et al.*'s algorithm. Unfortunately, we now face many new obstacles, and it took some time before we find an inductive hypothesis we feel comfortable analyzing. The idea is to force not just one attachment edge, but a path of attachment edges, to cost less, with factor 1 (instead of τ) in the analysis. Additionally, a technical complication arises because of the need to recurse on general trees, not necessarily subtrees of the MST.

4.1 The new approach

Interpret $\nwarrow$ as a right-to-left associative operator. Given a (finite) sequence of trees $T, T', T'', \ldots$, with roots $v, v', v'', \ldots$, we describe an algorithm that transforms $T \nwarrow T' \nwarrow T'' \nwarrow \cdots$ to a new tree, such that *the root v has degree at most 2 in the new tree, and the new tree has weight at most $|vv'| + |v'v''| + \cdots + \tau(w(T) + w(T') + w(T'') + \cdots)$.*

The algorithm works according to one of several schemes depicted in Figure 6. Pick a permutation $v_1, \ldots, v_k$ of the k children of v in T. Let $T_1, \ldots, T_k$ be their corresponding subtrees.

- If $|v_{k-1}v_k| \leq |vv_k|$, then we apply Scheme A: remove edge vv_k and insert $v_{k-1}v_k$ to lower the root's degree, and repeat.
- If $\min\{|vv_k| + |v_kv'|, |vv'| + \tau|v_kv'|\} \leq \tau|vv_k| + |vv'|$, then we apply Scheme B: pull out T_k from T to get a tree $\hat{T}$ with lower root degree, and depending on which of $|vv_k| + |v_kv'|$ and $|vv'| + \tau|v_kv'|$ is smaller, take either $\hat{T} \nwarrow T_k \nwarrow T' \nwarrow T'' \nwarrow \cdots$ or $\hat{T} \nwarrow (T' \nwarrow T_k) \nwarrow T'' \nwarrow \cdots$ and repeat.
- If the above schemes are not applicable for any permutation, we can consider two recursive schemes. In Scheme C, we recursively transform $T_1 \nwarrow \cdots \nwarrow T_k$ and $T' \nwarrow T'' \nwarrow \cdots$ and put in the edges vv_1 and vv'. By hypothesis, the weight of the resulting tree is bounded by cost $+ |vv'| + |v'v''| + \cdots + \tau\left(\sum_{i=1}^{k} w(T_i) + w(T') + w(T'') + \cdots\right)$, where we define
$$\text{cost} := |vv_1| + |v_1v_2| + \cdots + |v_{k-1}v_k|.$$
- In Scheme D, we recursively transform $T_1 \nwarrow \cdots \nwarrow T_{k-1}$, and depending on which of $|vv_k| + |v_kv'|$ and $|vv'| + \tau|v_kv'|$ is smaller, we either recursively transform $T_k \nwarrow T' \nwarrow T'' \nwarrow \cdots$ and put in vv_k, or recursively transform $(T' \nwarrow T_k) \nwarrow T'' \nwarrow \cdots$ and put in vv'. (This is best understood pictorially, with the aid of Figure 6.) By hypothesis, the weight of the resulting tree is bounded by cost$+ |vv'| + |v'v''| + \cdots + \tau\left(\sum_{i=1}^{k} w(T_i) + w(T') + w(T'') + \cdots\right)$, where we define
$$\begin{aligned}\text{cost} \;:=\; & |vv_1| + |v_1v_2| + \cdots + |v_{k-2}v_{k-1}| + \\ & \min\{|vv_k| + |v_kv'|, |vv'| + \tau|v_kv'|\} - |vv'|.\end{aligned}$$

It suffices to choose a permutation and a scheme such that the cost is at most $\tau \sum_{i=1}^{k} |vv_i|$. We will prove that such a choice exists whenever Schemes A and B are not applicable.

Note that if Schemes A and B are not applicable, then $k \leq 4$ (see Section 4.3). Testing/handling Schemes A and B requires time proportional to the degree k, which is at most n, and so the algorithm can be implemented in quadratic time. By being careful in how to apply Scheme A (details in full paper), it is possible to always keep the degree k below a constant, and thus the algorithm can actually be implemented in linear time given the MST.

4.2 Preliminaries for the proof

The degree-3 analysis requires a different set of tools. At some point we make use of an alternative triangle inequality—Lemma 4.1 below—which refines Lemma 3.1 when $\theta < 90°$, provided that z is the largest side. Lemma 4.2 deals with an expression on triangle sides that occurs already in our definition of cost. Finally, Lemma 4.3 comes in handy in bounding linear expressions.

LEMMA 4.1. *If a triangle has sides x, y, z with $x \leq y \leq z$, and the angle opposite z is θ, then*
$$z \leq G(\theta)x + (F(\theta) - G(\theta))y,$$
where $F(\theta) := 2\sin(\theta/2)$ and $G(\theta) := 1/(F(\theta) + 1)$.

PROOF. $z \geq y$ implies that $x \geq 2y\cos\theta$. The above is an equality when $x = 2y\cos\theta$, because of the identity $2G(\theta)\cos\theta + F(\theta) - G(\theta) = 1$. We also have equality when $x = y$. Since z is a convex function of x for a fixed y, the inequality holds for all x between $2y\cos\theta$ and y. $\square$

LEMMA 4.2. *If a triangle has sides x, y, z, and the angle opposite z is θ, then*
$$\min\{x + z, y + \tau z\} \leq H(\theta)x + y,$$
where
$$\begin{aligned} H(\theta) &:= \frac{\tau}{\tau - 1}\left[1 - J(\theta) + \sqrt{J(\theta)^2 - 1}\right], \\ J(\theta) &:= \frac{1 - (\tau - 1)^2\cos\theta}{1 - (\tau - 1)^2}. \end{aligned}$$

PROOF. W.l.o.g., say $x = 1$. The maximum of
$$\min\left\{1 + \sqrt{y^2 + 1 - 2y\cos\theta},\, y + \tau\sqrt{y^2 + 1 - 2y\cos\theta}\right\} - y$$
occurs when the two min terms coincide; its value can be found by solving a quadratic equation. $\square$

LEMMA 4.3. *If $0 \leq x_1 \leq x_2 \leq \cdots \leq x_m$, then*
$$\begin{aligned} & a_1x_1 + \cdots + a_mx_m \\ \leq\; & \max\left\{a_m,\, \tfrac{1}{2}(a_{m-1} + a_m),\, \ldots,\, \tfrac{1}{m}(a_1 + \cdots + a_m)\right\} \\ & (x_1 + \cdots + x_m). \end{aligned}$$

PROOF. Trivially it is true when $x_1 = x_2 = \cdots = x_m$. It remains true when $x_2, \ldots, x_m$ increase simultaneously by a common amount. Etc. $\square$

4.3 The analysis

Suppose that Scheme A is not applicable. Then $|v_iv_j| > |vv_i|, |vv_j|$ for all i, j. In particular, this implies that all angles $\angle v_ivv_j$ among children of v in T exceed $60°$. In addition, it validates the subsequent applications of Lemma 4.1.

Suppose further that Scheme B is not applicable. This implies that the angle $\angle v_ivv'$ between any child of v in T and

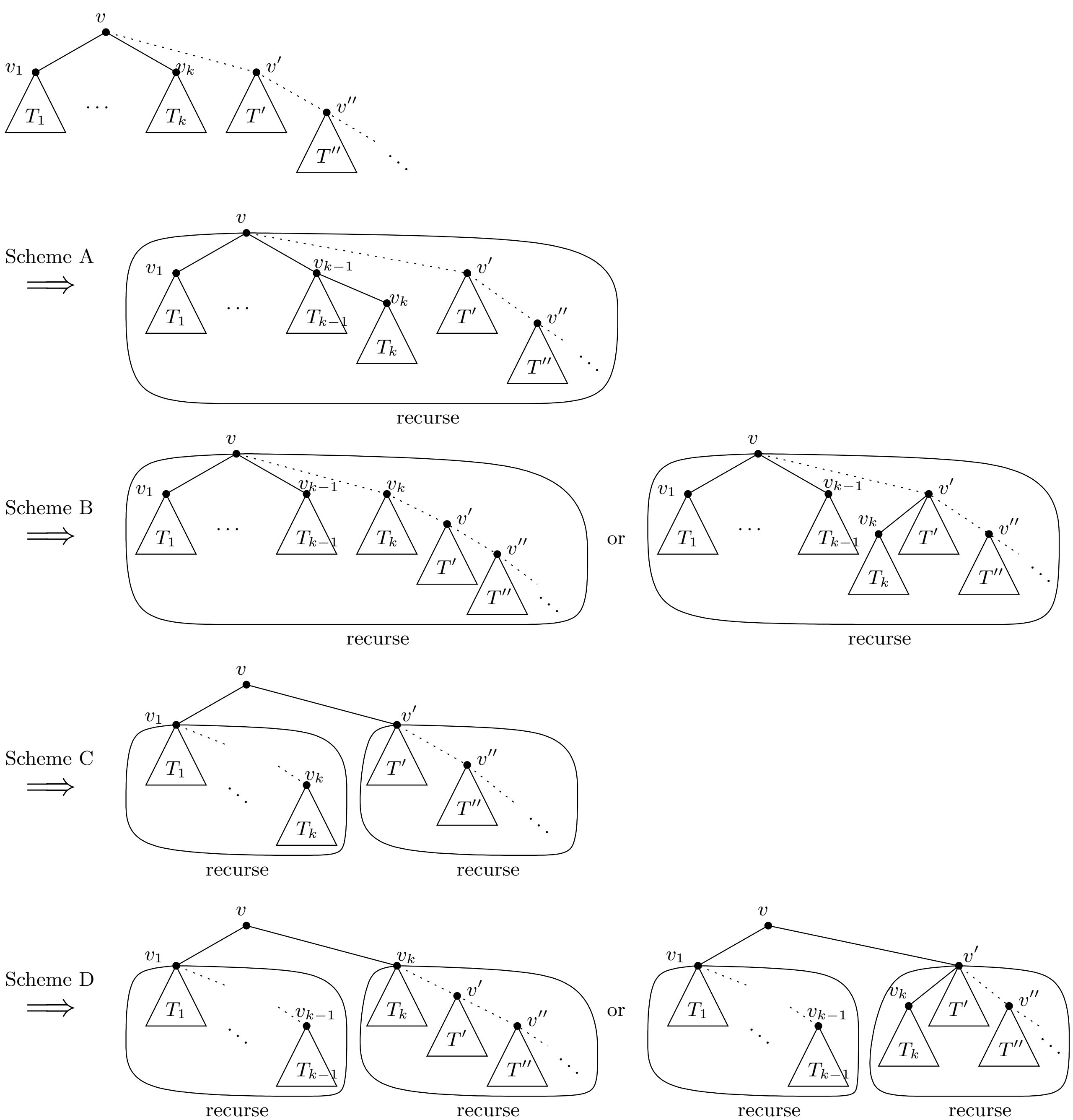


Figure 6: The new degree-3 algorithm in picture form.

v' must exceed 72°, because of Lemma 4.2, since $H(72^\circ) < 1.395$.

Thus, $k \le 4$. The $k = 1$ case is trivial. We consider the $k = 2$, $k = 3$, and $k = 4$ cases separately. Before plunging into the details, some words of caution: The present proof is less elegant than the previous proof, because the algorithm is now more involved and there is less symmetry. On the other hand, when required, we only take the simplest convex combinations, i.e., averages, in contrast to the fancier ones used in Lemma 3.2. The $k = 2$ case turns out to be the "critical" case, and so we can afford to be looser in our estimates for $k = 3$ and $k = 4$—otherwise, a complete analysis, especially for the $k = 4$ case, would be even more daunting.

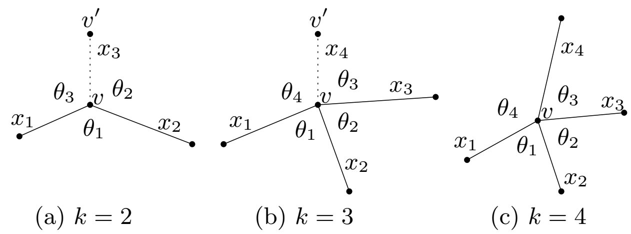


Figure 7: Notation for the degree-3 analysis.

Case $k = 2$. Let v_a, v_b be the children of v in T. Let $x_1 = |vv_a|$, $x_2 = |vv_b|$, $\theta_1 = \angle v_a v v_b$, $\theta_2 = v_b v v'$, and $\theta_3 = v' v v_a$, as in Figure 7(a). W.l.o.g., say $x_1 \le x_2$.

We want to show that Scheme C or D under some permutation has cost at most $\tau(x_1 + x_2)$.

- Subcase $\theta_1 \le 128.6^\circ$. By Lemma 3.1, Scheme C yields cost bounded by

$$F(\theta_1)x_1 + x_2 \;\le\; \max\left\{1, \tfrac{1}{2}(F(\theta_1)+1)\right\}(x_1 + x_2),$$

using Lemma 4.3. The above coefficient is at most $\frac{1}{2}(F(128.6^\circ)+1) < 1.402$.

- Subcase $\theta_3 \le 115.7^\circ$. By Lemma 4.2, Scheme D yields cost bounded by

$$x_2 + H(\theta_3)x_1 \;\le\; \max\left\{1, \tfrac{1}{2}(H(\theta_3)+1)\right\}(x_1 + x_2),$$

using Lemma 4.3. The above coefficient is at most $\frac{1}{2}(H(115.7^\circ)+1) < 1.402$.

- Subcase $\theta_1 \ge 128.6^\circ$, $\theta_3 \ge 115.7^\circ$. By Lemma 4.2, Scheme D yields costs at most

$$x_2 + H(\theta_3)x_1, \quad x_1 + H(\theta_2)x_2.$$

Taking the average and applying Lemma 4.3 bounds the minimum cost by

$$\max\left\{\tfrac{1}{2}(H(\theta_2)+1),\, \tfrac{1}{4}(H(\theta_2)+H(\theta_3)+2)\right\}(x_1 + x_2).$$

Because $\theta_2 \le 115.7^\circ$ and $\theta_2 + \theta_3 \le 231.4^\circ$, both max terms are again at most $\frac{1}{2}(H(115.7^\circ)+1) < 1.402$.

Case $k = 3$. This case is the lengthiest.

Let v_a, v_b, v_c be the children of v in T, with v' between v_a and v_c in angle around v. Let $x_1 = |vv_a|$, $x_2 = |vv_b|$, $x_3 = |vv_c|$, $\theta_1 = \angle v_a v v_c$, $\theta_2 = \angle v_b v v_c$, $\theta_3 = \angle v_c v v'$, and $\theta_4 = \angle v' v v_a$, as in Figure 7(b). W.l.o.g., say $x_1 \le x_3$.

We want to show that Scheme C or D under some permutation has cost at most $\tau(x_1 + x_2 + x_3)$.

- Subcase $x_1 \le x_3 \le x_2$. By Lemmas 3.1 and 4.2, Scheme D yields costs at most

$$F(\theta_1)x_1 + x_2 + H(\theta_3)x_3, \quad F(\theta_2)x_3 + x_2 + H(\theta_4)x_1.$$

Taking the average and applying Lemma 4.3 bounds the minimum cost by

$$\max\left\{1,\, \tfrac{1}{4}(F(\theta_2)+H(\theta_3)+2),\right.$$
$$\left.\tfrac{1}{6}(F(\theta_1)+F(\theta_2)+H(\theta_3)+H(\theta_4)+2)\right\}(x_1+x_3+x_2).$$

Since $\theta_2 + \theta_3 \le 360^\circ - 60^\circ - 72^\circ = 228^\circ$, it can be confirmed numerically that the second max term is maximized near $\theta_2 \approx 125^\circ$, $\theta_3 \approx 103^\circ$ and has value below 1.372. Since $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 360^\circ$, it can confirmed numerically that the third max term is maximized near $\theta_1, \theta_2 \approx 100^\circ$, $\theta_3, \theta_4 \approx 80^\circ$ and has value below 1.341.

- Subcase $x_1 \le x_2 \le x_3$.

  – *Subsubcase* $\theta_3 \le 115^\circ$. By Lemmas 3.1 and 4.2, Scheme D yields costs at most

$$F(\theta_1)x_1 + x_2 + H(\theta_3)x_3, \quad F(\theta_2)x_2 + x_3 + H(\theta_4)x_1.$$

  Taking the average and applying Lemma 4.3 bounds the minimum cost by

$$\max\left\{\tfrac{1}{2}(H(\theta_3)+1),\, \tfrac{1}{4}(F(\theta_2)+H(\theta_3)+2),\right.$$
$$\left.\tfrac{1}{6}(F(\theta_1)+F(\theta_2)+H(\theta_3)+H(\theta_4)+2)\right\}(x_1+x_2+x_3).$$

  The first max term is at most $\frac{1}{2}(H(115^\circ)+1) < 1.399$. As in the previous subcase, the second and third max term are bounded by 1.372 and 1.341.

  – *Subsubcase* $\theta_3 \ge 115^\circ$. By Lemma 3.1, Scheme C yields cost at most

$$\begin{aligned} &F(\theta_1)x_1 + F(\theta_2)x_2 + x_3 \\ &\le\; \max\left\{1,\, \tfrac{1}{2}(F(\theta_2)+1),\, \tfrac{1}{3}(F(\theta_1)+F(\theta_2)+1)\right\} \\ &\qquad (x_1 + x_2 + x_3), \end{aligned}$$

  using Lemma 4.3. Since $\theta_2 \le 360^\circ - 115^\circ - 72^\circ - 60^\circ = 113^\circ$, the second max term is at most $\frac{1}{2}(F(113^\circ)+1) < 1.334$. Since $\theta_1 + \theta_2 \le 360^\circ - 115^\circ - 72^\circ = 173^\circ$, the third max term is at most $\frac{1}{3}(2F(86.5^\circ)+1) < 1.247$.

- Subcase $x_2 \le x_1 \le x_3$.

  – *Subsubcase* $\theta_3 \le 115^\circ$, $\theta_3 + \theta_4 \le 195^\circ$. By Lemmas 3.1 and 4.2, Scheme D yields costs at most

$$F(\theta_1)x_2 + x_1 + H(\theta_3)x_3, \quad F(\theta_2)x_2 + x_3 + H(\theta_4)x_1.$$

  Taking the average and applying Lemma 4.3 bounds the minimum cost by

$$\max\left\{\tfrac{1}{2}(H(\theta_3)+1),\, \tfrac{1}{4}(H(\theta_3)+H(\theta_4)+2),\right.$$
$$\left.\tfrac{1}{6}(F(\theta_1)+F(\theta_2)+H(\theta_3)+H(\theta_4)+2)\right\}(x_2+x_1+x_3).$$

  The first max term is at most $\frac{1}{2}(H(115^\circ)+1) < 1.399$. Since $\theta_3 + \theta_4 \le 195^\circ$, it can be verified that the second max term is bounded by 1.333. As in the earlier subcases, the third max term is bounded by 1.341.

  – *Subsubcase* $\theta_3 \ge 115^\circ$, $\theta_3 + \theta_4 \le 195^\circ$. Here, just take the upper bound

$$\begin{aligned} &F(\theta_2)x_2 + x_3 + H(\theta_4)x_1 \\ &\le\; \max\left\{1,\, \tfrac{1}{2}(H(\theta_4)+1),\, \tfrac{1}{3}(F(\theta_2)+H(\theta_4)+1)\right\} \\ &\qquad (x_2 + x_1 + x_3), \end{aligned}$$

using Lemma 4.3. Since $\theta_4 \le 80°$, the second max term is at most $\frac{1}{2}(H(80°)+1) < 1.245$. Since $\theta_2 \le 360° - 115° - 72° - 60° = 113°$, the third max term is at most $\frac{1}{3}(F(113°)+H(80°)+1) < 1.386$.

– *Subsubcase* $\theta_3 + \theta_4 \ge 195°$. By Lemma 4.1, Scheme C yields cost at most

$$\begin{aligned} & x_1 + [G(\theta_1)x_2 + (F(\theta_1)-G(\theta_1))x_1] \\ & \quad + [G(\theta_2)x_2 + (F(\theta_2)-G(\theta_2))x_3] \\ \le\ & \max\{\, F(\theta_2)-G(\theta_2), \\ & \qquad \tfrac{1}{2}(F(\theta_1)-G(\theta_1)+F(\theta_2)-G(\theta_2)+1), \\ & \qquad \tfrac{1}{3}(F(\theta_1)+F(\theta_2)+1)\,\}\,(x_2+x_1+x_3), \end{aligned}$$

using Lemma 4.3. Since $\theta_2 \le 360° - 195° - 60° = 105°$, the first max term is at most $F(105°)-G(105°) < 1.201$. Since $\theta_1 + \theta_2 \le 360° - 195° = 165°$, it can be verified that the second max term is at most $F(82.5°) - G(82.5°) + \frac{1}{2} < 1.388$ and the third max term is at most $\frac{1}{3}(2F(82.5°)+1) < 1.213$.

Case $k = 4$**.** Having experienced the $k = 3$ case, we are happy to report, counter to intuition, that $k = 4$ case can be disposed of more quickly. As it turns out, Scheme C alone is enough to provide the desired bound here.

Let v_a, v_b, v_c, v_d be the children of v in T sorted by angle. Let $x_1 = |vv_a|$, $x_2 = |vv_b|$, $x_3 = |vv_c|$, $x_4 = |vv_d|$, $\theta_1 = \angle v_a v v_b$, $\theta_2 = \angle v_b v v_c$, $\theta_3 = \angle v_c v v_d$, and $\theta_4 = \angle v_d v v_a$, as in Figure 7(c). W.l.o.g., say $x_4 \ge x_1, x_2, x_3$.

We know that one of these four angles, depending on the placement of v' around v, exceeds $2(72°) = 144°$. W.l.o.g., say $\max\{\theta_2, \theta_4\} \ge 144°$. Note that $\theta_1, \theta_3 \le 360° - 144° - 60° - 60° = 96°$, and so $G(\theta_1), G(\theta_3) > 0.402$.

We want to show that Scheme C under some permutation has cost at most $\tau(x_1+x_2+x_3+x_4)$.

- Subcase $x_1 \le x_2$. By Lemmas 3.1 and 4.1, we get cost bounded by

$$\begin{aligned} & x_1 + [0.4x_1 + (F(\theta_1)-0.4)x_2] + [x_2 + x_3] \\ & \quad + [0.4x_3 + (F(\theta_3)-0.4)x_4] \\ \le\ & 1.4(x_1+x_3) + \\ & \max\left\{F(\theta_3)-0.4,\ \tfrac{1}{2}(F(\theta_1)+F(\theta_3)+0.2)\right\}(x_2+x_4), \end{aligned}$$

using Lemma 4.3. The first max term is at most $F(96°) - 0.4 < 1.087$. Since $\theta_1 + \theta_3 \le 360° - 144° - 60° = 156°$, the second max term is at most $F(78°) + 0.1 < 1.359$.

- Subcase $x_2 \le x_1$. By Lemmas 3.1 and 4.1, we get cost bounded by

$$\begin{aligned} & x_1 + [0.4x_2 + (F(\theta_1)-0.4)x_1] + [x_2 + x_3] \\ & \quad + [0.4x_3 + (F(\theta_3)-0.4)x_4] \\ \le\ & 1.4(x_2+x_3) + \\ & \max\left\{F(\theta_3)-0.4,\ \tfrac{1}{2}(F(\theta_1)+F(\theta_3)+0.2)\right\}(x_1+x_4), \end{aligned}$$

using Lemma 4.3. Again, the max terms are bounded by 1.087 and 1.359.

A ratio of 1.402 has thus been established.

Remark: The constant is tight in following sense: if $\tau = 1.401$ instead, there exists placements of vertices v, v_1, v_2, v' for $k = 2$, such that neither Schemes C nor D can yield cost less than $\tau(|vv_1| + |vv_2|)$, under the present definition of cost.

5. BETTER DEGREE-3 SPANNING TREES IN ARBITRARY DIMENSIONS

We close with a preliminary discussion on higher-dimensional spanning trees. Khuller *et al.* [10] showed that a nontrivial result can be obtained already for the degree-3 case: the same algorithm described in Section 2 always produces a spanning tree of weight $(5/3)w(T)$. This is a consequence of the following fact: given arbitrary points $v, v_1, \ldots, v_k \in \mathbb{R}^d$, there exists a path that starts at v and visits $v_1, \ldots, v_k$ in some order, with weight at most $\frac{5}{3}\sum_{i=1}^k |vv_i|$. We observe that by a slightly more careful analysis, the constant $5/3$ can be reduced to $\frac{2}{3}\sqrt{6} < 1.633$.

The proof is similar to Khuller *et al.*'s but uses a different geometric inequality:

LEMMA 5.1. *Given points* $v, v_0, v_1, v_2, v_3, v_4 \in \mathbb{R}^d$ *such that* $|vv_0| \le |vv_1| \le \cdots \le |vv_4|$*,*

$$\begin{aligned} & \min\{\, |v_0v_1| + |v_1v_2| + |v_2v_3| + |v_3v_4|, \\ & \qquad |v_0v_2| + |v_2v_3| + |v_3v_1| + |v_1v_4|, \\ & \qquad |v_0v_3| + |v_3v_1| + |v_1v_2| + |v_2v_4|\,\} \\ \le\ & \tfrac{2}{3}\sqrt{6}\,(|vv_1| + |vv_2| + |vv_3| + |vv_4|). \end{aligned}$$

PROOF. By bounding the minimum with the average, it suffices to prove that

$$\begin{aligned} & |v_0v_1| + |v_0v_2| + |v_0v_3| + |v_4v_1| + |v_4v_2| + |v_4v_3| \\ & \qquad + 2|v_1v_2| + 2|v_2v_3| + 2|v_3v_1| \\ \le\ & 2\sqrt{6}\,(|vv_1| + |vv_2| + |vv_3| + |vv_4|). \qquad (2) \end{aligned}$$

First consider the case where $|vv_1| = |vv_2| = |vv_3| = |vv_4| = r$. The expression

$$|v_0v_1| + |v_0v_2| + |v_0v_3| + |v_1v_2| + |v_2v_3| + |v_3v_1|$$

is bounded by $4\sqrt{6}r$ (in geometric terms, the total side length of a tetrahedron inside a sphere is maximized when the tetrahedron is regular); Khuller *et al.* cited Lillington [12], although it is not hard to establish this fact algebraically, by the Cauchy-Schwarz inequality. Similarly, $|v_4v_1| + |v_4v_2| + |v_4v_3| + |v_1v_2| + |v_2v_3| + |v_3v_1| \le 4\sqrt{6}r$. Therefore, (2) is true, since the L.H.S. is at most $8\sqrt{6}r$ and the R.H.S. is exactly $8\sqrt{6}r$.

Now, suppose v_2, v_3, v_4 are moved radially outward so that $|vv_2|, |vv_3|, |vv_4|$ all increase by αr. Then the L.H.S. of (2) increases by at most $7\alpha r + \alpha|v_4v_2| + \alpha|v_4v_3| + 2\alpha|v_2v_3|$; here, the $|v_iv_j|$'s refer to the old distances. The expression

$$|v_2v_3| + |v_3v_4| + |v_4v_2|$$

is bounded by $3\sqrt{3}r$ (in geometric terms, the perimeter of a triangle inside a circle is maximized when the triangle is equilateral). Therefore, the amount of change to the L.H.S. is bounded by $(9 + 3\sqrt{3})\alpha r \le 14.2\alpha r$, but the amount of change to the R.H.S. is $6\sqrt{6}\alpha r > 14.6\alpha r$. So, (2) still holds.

Now, suppose $|vv_3|, |vv_4|$ increase further by δ. Then the L.H.S. of (2) increases by at most 9δ, while the R.H.S. increases by $4\sqrt{6}\delta > 9.7\delta$. So, (2) is still true.

Finally, suppose $|vv_4|$ increases further yet by ε. Then the L.H.S. of (2) increases by at most 3ε, while the R.H.S. increases by $2\sqrt{6}\varepsilon > 4.8\varepsilon$. We conclude that (2) holds for all values of $|vv_1| \le |vv_2| \le |vv_3| \le |vv_4|$. $\square$

Sort $v_1, \ldots, v_k$ by distances, so that $|vv_1| \le \cdots \le |vv_k|$. The main idea is to divide the sequence $v_1, \ldots, v_k$ into blocks of 4 and rearrange each block separately (Khuller *et al.* in contrast divides into blocks of 3). Lemma 5.1 implies that there is a path from v_{k-4j-4} to v_{k-4j} via $v_{k-4j-3}, \ldots, v_{k-4j-1}$, of weight at most $\frac{2}{3}\sqrt{6}\sum_{i=k-4j-3}^{k-4j} |vv_i|$, for all $j = 0, \ldots, \lfloor (k-5)/4 \rfloor$. Special cases of Lemma 5.1 imply that for $\ell \in \{1, \ldots, 4\}$, there is a path from v to v_ℓ via $v_1, \ldots, v_{\ell-1}$, of weight at most $\frac{2}{3}\sqrt{6}\sum_{i=1}^{\ell} |vv_i|$. Our result follows immediately by concatenating these paths.

As Khuller *et al.* have observed, this algorithm can be implemented in linear time for any dimension d (possibly nonconstant) if the MST is given.

6. CONCLUSIONS

The obvious open problem is to improve the upper bounds further by designing better algorithms. Currently, the only published lower bounds [10] are $\frac{1}{4}(\sqrt{2} + 3) > 1.103$ and $\frac{1}{5}(F(72°) + 4) > 1.035$ respectively for the worst-case ratio of the minimum degree-3 and degree-4 spanning tree to the MST in the plane (achieved by the center plus vertices of a square and a regular pentagon respectively); Fekete *et al.* [8] boldly conjectured that these lower bounds are tight. If we insist that a designated root has degree at most 2 and 3 respectively, then the lower bounds increase to $\frac{1}{3}(\sqrt{3}+2) > 1.244$ and $\frac{1}{4}(\sqrt{2} + 3) > 1.103$ (this time, by the center plus vertices of an equilateral triangle and a square).

We hope that our work would inspire more progress on the determination of these fascinating constants.

Acknowledgement. I thank Alex Golynski for asking whether the 1.5 bound for planar degree-3 trees is improvable.

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