Addenda

Automatic Sequences: Theory, Applications, Generalizations

by Jean-Paul Allouche and Jeffrey Shallit

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One of the biggest gaps in our book was the omission (for all practical purposes) of the applications of logic to automatic sequences. This has now been remedied by the following book: J. Shallit, *The Logical Approach to Automatic Sequences: Exploring Combinatorics on Words with* `Walnut`, Cambridge University Press, 2023.

Page 30: Open Problem 3 in section 1.10 has been solved by Ron Clark, in his 2001 Ph. D. thesis, available at `http://www.math.ucla.edu/~rclark/thesis.pdf`.

Page 36: The lower bound on the number of ternary squarefree words has been improved to 1.1184 by X. Sun, New lower bound on the number of ternary square-free words, *J. Integer Sequences*, Paper #03.3.2, `http://www.math.uwaterloo.ca/JIS/VOL6/Sun/sun.html`.

Chapter 8: Recently M. Peter [The asymptotic distribution of elements in automatic sequences, *Theor. Comput. Sci.* **301** (2003), 285–312] gave a necessary and sufficient condition for the limiting frequency of a symbol in an automatic sequence to exist.

Page 125: In the notes about Theorem 3.6.1, there are some earlier discoveries of bijective representation that merit being listed here. For example, J. E. Foster, A number system without a zero-symbol, *Math. Mag.* **21** (1947), 39–41; R. M. Smullyan, *Theory of Formal Systems*, Annals of Mathematics Studies, Vol. 47 (1961), pp. 34–36.

Page 208: Problem 3 asked for results about the subword complexity of k-context-free sequences. Yossi Moshe proved that if a sequence is the characteristic sequence of a context-free language generated by a DPDA without ϵ-transitions then its subword complexity is bounded by a polynomial in n. See Yossi Moshe, On some questions regarding k-regular and k-context-free sequences, *Theoret. Comput. Sci.* **400** (2008), 62–69.

Page 245: In the notes to this chapter, we should mention the D0L ω-equivalence problem. This was originally solved by Culik and Harju in *J. Assoc. Comput. Mach.* **31** (1984), 282–298. Also see J. Honkala, *Internat. J. Found. Comput. Sci.* **18** (2007), 181–194, and J. Honkala, The equality problem for purely substitutive words, in *Combinatorics, automata and number theory*, pp. 505–529, Encyclopedia Math. Appl., 135, Cambridge Univ. Press, 2010. And J. Honkala, The equality problem for infinite words generated by primitive morphisms. *Inform. and Comput.* **207** (2009), 900–907 and J. Honkala, A bound for the

ω-equivalence problem of polynomial D0L systems. *Theor. Inform. Appl.* **37** (2003), 149–157. The corresponding problem for CD0L systems, or even two k-automatic sequences with multiplicatively independent k, seems to be still open.

Page 245: There are additional examples of automatic and morphic sequences in music. For example:

Y. H. Au et al., Notes and note-pairs in Nørgård's infinity series, *J. Math. Music* **11** (2017), 1–19.

J.-P. Allouche and T. Johnson, Combinatorics of words and morphisms in some pieces of Tom Johnson, *J. Math. Music* **12** (2018), 248–257.

J.-P. Allouche, Marcel Frémiot, determinism versus chaos, and the tower of Hanoi, *J. Math. Music* **12** (2018), 258–264.

R. Gómez and L. Nasser, Symbolic structures in music theory and composition, binary keyboards, and the Thue-Morse shift, *J. Math. Music* **15** (2021), 247–266.

Page 282: Open problem 8.8.1 has been solved by Jason P. Bell, Logarithmic frequency in morphic sequences, *J. Théorie Nombres Bordeaux* **20** (2008), 227–241.
Page 295: Open Problem 1 in section 9.5 has recently been studied by Kevin O'Bryant. See his preprint `http://xxx.lanl.gov/abs/math.CO/0211200`.

Page 340: Problem 3 asked if there is a k-context-free sequence with bounded gaps that is not k-automatic. Yossi Moshe constructed an example of such a sequence based on the language $\{w1^i01^i \ : \ i \geq 1, w \in \{0,1\}^*\}$. See Yossi Moshe, On some questions regarding k-regular and k-context-free sequences, *Theoret. Comput. Sci.* **400** (2008), 62–69.

Pages 375–6: Open Problem 3 in section 12.9 was solved by Joe Kramer-Miller. See Transcendence properties of the Artin-Hasse exponential modulo p, arxiv:2404.06968, April 10 2024.

Page 391, §13.5. It is now known that all automatic numbers are either rational or transcendental. See, for example, B. Adamczewski and Y. Bugeaud, On the complexity of algebraic numbers. I. Expansions in integer bases. *Ann. of Math.* (2) **165** (2007), no. 2, 547–565.

Page 426: The notes to Chapter 14 should refer to Jean Moulin Ollagnier, Sur quelques classes d'applications de N^2 dans les ensembles finis, *RAIRO Info. Théor.* **23** (1989), 461–492.

Page 453: Open Problem 16.7.2 was solved by Zhao Shen, On the 3-adic valuation of a class of Apery-like numbers. See https://arxiv.org/abs/2112.11135 and *Int. Math. Research Notices* **2023** (8), 6691–6702.

Page 453: Open Problem 16.7.3 was solved by J. P. Bell [*Discrete Math.* **307** (2007), 3070–3075] and Z. Shu & J.-Y. Yao [*C. R. Acad. Sci. Paris* **349** (2011), 947–952].

Page 453: Open Problem 16.7.4 has been solved by Schlage-Puchta, *Bull. Belg. Math. Soc.* **18** (2011), 375–377.

Page 453: Open Problem 16.7.9 has been solved by J. P. Bell, *Advances in Applied Math.* **34** (2005), 634–643.

Page 454: Open Problem 16.7.10 has been solved by Yossi Moshe, On some questions regarding k-regular and k-context-free sequences, *Theoret. Comput. Sci.* **400** (2008), 62–69. Also see Eric Rowland, Non-regularity of $\lfloor \alpha + \log_k n \rfloor$, *INTEGERS* **10** (1), #A03, pp. 19–23.

Page 482: The paper of Allouche, Baake, Cassaigne, and Damanik has now appeared: *Theor. Comput. Sci.* **292** (2003), 9–31.

Page 496: The paper of Cassaigne and Karhumäki [1995a, 1995b] appeared in *European J. Combinatorics* **18** (1997), 497–510. (J. Berstel)

Page 507: The paper of Epifanio, Koskas, and Mignosi has now appeared: *Theor. Comput. Sci.* **299** (2003), 123–150.