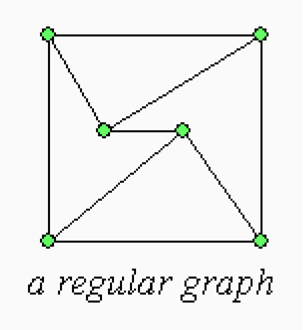
Highly Irregular Graphs

Shahin Kamali
Ararat Harutyunyan

Motivation

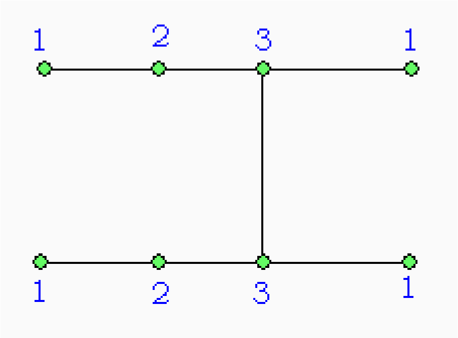


- Inspired by “regular graphs”, How to define an “Irregular Graph”?
- Maybe define an irregular graph as a graph whose degrees of vertices are distinct.
- There is a little problem in this definition!
- In regular graphs all neighbors of a vertex have the same degree.

Highly Irregular Graphs

- A connected graph is *Highly Irregular* if its vertices are adjacent only to vertices with distinct degrees.

$$u, w \in N(v), u \neq w \Rightarrow \deg(u) \neq \deg(w)$$



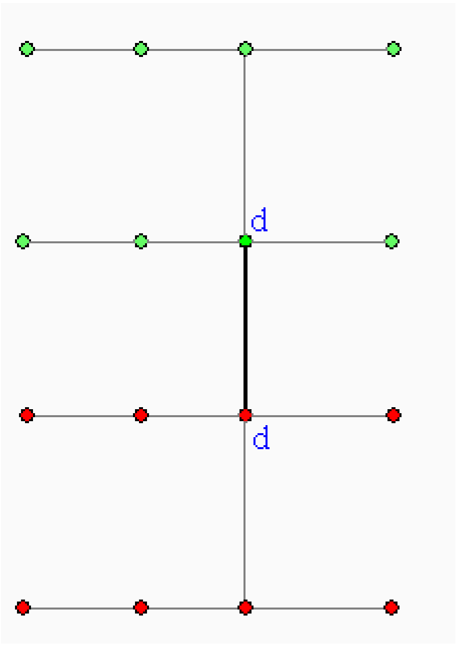
- First introduced by *Alavi, Chartrand, Chung, Erdos, Graham and Oellermann in 1987*

HI-graphs

k_1

- Theorem:
  - For all *d,* there exists a highly irregular graph with maximum degree *d.*

- Proof by construction:
  - There is a highly irregular graph of degree 0.
  - Take two copies of a graph with maximum degree *d* and connect two vertices of degree *d*

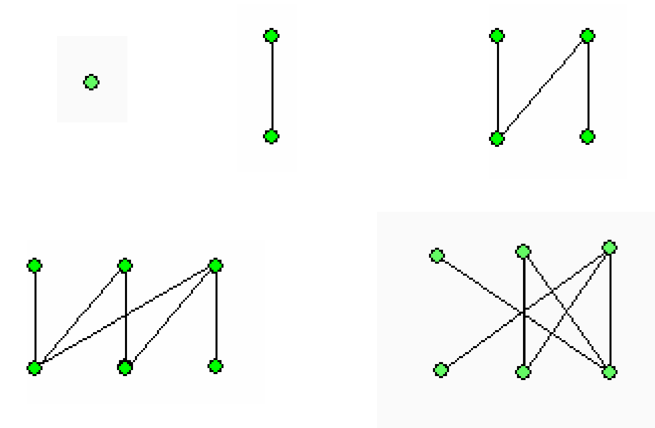


HI-graphs

Is there an ‘*effective*’ algorithm for generating <u>all</u> HI-graphs of order *n* ?

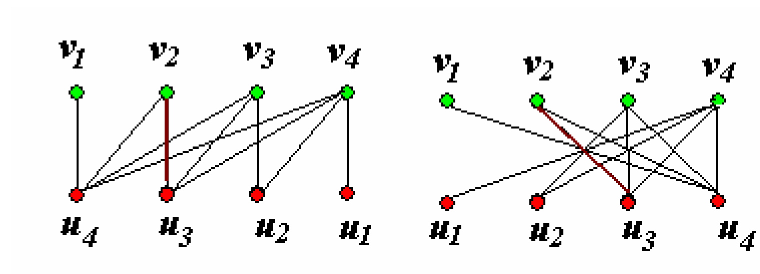
- The answer looks to be ‘No’ !
- ‘HI-graphs” are easy to distinguish but difficult to generate.
  - In this sense HI-Graphs looks to be similar to Planar Graphs

HI-graphs Order



- Theorem
  - If $n \neq 3,5,7$ there exist a HI-graph of order *n.*

- Proof:
  - For *$n=2d \geq 8$ just consider the bipartite graph with:*

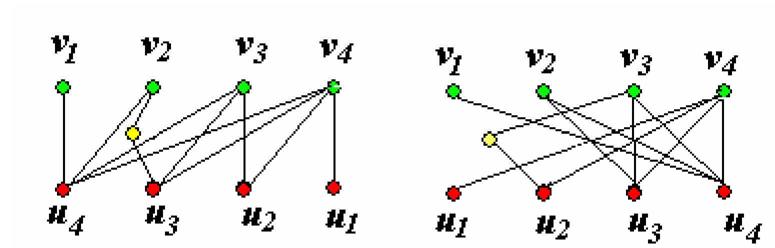


$$V = \{v_1, v_2, \ldots, v_d\}$$

$$U = \{u_1, u_2, \ldots, u_d\}$$

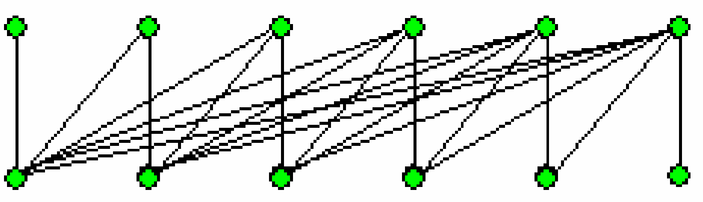
$$E(H) = \bigcup_{i=1}^{d} E_i \qquad where \qquad E_i = \{v_i u_j \mid d - i + 1 \leq j \leq d\}$$

  - *For n=2d+1, just subdivide the edge $v_2 u_{d-1}$*



HI-graphs Size

- Theorem
  - The size of HI-graph of order n is at most $n(n+2)/8$ with equality possible for n even.



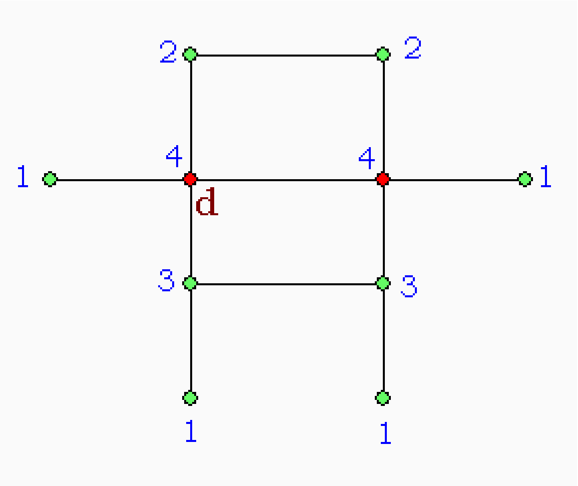
- Proof:
  - The last construction shows the tightness of the bound.
  - We just need to show that there is no highly irregular graph with more than $n(n+2)/8$ *edges*

HI-graphs Size!

- Theorem
  - The size of highly irregular graph of order *n* is at most *n(n+2)/8* with equality possible for *n* even.

- If *v* is a vertex of maximum degree *d*, then *v* is adjacent to exactly one vertex of degree k for
- $1 \le k \le d$.
  - → There are at least two vertices of degree k

- A highly irregular graph with maximum degree *d* has at least *2d vertices*
  - → the maximum degree of a HIG of order *n* is at most *n/2*



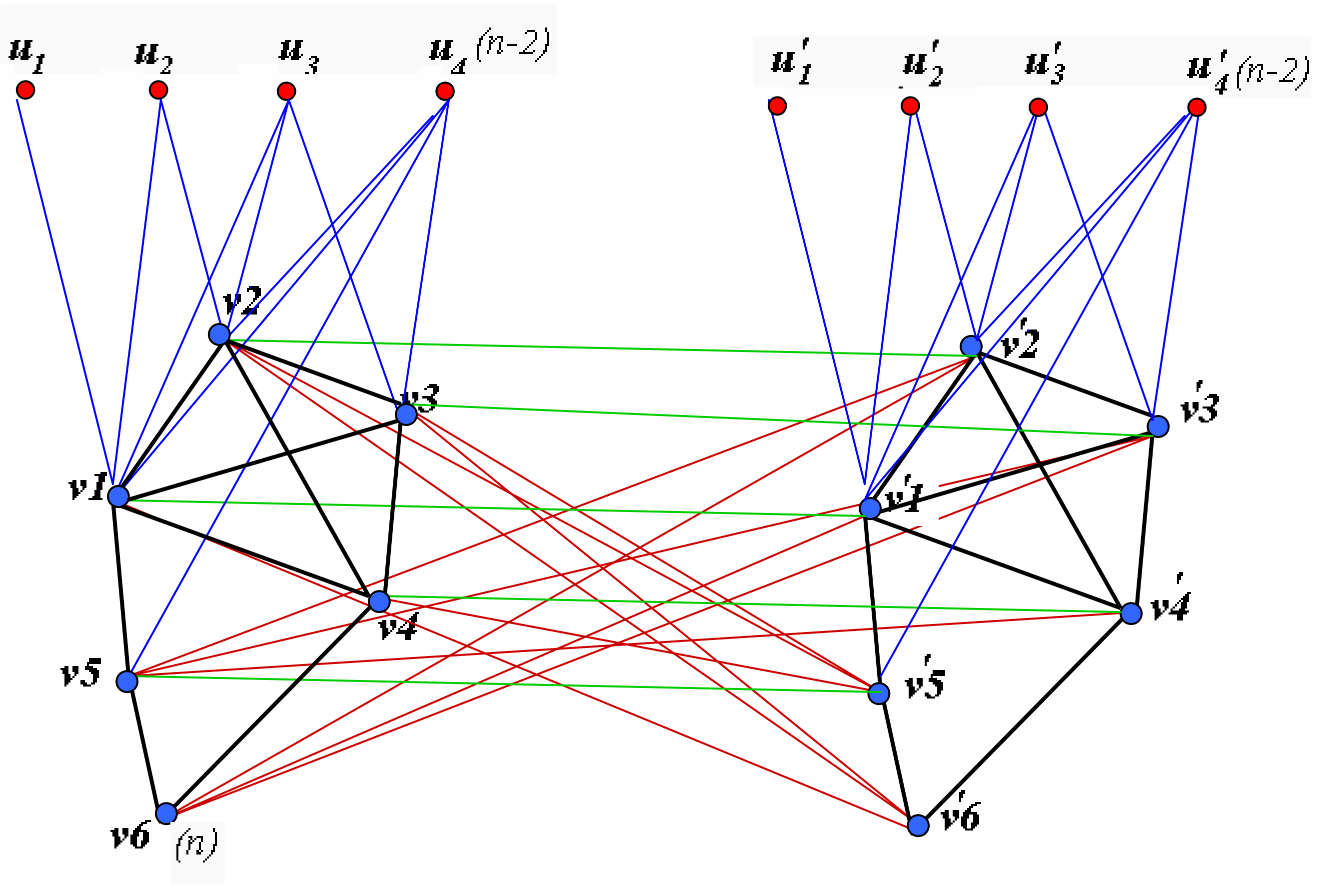
$$size(H) \le (1/2)2(1+2+\ldots+n/2) = n/2(n/2+1)/2 = n(n+2)/8$$

HI-Graphs containing a given graph as induced subgraph

- Theorem
  - Every graph of order $n \geq 2$ is an induced subgraph of a highly irregular graph of order $4n-4$.

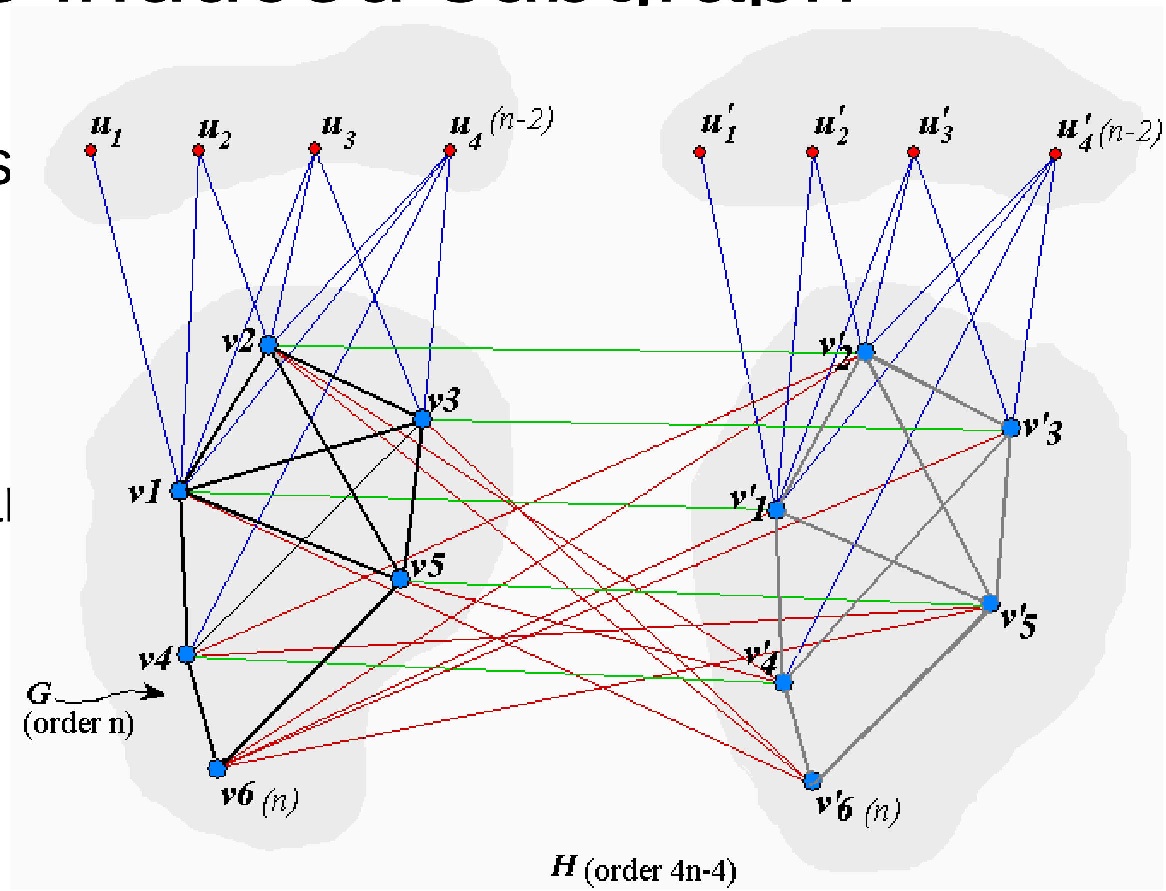
- Proof by Construction:
  - Construct a highly irregular graph *H* containing *G.*

HI-Graphs containing a given graph as induced subgraph



HI-Graphs containing a given graph as induced subgraph

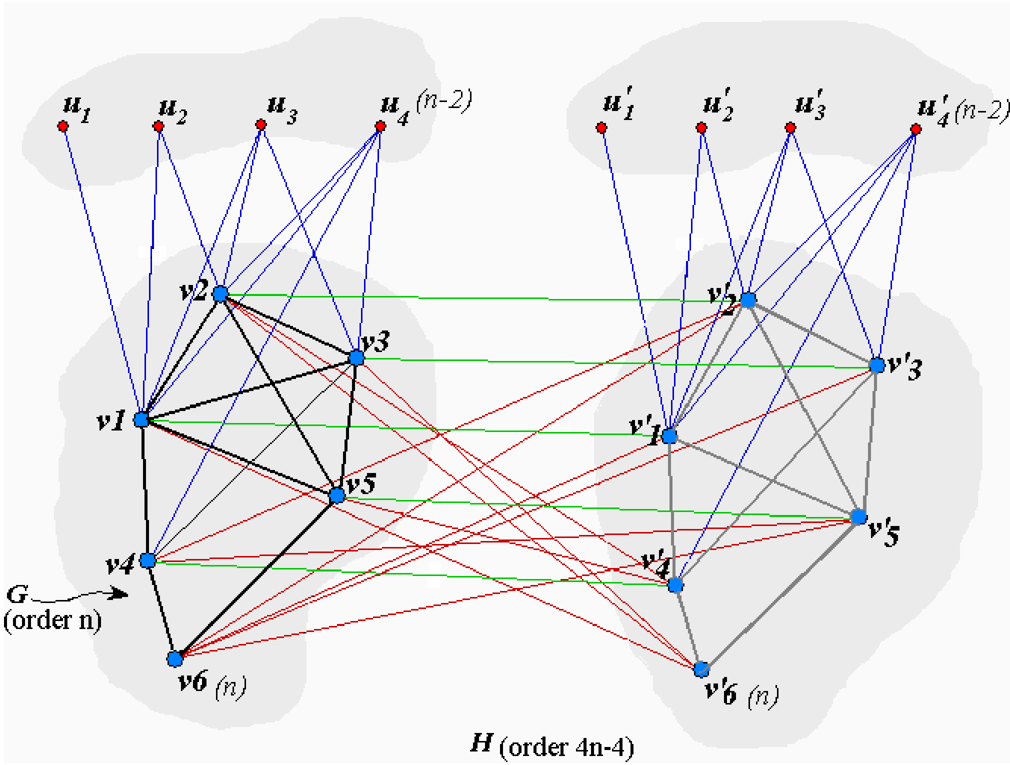
- *H* has *4n-4* vertices
- *H* Contains *G*
- *H* is Highly Irregular
- $deg_H\, u_i = deg_H\, u'_i = i$
- $deg_H v_i = deg_H v'_i = 2n-1-i$



HI-Graphs containing a given graph as induced subgraph

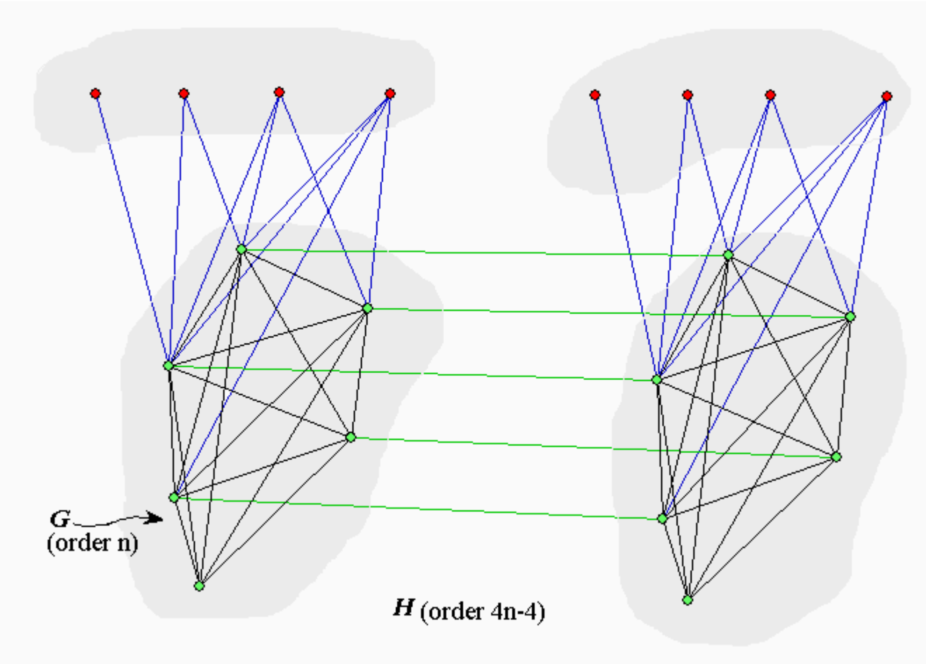
- Having *H* we can uniquely recover *G*

$$\text{numer of HI-Graphs of order } 4n-4$$
$$\geq$$
$$\text{number of graphs of order } n$$



HI-graphs and cliques

- Theorem:
  - The smallest order of a HI-Graph with clique number *n* is *4n-4*
- Proof:
  - Such a HI-Graph exists →
  - The degrees of vertices of a potential clique should be distinct.
  - Maximum degree ≥ 2n-2
  - *Order(H)* ≥ *4n-4*



How many HI-Graphs exist?

- *HI(n)*← the number of (nonisomorphic) HI-graphs of order *n*
- *G(n)*← total number of graphs of order *n*

- **Theorem**

$$\frac{1}{16} + o(1) < \frac{\log HI(n)}{\log G(n)} < 2 - \frac{3}{4}\log_2 3 + o(1) = 0.8112..$$

How many HI-Graphs exist?

- To see (left hand)

$$\frac{1}{16} + o(1) < \frac{\log HI(n)}{\log G(n)}$$

- We previously shown:

$$HI(4n-4) \geq G(n)$$

since

$$G(t) = (1+o(1))2^{\binom{t}{2}}$$

then

$$HI(n) \geq (1+o(1))2^{n^2/32}$$

How many HI-Graphs exist?

To show (right hand):

$$\frac{\log HI(n)}{\log G(n)} < 2 - \frac{3}{4}\log_2 3 + o(1)$$

a HI-Graph with n vertices has at most $\frac{n^2}{8} + O(n)$ edges. So

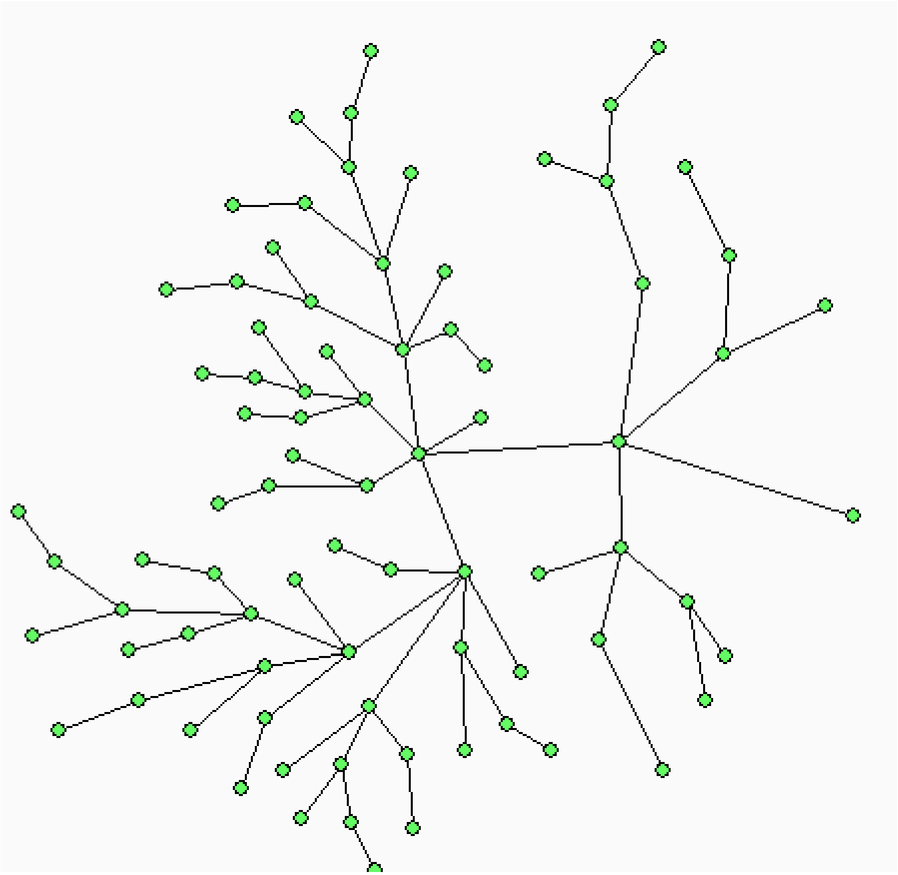
$$HI(n) \le \sum_{1 \le j \le n^2/8 + O(n)} \binom{\binom{n}{2}}{j} \quad \sim \quad 2^{(n^2/2)H(1/4)}$$

H(x) is the binary entropy function:

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x) \quad \text{and} \quad \mathrm{H}(1/4) = 2 - (3/4)\log_2 3$$

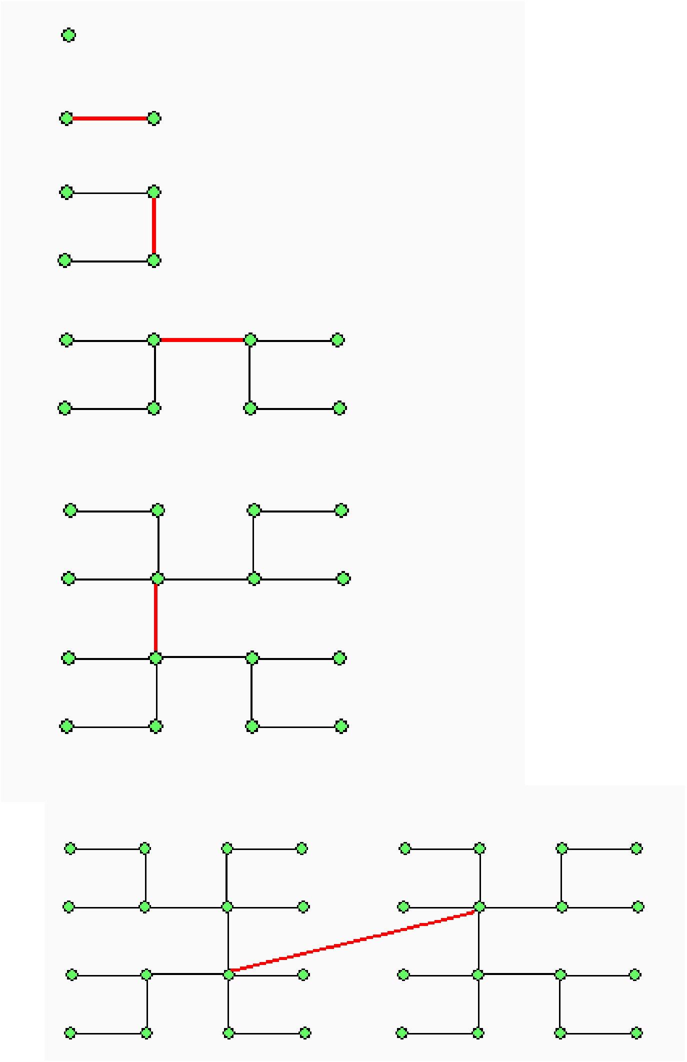
since $G(n) = 2^{(n^2/2)(1+o(1))}$ the theorem follows.

Highly Irregular Trees



Highly Irregular Trees (HI_ Trees)

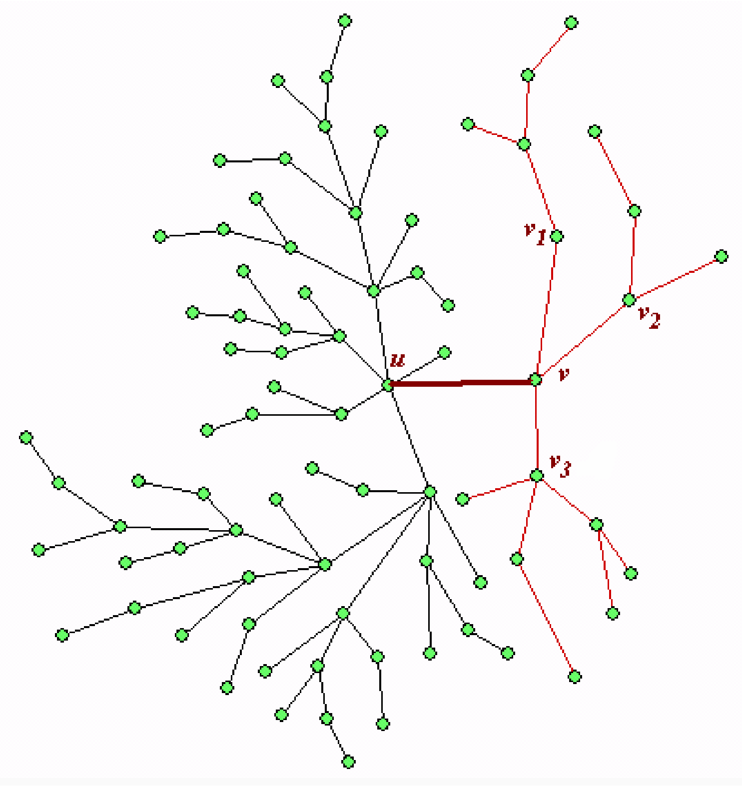
- Theorem
  - For all d, there exists a HI-tree with maximum degree *d* and order 2^d
- Proof by Construction
- This is the smallest possible order for a such a tree!



Order of HI-Trees

- Theorem
  - The order of a highly irregular tree with maximum degree *d* is at least 2^d.

- Proof:
  - Let *v* be an arbitrary vertex of tree and u an arbitrary neighbor of *v*.
  - If *f(k)* is the number of edges in the split of uv containing *v*, then $f(k) \geq 2^{k-1}$ (proof straightforward by induction on *k*)
  - Considering the vertex with maximum degree we have:
    - $n \geq 2f(d) \geq 2^d$



Existence of HI-Trees

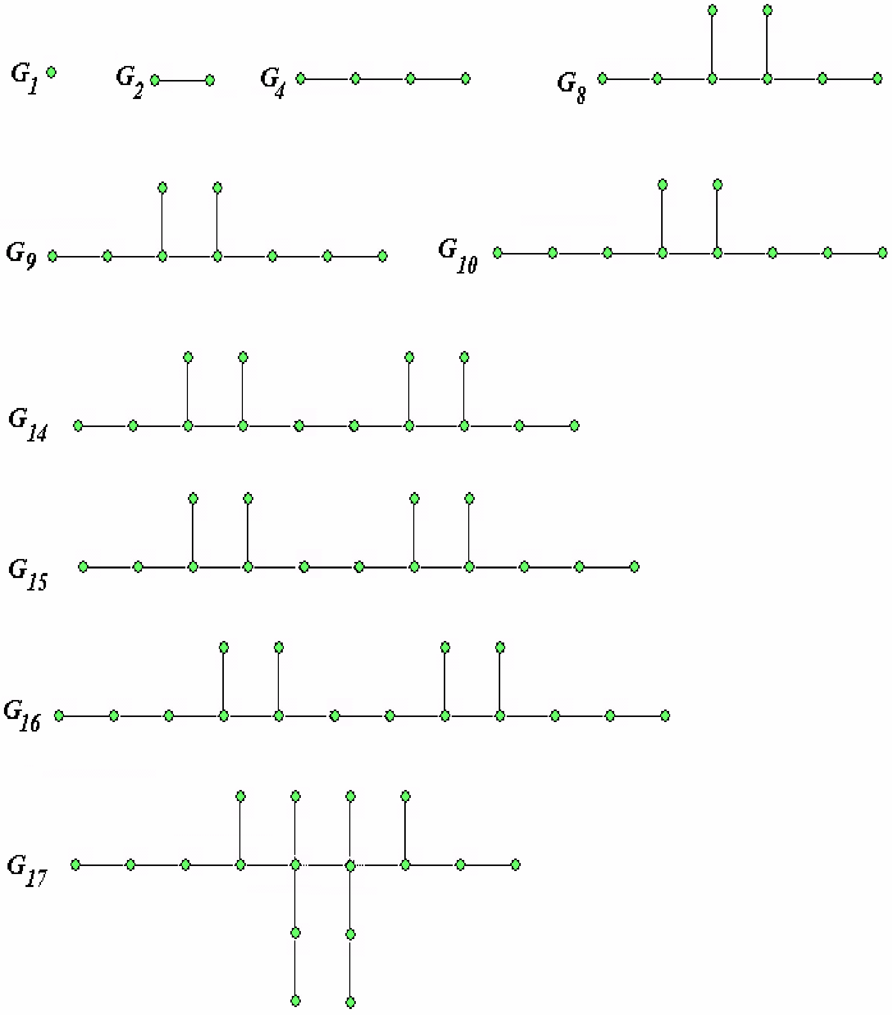
- Theorem:
  - For each

    $n \in N / \{3,5,6,7,11,12,13\}$

    there exist a HI-tree of order n.
- Proof:
  - For

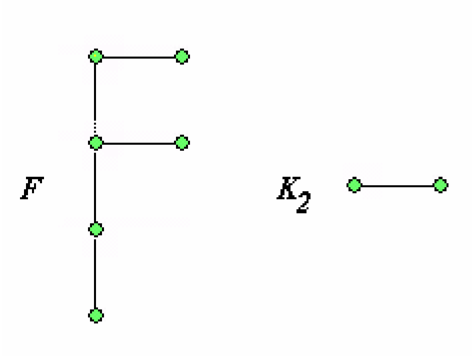
    $n \in N / \{1,2,4,8,9,10,14,15,16,17\}$

    there exists HI-tree of order *n*

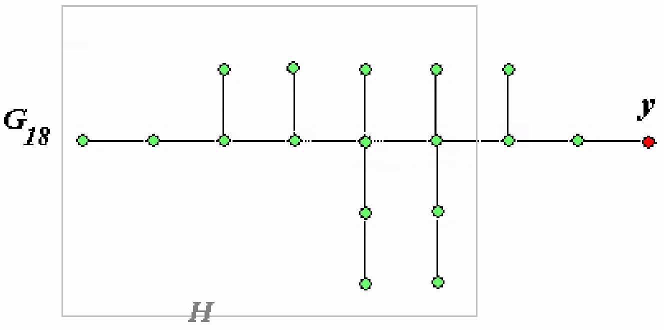


Existence of HI-Trees

- Proof for n>17 by construction.
- We start by G_{18} and two '*magic*' trees F and K_2.



- If *H* is a subgraph of a graph G, then vertex y of G will be called **(G - H)-*pendant*** if:
  - $y \notin V(H)$
  - *y* is a pendant vertex in G
  - *y* is joined in G with a vertex of degree 2



Existence of HI-Trees

- Construction:
  - Step 0:

$$G_{19} = G_{18}(y \text{ id } x)K_2 \text{ where } y \text{ is } (G_{18} - T)\text{-pendant}$$

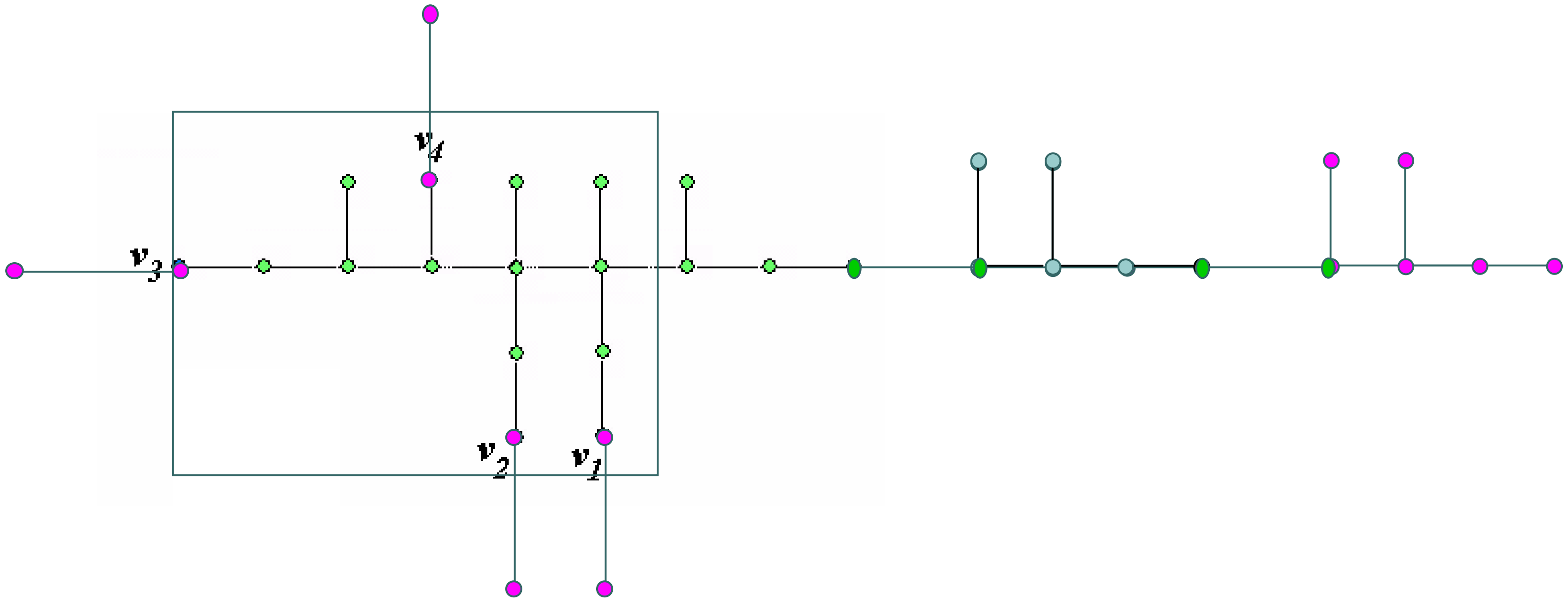
$$\mathrm{G}_{19+\mathrm{j}} = \begin{cases} G_{19} \ (v_j \text{id } x) \ K_2 & \text{for } j = 1,2,3,4 \\ G_{19} \ (y \text{ id } x) \ F, \text{where } y \text{ is } (G_{19} - T)\text{-pendant} & \text{for j} = 5 \end{cases}$$

  - Step k:

$$G_{19+6k} = G_{19+6k-1}(y \,\text{id}\, x)K_2 \text{ where } y \text{ is } (G_{19+6k-1} - T)\text{-pendant}$$

$$\mathrm{G}_{19+6\mathrm{k}+\mathrm{j}} = \begin{cases} G_{19+6k} \ (v_j \text{id}\, x) \ K_2 & \text{for } j = 1,2,3,4 \\ G_{19+6k} \ (y \,\text{id}\, x) \ F, \text{where } y \text{ is} (G_{19+6k} - T)\text{-pendant} & \text{for j} = 5 \end{cases}$$

Existence of HI-Trees



- It is easy to verify the constructed graphs are HI-trees

References

- Y. Alavi, G. Chartrand, F.R.K. Chung, Paul Erd˝os, R.L. Graham, O.R. Oellermann, **Highly irregular graphs**,
- Y. Alavi, J. Liu, J. Wang, **Highly irregular digraphs**, Discrete Math. 111 (1993) 3–10.
- Y. Alavi, A. Boals, G. Chartrand, O.R. Oellermann, **K-path Irregular graphs**
- Y. Alavi, F.Buckley, M. Shamula, S Ruiz. **Highly irregular *m-chromatic* graphs**
- Z. Majcher, J. Michael, **Degree sequences of highly irregular graphs**,
- Z. Majcher, J. Michael, **Highly irregular graphs with extreme numbers of edges**,
- Z. Majcher, J. Michael, **Degree sequences of digraphs with highly irregular properties**,
- J. Gorska, Z. Skupien, Z. Majcher, J Michael, **A smallest irregular oriented graph containing a given**
- **diregular one**
- Z. Majcher, J Michaela, Joanna Gorskab, Z Skupie, **The minimum size of fully irregular oriented graphs**
- G. Chartrand, Paul Erdos, O.R. Oellermann, **How to Define an Irregular Graph.**
- Y. Alavi, **An introduction to highly irregular graphs**