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Solving #SAT Using Vertex Covers

Abstract We propose an exact algorithm for counting the models of propositional formulas in conjunctive normal form (CNF). Our algorithm is based on the detection of strong backdoor sets of bounded size; each instantiation of the variables of a strong backdoor set puts the given formula into a class of formulas for which models can be counted in polynomial time. For the backdoor set detection we utilize an efficient vertex cover algorithm applied to a certain "obstruction graph" that we associate with the given formula. This approach gives rise to a new hardness index for formulas, the clustering-width. Our algorithm runs in uniform polynomial time on formulas with bounded clustering-width.

It is known that the number of models of formulas with bounded clique-width, bounded treewidth, or bounded branchwidth can be computed in polynomial time; these graph parameters are applied to formulas via certain (hyper)graphs associated with formulas. We show that clustering-width and the other parameters mentioned are incomparable: there are formulas with bounded clustering-width and arbitrarily large clique-width, treewidth, and branchwidth. Conversely, there are formulas with arbitrarily large clustering-width and bounded clique-width, treewidth, and branchwidth.

Keywords: Model counting, fixed-parameter tractability, backdoor set, treewidth, clique-width.

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1 Introduction

1.1 Background

#SAT is the problem of determining the number of satisfying truth assignments or models of a given propositional formula in conjunctive normal form (CNF). This problem arises in several areas of artificial intelligence, in particular in the context of probabilistic reasoning [1, 23]. However, since the problem is #P-complete (Valiant [28]), it is very unlikely that it can be solved in polynomial time. #SAT remains #P-hard even for monotone 2CNF formulas and Horn 2CNF formulas, and it is NP-hard to approximate the number of models of a formula with n variables within $2^{n^{1-\epsilon}}$ for $\epsilon > 0$. This approximation hardness holds also for monotone 2CNF formulas and Horn 2CNF formulas [23].

An alternative to restricting the language of formulas is to impose *structural restrictions* in terms of certain (hyper)graphs associated with formulas. In particular, graph parameters that restrict the structure of associated primal graphs, incidence graphs, and formula hypergraphs have been considered; see Section 4 for definitions of the various graphs and graph parameters. Bacchus, Dalmao, and Pitassi [1] propose an algorithm that solves #SAT in time $n^{O(1)}2^{O(k)}$ for formulas with n variables whose formula hypergraphs have *branchwidth* k. The algorithm is based on the DPLL procedure and uses caching techniques for an efficient reuse of solutions for subproblems. A similar time complexity can be achieved by restricting the *treewidth* of primal graphs and by dynamic programming on tree-decompositions; this approach is described by Gottlob, Scarcello, and Sideri [12] for SAT and can be extended to #SAT as explicated by Samer and Szeider [25]. Bounding the *clique-width* of directed incidence graphs yields larger classes of formulas for which #SAT is tractable: by combining Oum and Seymour's approximation algorithm for clique-width [22] with a general result of Courcelle, Makowsky, and Rotics [5] on counting problems expressible in a certain fragment of Monadic Second Order Logic, it can be shown that #SAT is fixed-parameter tractable for formulas of clique-width at most k. Fischer, Makowsky, and Ravve [8] improve the constants to obtain an algorithm that solves #SAT in time $n^{O(1)}O(f(k))$ for formulas with n variables whose directed incidence graphs have clique-width k, where f is a simply exponential function. The latter result is more general than the results for bounded treewidth and branchwidth in the sense that every class of formulas with bounded treewidth or bounded branchwidth also has bounded clique-width; however, there are classes of formulas with bounded clique-width but unbounded treewidth and unbounded branchwidth; see Section 4. Practical application of the clique-width based algorithm is, however, very limited due to a huge hidden constant in the estimation of its running time.

Note that the algorithms considered above are so-called *fixed-parameter algorithms*, since the bound on the running time is, although exponential in the parameter k, uniformly polynomial in n. The main advantage of fixed-parameter algorithms is that the running time increases moderately when n

becomes large, in contrast to algorithms with running time $n^{O(k)}$. We will review the basic concepts of parameterized complexity in Section 2.2.

1.2 Our Approach: Backdoor Sets

The concept of strong backdoor sets with respect to a base class $\mathcal{C}$ of formulas was introduced by Williams, Gomes, and Selman [29] as a tool for analyzing the performance of local search SAT algorithms. Backdoor sets have recently received a lot of attention in satisfiability research [15,17,19,21,24,27].

A set B of variables of a formula F is a *strong $\mathcal{C}$-backdoor set* if for all truth assignments $\tau : B \to \{0,1\}$, the restriction $F[\tau]$ of F to τ belongs to the base class $\mathcal{C}$. Note that if a strong $\mathcal{C}$-backdoor set of size k is found, then we can decide the satisfiability of the given formula by deciding the satisfiability of 2^k formulas that belong to the base class $\mathcal{C}$. Based on this concept, Nishimura, Ragde, and Szeider [21] propose algorithms for SAT that search for strong backdoor sets of bounded size with respect to the base classes HORN and 2CNF.

Another type of backdoor set can be defined by removing literals from a formula associated with a set B of variables. We say that B is a *deletion $\mathcal{C}$-backdoor set* if $F - B \in \mathcal{C}$, where $F - B$ denotes the formula obtained from F by removing all the literals $x, \overline{x}$ for $x \in B$ from the clauses of F. In fact, this definition forms the basis of the detection of strong HORN-backdoors sets and strong 2CNF-backdoor sets, since B is a strong HORN-backdoor set (strong 2CNF-backdoor set) of a formula F if and only if B is a deletion HORN-backdoor set (deletion 2CNF-backdoor set, respectively). In general, deletion $\mathcal{C}$-backdoor sets are not necessarily strong $\mathcal{C}$-backdoor sets. However, if all subsets of a formula in $\mathcal{C}$ also belong to $\mathcal{C}$ ($\mathcal{C}$ is *clause-induced*), then indeed deletion $\mathcal{C}$-backdoor sets are strong $\mathcal{C}$-backdoor sets.

In this paper we extend the algorithmic use of backdoor sets for SAT to the counting problem #SAT. It is easy to see that the number of models of a formula F equals the sum over the number of models of the restrictions $F[\tau]$ for all truth assignments $\tau : B \to \{0,1\}$ for a set B of variables of F. Hence, if we can solve #SAT for the elements of a base class $\mathcal{C}$ in polynomial time, then we can solve #SAT for a formula F in time $2^k n^{O(1)}$ provided that we know a strong $\mathcal{C}$-backdoor set of F of size at most k. Hence, to convert the above considerations into an algorithm for #SAT, we need to identify a base class $\mathcal{C}$ for which the following holds:

1. #SAT can be solved in polynomial time for formulas in $\mathcal{C}$, and
2. for a given formula F we can find strong $\mathcal{C}$-backdoor sets of bounded size efficiently.

The second condition can be relaxed to deletion $\mathcal{C}$-backdoor sets if $\mathcal{C}$ is clause-induced.

To this end, we introduce the clause-induced class CLU of *cluster formulas*. A cluster formula is a variable-disjoint union of so-called *hitting formulas*; any two clauses of a hitting formula clash in at least one literal. The known

polynomial-time algorithm for computing the number of models of a hitting formula [16] can be extended in a straight-forward way to compute the number of models of a cluster formula.

A strong CLU-backdoor set of size k of a formula F with n variables can obviously be found by exhaustive search, considering all $O(n^k)$ sets of k variables. This approach does not yield a fixed-parameter algorithm and becomes inefficient for large n even if k is small. We show in Section 3.1 that under a certain complexity theoretic assumption, there is no algorithm that is significantly faster than exhaustive search (Theorem 4). We overcome this limitation by restricting by k the size of a smallest deletion CLU-backdoor set. In Theorem 10 we propose a fixed-parameter algorithm that for a given formula either finds a deletion CLU-backdoor set of size at most k or decides that the given formula has no deletion CLU-backdoor set of size at most k.

To develop such an algorithm, we proceed as follows. We associate with every formula F a certain graph $G(F)$, the *obstruction graph* of F, which can be obtained in polynomial time. The vertex set of $G(F)$ is the set of variables of F. We show that every *vertex cover* of $G(F)$ is a strong CLU-backdoor set of F; recall that a vertex cover is a set S of vertices such that every edge is incident with a vertex in S. Now we can apply known vertex cover algorithms, e.g., the algorithm of Chen, Kanj, and Xia [4] for the detection of strong CLU-backdoor sets. Of related interest is Gramm et al.'s work [14] on a graph editing problem involving *cluster graphs* (i.e., disjoint unions of cliques).

1.3 Clustering-Width

We define the *clustering-width* of a formula F as the size of a smallest vertex cover of the obstruction graph of F. It follows from our results that the clustering-width of a formula F is a lower bound on the size of a smallest deletion CLU-backdoor set of F and an upper bound on the size of a smallest strong CLU-backdoor set of F.

Finally, we exhibit a class of formulas of bounded clustering-width for which all the parameters clique-width, branchwidth, and treewidth are unbounded. We also exhibit a class of formulas with unbounded clustering-width for which all the parameters clique-width, branchwidth, and treewidth are bounded. Theorem 16 establishes the incomparability of various parameters with clustering-width.

2 Preliminaries

2.1 SAT and #SAT

We consider propositional formulas in conjunctive normal form (CNF), represented as sets of clauses. That is, a *literal* is a (propositional) variable x or a negated variable $\overline{x}$; a *clause* is a finite set of literals not containing a complementary pair x and $\overline{x}$; a *formula* is a finite set of clauses. For a literal $\ell = \overline{x}$ we write $\overline{\ell} = x$; for a clause C we set $\overline{C} = \{\, \overline{\ell} : \ell \in C \,\}$. For a clause C,

$\mathrm{var}(C)$ denotes the set of variables x with $x \in C$ or $\overline{x} \in C$. Similarly, for a formula F we write $\mathrm{var}(F) = \bigcup_{C \in F} \mathrm{var}(C)$.

We say that two clauses C, D *overlap* if $C \cap D \neq \emptyset$; we say that C and D *clash* if C and $\overline{D}$ overlap. Note that two clauses can clash and overlap at the same time.

A *truth assignment* (or *assignment*, for short) is a mapping $\tau : X \to \{0,1\}$ defined on some set X of variables. We extend τ to literals by setting $\tau(\overline{x}) = 1 - \tau(x)$ for $x \in X$. $F[\tau]$ denotes the formula obtained from F by removing all clauses that contain a literal x with $\tau(x) = 1$ and by removing from the remaining clauses all literals y with $\tau(y) = 0$; $F[\tau]$ is the *restriction* of F to τ. Note that $\mathrm{var}(F[\tau]) \cap X = \emptyset$ holds for every assignment $\tau : X \to \{0,1\}$ and every formula F. A truth assignment $\tau : X \to \{0,1\}$ *satisfies* a formula F if $F[\tau] = \emptyset$. A truth assignment $\tau : \mathrm{var}(F) \to \{0,1\}$ that satisfies F is a *model* of F. We denote by $\#(F)$ the number of models of F. A formula F is *satisfiable* if $\#(F) > 0$. The satisfiability problem SAT is the problem of deciding whether a given formula is satisfiable. #SAT, the counting version of SAT, is the problem of determining $\#(F)$ for a given formula F. SAT and #SAT are complete problems for the complexity classes NP and #P, respectively.

The following concept of connectedness of formulas will be useful below. We call a formula F *connected* if for any two clauses $C, D \in F$ there exists a sequence of clauses $C_1, \ldots, C_r \in F$ such that $C_1 = C$, $C_r = D$, and $\mathrm{var}(C_i) \cap \mathrm{var}(C_{i+1}) \neq \emptyset$ holds for all $i \in \{1, \ldots, r-1\}$. A maximal connected subset of a formula is a *connected component*.

2.2 Parameterized Complexity

Next we give a brief and rather informal review of the most important concepts of parameterized complexity. For an in-depth treatment of the subject we refer the reader to other sources [7,20].

The instances of a parameterized problem can be considered as pairs (I, k) where I is the *main part* of the instance and k is the *parameter* of the instance; the latter is usually a non-negative integer. A parameterized problem is *fixed-parameter tractable* if instances (I, k) of size n (with respect to some reasonable encoding) can be solved in time $O(f(k)n^c)$ where f is a computable function and c is a constant independent of k.

The framework of parameterized complexity offers a *completeness theory*, similar to the theory of NP-completeness, that allows the accumulation of strong theoretical evidence that a parameterized problem is *not* fixed-parameter tractable. This completeness theory is based on the *weft hierarchy* of equivalence classes $\mathrm{W}[1], \mathrm{W}[2], \ldots, \mathrm{W}[\mathrm{P}]$ of certain parameterized decision problems under *parameterized reductions*. A parameterized reduction is a straightforward extension of a polynomial-time many-one reduction that ensures the parameter for one problem maps into the parameter for another (see [7] for details).

Below we will refer to the following parameterized decision problem, which is known to be W[2]-complete [7].

HITTING SET
Instance: A family $\mathcal{S}$ of finite sets $S_1, \ldots, S_m$ and an integer $k \geq 0$.
Parameter: The integer $k \geq 0$.
Question: Is there a subset $R \subseteq \bigcup_{i=1}^{m} S_i$ of size at most k such that $R \cap S_i \neq \emptyset$ for all $i = 1, \ldots, m$? (R is a *hitting set* of $\mathcal{S}$)

2.3 Backdoor Sets

Consider a formula F and a set B of variables of F. A set $B \subseteq \mathrm{var}(F)$ is a *strong backdoor set* of F with respect to $\mathcal{C}$ (or *strong $\mathcal{C}$-backdoor set*, for short) if $B \subseteq \mathrm{var}(F)$ and for every truth assignment $\tau : B \to \{0,1\}$ we have $F[\tau] \in \mathcal{C}$. For every formula F and every set $B \subseteq \mathrm{var}(F)$ we have

$$\#(F) = \sum_{\tau : B \to \{0,1\}} \#(F[\tau]). \tag{1}$$

Thus, if B is a strong $\mathcal{C}$-backdoor set of a formula F, then determining $\#(F)$ reduces to determining the number of satisfying assignments for $2^{|B|}$ formulas of the base class $\mathcal{C}$.

Now consider a *base class* $\mathcal{C}$ of formulas for which the problems #SAT and recognition can be solved in polynomial time. Thus, when we have found a small strong $\mathcal{C}$-backdoor set of F, we can compute $\#(F)$ efficiently. A key question is whether we can find a small backdoor set if it exists. To study this question, we define for every base class $\mathcal{C}$ the following parameterized problem.

STRONG $\mathcal{C}$-BACKDOOR
Input: A formula F and an integer $k > 0$.
Parameter: The integer k.
Question: Does F have a strong $\mathcal{C}$-backdoor set of size at most k?

For base classes that have a certain property, we can relax the problem STRONG $\mathcal{C}$-BACKDOOR as follows. For a formula F and a set X of variables let $F - X$ denote the formula obtained from F by removing all literals x and $\overline{x}$ from the clauses of F. We call a set $B \subseteq \mathrm{var}(F)$ a *deletion backdoor set* with respect to a base class $\mathcal{C}$ (or *deletion $\mathcal{C}$-backdoor set*, for short) if $F - B \in \mathcal{C}$. Furthermore, we define a base class $\mathcal{C}$ to be *clause-induced* if for every $F \in \mathcal{C}$ and every $F' \subseteq F$, also $F' \in \mathcal{C}$.

Lemma 1 *Let F be a formula and $\mathcal{C}$ a clause-induced base class. Every deletion $\mathcal{C}$-backdoor set of F is also a strong $\mathcal{C}$-backdoor set.*

Proof The result follows directly from the fact that $F[\tau] \subseteq F - X$ holds for every truth assignment $\tau : X \to \{0,1\}$. □

For a base class $\mathcal{C}$, smallest deletion backdoor sets can be larger than smallest strong backdoor sets. However, if the detection of strong $\mathcal{C}$-backdoor sets is fixed-parameter intractable, we can still hope that the detection of deletion $\mathcal{C}$-backdoor sets is fixed-parameter tractable. We state the corresponding parameterized problem:

DELETION $\mathcal{C}$-BACKDOOR
Input: A formula F and an integer $k > 0$.
Parameter: The integer k.
Question: Does F have a deletion $\mathcal{C}$-backdoor set of size at most k?

2.4 Hitting Formulas and Cluster Formulas

A formula is a *hitting formula* if any two of its clauses clash (see [18]). A *cluster formula* is the variable-disjoint union of hitting formulas. In other words, a formula is a cluster formula if and only if all its connected components are hitting formulas. We denote the class of all hitting formulas by HIT and the class of all cluster formulas by CLU.

The next lemma is due to an observation of Iwama [16].

Lemma 2 *A hitting formula F with n variables has exactly $2^n - \sum_{C \in F} 2^{n-|C|}$ models.*

Proof Let F be a hitting formula with n variables. For a clause $C \in F$ let T_C denote the set of all truth assignments $\tau : \mathrm{var}(F) \to \{0,1\}$ that *do not* satisfy C. Obviously $|T_C| = 2^{n-|C|}$ since T_C contains exactly those assignments that set all literals in C to 0. Since F is a hitting formula, the sets T_C and $T_{C'}$ are disjoint for any two distinct clauses $C, C' \in F$. Hence the lemma follows. □

Lemma 3 #SAT *can be solved in polynomial time for cluster formulas.*

Proof If a formula F is the variable-disjoint union of formulas $F_1, \ldots, F_q$, then $\#(F) = \prod_{i=1}^{q} \#(F_i)$. Thus the result follows directly from Lemma 2. □

By means of the previous lemma we can consider CLU as the base class for a backdoor set approach to #SAT. Observe that CLU is clause-induced.

3 Clustering Formulas and Backdoor Sets

3.1 Finding Smallest Strong CLU-Backdoor Sets

In this section we show that the detection of strong CLU-backdoor sets is fixed-parameter intractable.

We shall use the following construction. Let D be a directed graph. We associate with D a formula F_D where every arc a of D corresponds to a variable x_a of F, and every vertex v of D corresponds to a clause C_v of F. There is an outgoing arc a from v if and only the clause C_v contains the literal x_a and there is an incoming arc b to v if and only if the clause C_v contains the literal $\overline{x_b}$. Note that if D is the orientation of a complete graph, then F_D is a hitting formula.

Theorem 4 *The problem* STRONG CLU-BACKDOOR is W[2]-hard.

Proof We give a parameterized reduction from the W[2]-complete problem HITTING SET as defined in Section 2.2. Let $\mathcal{S} = S_1, \ldots, S_m$ be an instance of HITTING SET such that $\bigcup_{i=1}^{m} S_i = \{x_1, \ldots, x_n\}$. Let D be an orientation of a complete graph with $r = (m+1)(k+1)$ vertices. Consider the hitting formula F_D. We partition F_D into formulas $F_1, \ldots, F_m, H$ such that each of the partite sets contains exactly $k+1$ clauses. For $i = 1, \ldots, m$ we let

$$F_i' = \{\, C \cup S_i : C \in F_i \,\}.$$

Finally, we set $C^* = \{\overline{x_1}, \ldots, \overline{x_n}\}$ and

$$F = \{C^*\} \cup \bigcup_{i=1}^{m} F_i' \cup H.$$

We claim that $\mathcal{S}$ has a hitting set of size at most k if and only if F has a strong CLU-backdoor set of size at most k.

Let $R \neq \emptyset$ be a hitting set of $\mathcal{S}$ and consider any truth assignment $\tau : R \to \{0,1\}$. If τ sets at least one variable to 0, then τ satisfies the clause C^*; obviously $F[\tau]$ is then a hitting formula. Now assume that τ sets all variables of R to 1. Since R is a hitting set, it follows by definition of the sets F_i' that τ satisfies the formula $\bigcup_{i=1}^{m} F_i'$. Hence $F[\tau] = \{C\} \cup H$ for some subset C of C^*. Thus $F[\tau]$ is the variable disjoint union of the hitting formulas $\{C\}$ and H. We have shown that $F[\tau] \in$ CLU for every truth assignment $\tau : R \to \{0,1\}$; i.e., R is a strong CLU-backdoor set of F.

Conversely, let B be a strong CLU-backdoor set of F with $|B| \leq k$. We show that $R = B \cap \{x_1, \ldots, x_n\}$ is a hitting set of $\mathcal{S}$. Assume to the contrary that there is some $i_0 \in \{1, \ldots, m\}$ such that $R \cap S_{i_0} = \emptyset$. Consider the truth assignment $\tau : B \to \{1\}$. The restriction $F[\tau]$ contains the clause $C = C^* \setminus \overline{B}$ with $\overline{S_{i_0}} \subseteq C$. Since $|B| \leq k$ and $|F_{i_0}'| = k+1$, there is at least one clause $C_{i_0} \in F_{i_0}'$ with $\mathrm{var}(C_{i_0}) \cap B = \emptyset$. Hence $C_{i_0} \in F[\tau]$. We conclude similarly that there is at least one clause $C_H \in H$ with $\mathrm{var}(C_H) \cap B = \emptyset$; thus $C_H \in F[\tau]$. C_{i_0} contains a variable x_i for $i \in \{1, \ldots, n\}$, hence C_{i_0} and C^* clash. However, since $B \cap \mathrm{var}(C_{i_0}) = \emptyset$, $x_i \notin B$ and so $\overline{x_i} \in C$. Moreover, C_{i_0} and C_H clash by the definition of F; since $\mathrm{var}(C_{i_0}) \cap B = \emptyset$ and $\mathrm{var}(C_H) \cap B = \emptyset$, C_{i_0} and C_H belong to $F[\tau]$. In summary, $F[\tau]$ contains three distinct clauses C, C_{i_0}, and C_H with $\mathrm{var}(C) \cap \mathrm{var}(C_{i_0}) \neq \emptyset$ and $\mathrm{var}(C_{i_0}) \cap \mathrm{var}(C_H) \neq \emptyset$, but $\mathrm{var}(C) \cap \mathrm{var}(C_H) = \emptyset$. Thus C and C_H are two clauses that do not clash but belong to the same connected component of $F[\tau]$. Hence $F[\tau]$ is not a cluster formula, a contradiction. Whence R is indeed a hitting set of $\mathcal{S}$. □

A parameterized problem gives rise to a traditional "non-parameterized" problem where the parameter is taken as part of the input. The proof of Theorem 4 gives a polynomial-time many-one reduction of the NP-hard non-parameterized version of HITTING SET [10] to the non-parameterized version of STRONG CLU-BACKDOOR. This shows that the non-parameterized version of STRONG CLU-BACKDOOR is NP-hard.

We will show in sections below that the concept of deletion backdoor sets can be used to find small strong backdoor sets with respect to CLU. Next we give an example that shows that for the base class CLU, smallest deletion backdoor sets can be larger that smallest strong backdoor sets.

Consider the formula

$$F = \{\{x_1, \ldots, x_n\}, \{\overline{x_1}, \ldots, \overline{x_n}, y_1, \ldots, y_n\}, \{\overline{y_1}, \ldots, \overline{y_n}\}\}.$$

Note that each of the variables of F forms a strong CLU-backdoor set of F; e.g., $B = \{x_1\}$ is a strong CLU-backdoor set. However, we need to delete at least n variables in order to obtain a cluster formula. Thus a smallest strong CLU-backdoor set of F has size 1, but every deletion CLU-backdoor set of F has size at least n.

3.2 Obstructions

In the following results, it is helpful to characterize cluster formulas in terms of obstructions. An *overlap obstruction* is a formula $\{C_1, C_2\}$ consisting of two clauses that overlap but do not clash. With an overlap obstruction we associate the following pair of sets of variables:

$$\{\mathrm{var}(C_1 \cap C_2),\ \mathrm{var}(C_1 \triangle C_2)\}.$$

Here $C_1 \triangle C_2$ denotes the symmetric difference $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ of C_1 and C_2. A *clash obstruction* is a formula $\{C_1, C_2, C_3\}$ where C_1 and C_2 clash such that $(C_1 \setminus C_3) \cap \overline{C_2} \neq \emptyset$, C_2 and C_3 clash such that $(C_3 \setminus C_1) \cap \overline{C_2} \neq \emptyset$, and C_1 and C_3 do not clash. (Any two of the three clauses may overlap.) With a clash obstruction we associate the following pair of sets of variables:

$$\{\mathrm{var}((C_1 \setminus C_3) \cap \overline{C_2}),\ \mathrm{var}((C_3 \setminus C_1) \cap \overline{C_2})\}.$$

We say that an overlap or clash obstruction F' is an *obstruction of* a formula F if F' is a subset of F. A pair $\{X, Y\}$ of sets of variables is a *deletion pair* of F if the pair is associated with an overlap or clash obstruction of F. It follows from the definitions of overlap and clash obstructions that the two sets in a deletion pair are nonempty and disjoint.

Lemma 5 *A formula is a cluster formula if and only if it has no overlap or clash obstruction.*

Proof If a formula F contains an overlap or clash obstruction, then there are two clauses $C, D \in F$ that belong to the same connected component of F but do not clash. Hence F is not a cluster formula.

Conversely, consider a formula F that does not contain any overlap or clash obstructions. We show that F is a cluster formula. Consider a connected component F' of F. If $|F'| = 1$ then F' is a hitting formula; hence assume $|F'| > 1$. We show that any two clauses of F' clash. Choose two arbitrary clauses $C, D \in F'$. Since F' is connected, there is a sequence of clauses $C_1, \ldots, C_r \in F$ such that $C_1 = C$, $C_r = D$, and $\mathrm{var}(C_i) \cap \mathrm{var}(C_{i+1}) \neq \emptyset$ holds for all $i \in \{1, \ldots, r-1\}$. We observe that C_i and C_{i+1} clash for all $i \in \{1, \ldots, r-1\}$ since otherwise C_i and C_{i+1} would form an overlap obstruction. It now follows inductively that the clauses C_1 and C_i clash for all $i \in \{3, \ldots, r\}$ since otherwise either C_1 and C_i would form an overlap obstruction or the clauses C_1, C_{i-1}, and C_i would form a clash obstruction. Thus, indeed, C and D clash. Whence F' is a hitting formula. □

Lemma 6 *Let F be a formula and $B \subseteq \mathrm{var}(F)$. If $F-B$ is a cluster formula, then $X \subseteq B$ or $Y \subseteq B$ holds for every deletion pair $\{X,Y\}$ of F.*

Proof Assume that $F - B$ is a cluster formula and suppose to the contrary that there is a deletion pair $\{X,Y\}$ of F such that $X \setminus B \neq \emptyset$ and $Y \setminus B \neq \emptyset$.

First we consider the case that $\{X,Y\}$ is the deletion pair of an overlap obstruction $\{C_1,C_2\} \subseteq F$ with $X = \mathrm{var}(C_1 \cap C_2)$ and $Y = \mathrm{var}(C_1 \Delta C_2)$. Let $C_1' = C_1 \setminus (B \cup \overline{B})$ and $C_2' = C_2 \setminus (B \cup \overline{B})$ and observe that $C_1', C_2' \in F - B$. Since C_1 and C_2 do not clash, also C_1' and C_2' do not clash. Since $Y \setminus B \neq \emptyset$, C_1' and C_2' are distinct; since $X \setminus B \neq \emptyset$, C_1' and C_2' overlap. Hence $\{C_1', C_2'\}$ is an overlap obstruction of $F - B$, and so $F - B$ is not a cluster formula by Lemma 5, a contradiction.

Next we consider the case that $\{X,Y\}$ is the deletion pair of a clash obstruction $\{C_1,C_2,C_3\} \subseteq F$ with $X = \mathrm{var}((C_1 \setminus C_3) \cap \overline{C_2})$ and $Y = \mathrm{var}((C_3 \setminus C_1) \cap \overline{C_2})$. For $i = 1,2,3$ we consider $C_i' = C_i \setminus (B \cup B') \in F - B$. It follows, similarly as in the first case above, that $\{C_1', C_2', C_3'\}$ is a clash obstruction of $F - B$, again a contradiction with Lemma 5.

We conclude that for every deletion pair $\{X,Y\}$ of F either $X \subseteq B$ or $Y \subseteq B$ must be the case. □

3.3 Finding Backdoor Sets Using Vertex Covers

For a formula F let G_F denote the graph with vertex set $\mathrm{var}(F)$; two variables x and y are joined in G_F by an edge if and only if there is a deletion pair $\{X,Y\}$ of F with $x \in X$ and $y \in Y$. We call G_F the *obstruction graph* of F. Note that the obstruction graph of a formula can be constructed in polynomial time.

We consider vertex covers of obstruction graphs. Recall that a *vertex cover* of a graph is a set of vertices that contains at least one end of every edge of the graph. It is NP-hard to determine, given a graph and an integer k, whether the graph has a vertex cover of size at most k. Parameterized by the size of the vertex cover, however, the problem is fixed-parameter tractable. In fact, vertex cover is the best-studied problem in parameterized complexity with a long history of improvements [3]. The current best worst-case time complexity for the parameterized vertex cover problem is due to Chen, Kanj, and Xia [4]:

Theorem 7 *[4] Given a graph G on n vertices, one can find in time $O(1.273^k + nk)$ (and in polynomial space) a vertex cover of G of size at most k, or determine that no such vertex cover exists.*

The next two lemmas relate backdoor sets and vertex covers of obstruction graphs.

Lemma 8 *Every deletion* CLU*-backdoor set of a formula F is a vertex cover of the obstruction graph of F.*

Proof Let B be a deletion CLU-backdoor set of F. Let xy be any edge of the obstruction graph G_F. We show that either x or y is in B. By definition of G_F there is a deletion pair $\{X, Y\}$ of F with $x \in X$ and $y \in Y$. Since $F - B$ is a cluster formula, $X \subseteq B$ or $Y \subseteq B$ follows from Lemma 6. Thus $x \in B$ or $y \in B$ follows. Whence B is indeed a vertex cover of G_F. □

Lemma 9 *Every vertex cover of the obstruction graph of a formula F is a strong* CLU*-backdoor set of F.*

Proof Let B be a vertex cover of the obstruction graph of a formula F. Assume to the contrary that B is not a strong CLU-backdoor set of F. Thus, there is an assignment $\tau : B \to \{0, 1\}$ such that $F[\tau] \notin$ CLU. Let $B_0 = \{\, y \in B \cup \overline{B} : \tau(y) = 0 \,\}$; i.e., B_0 is the set of all literals over variables of B that are mapped to 0 under τ. By Lemma 5, $F[\tau]$ contains overlap or clash obstructions.

First assume that $F[\tau]$ contains an overlap obstruction. Thus $F[\tau]$ contains two clauses C_1, C_2 that overlap but do not clash. For the associated obstruction pair $\{X, Y\}$ with $X = \mathrm{var}(C_1 \cap C_2)$ and $Y = \mathrm{var}(C_1 \triangle C_2)$ choose $x \in X$ and $y \in Y$. By definition of $F[\tau]$ it follows that F contains clauses C_1', C_2' with $C_1 = C_1' \setminus B_0$ and $C_2 = C_2' \setminus B_0$. It follows that C_1' and C_2' overlap but do not clash, thus $\{C_1', C_2'\}$ is an overlap obstruction of F. We have $x \in X \subseteq \mathrm{var}(C_1' \cap C_2')$ and $y \in Y \subseteq \mathrm{var}(C_1' \triangle C_2')$. Thus xy is an edge of G_F. Since B is a vertex cover of G_F, either x or y must belong to B. This contradicts the fact that $\mathrm{var}(F[\tau]) \cap B = \emptyset$.

Next assume that $F[\tau]$ contains a clash obstruction. Thus $F[\tau]$ contains three clauses C_1, C_2, C_3 where C_1 and C_2 clash, C_2 and C_3 clash, and C_1 and C_3 do not clash. The corresponding obstruction pair is $\{X, Y\}$ with $X = \mathrm{var}((C_1 \setminus C_3) \cap \overline{C_2})$ and $Y = \mathrm{var}((C_3 \setminus C_1) \cap \overline{C_2})$. We choose $x \in X$ and $y \in Y$. Similarly as in the first case we conclude that F contains clauses C_1', C_2', C_3' with $C_i = C_i' \setminus B_0$ for $i \in \{1, 2, 3\}$. Obviously C_1' and C_2' clash, C_2' and C_3' clash, but C_1' and C_3' do not clash. Thus $\{C_1', C_2', C_3'\}$ is a clash obstruction of F. We have $x \in X \subseteq \mathrm{var}((C_1' \setminus C_3') \cap \overline{C_2'})$ and $y \in Y \subseteq \mathrm{var}((C_3' \setminus C_1') \cap \overline{C_2'})$. Thus xy is an edge of G_F. Since B is a vertex cover of G_F either x or y must belong to B. This contradicts the fact that $\mathrm{var}(F[\tau]) \cap B = \emptyset$.

Whence it follows that B is indeed a strong CLU-backdoor set of F. □

From Theorem 7 and the previous two lemmas we get immediately the main result of this section.

Theorem 10 *Given a formula with n variables together with its obstruction graph and an integer k, in time $O(1.273^k + nk)$ we can find a strong* CLU*-backdoor set of F of size at most k, or decide that the size of every deletion* CLU*-backdoor set of F exceeds k.*

3.4 Clustering-width

In the following, we consider as a parameter any computable function p that assigns to each formula F a non-negative integer $p(F)$. We assume that the parameter is invariant under changing the names of variables.

The following three parameters arise from the considerations of this paper. We denote by $\mathrm{str}_{\mathrm{CLU}}(F)$ the size of a smallest strong backdoor set of a formula F with respect to CLU, and we denote by $\mathrm{del}_{\mathrm{CLU}}(F)$ the size of a smallest deletion backdoor set of F with respect to CLU. The *clustering-width* $\mathrm{clu}(F)$ of F is the size of a smallest vertex cover of the obstruction graph of F. Consequently, HIT is the class of formulas with clustering-width 0. From Lemmas 1 and 8 we know that for every formula F the following holds:

$$\mathrm{str}_{\mathrm{CLU}}(F) \leq \mathrm{clu}(F) \leq \mathrm{del}_{\mathrm{CLU}}(F). \tag{2}$$

For a parameter p we consider the following generic parameterized problem.

#SAT(p)
Instance: A formula F and a non-negative integer k such that $p(F) \leq k$.
Parameter: The integer k.
Question: What is the total number of models of F? (I.e., what is the number $\#(F)$?)

The definition of fixed-parameter tractability carries over from decision problems to counting problems in a natural way. Flum and Grohe [9] provide a framework of intractability of parameterized counting problems.

Note that the above formulation of #SAT(p) is a "promise problem" in the sense that we only need to consider instances (F, k) for which we can take as granted that $p(F) \leq k$ holds. However, for most parameters p considered in the sequel for which #SAT(p) is fixed-parameter tractable, deciding whether $p(F) \leq k$ actually holds is also fixed-parameter tractable with respect to the parameter k. An exception is the parameter $\mathrm{del}_{\mathrm{CLU}}$; however, also in that case we do not depend on the promise as will be discussed below.

By Theorem 10, deciding whether $\mathrm{clu}(F) \leq k$ is fixed-parameter tractable; if $\mathrm{clu}(F) \leq k$, then it is also fixed-parameter tractable to produce a strong CLU-backdoor set B of F of size at most k. We then compute $\#(F)$ as the sum of $\#(F[\tau])$ over all truth assignments $\tau : B \to \{0,1\}$. Whence we have the following corollary to Theorem 10. The algorithm is summarized below. The first step can be executed in polynomial time. The fixed-parameter tractability of computing a vertex cover follows from Theorem 7. Finally, each $\#F[\tau]$ can be computed in polynomial time, as each is a clustering formula.

Algorithm: Input F, k
Step 1: Compute the obstruction graph G_F.
Step 2: Compute a vertex cover B of size at most k of G_F: if such a vertex cover does not exist, stop and output "$\mathrm{clu}(F) > k$".
Step 3: Compute $s = \sum_{\tau:B\to\{0,1\}} \#F[\tau]$ and output s.

Corollary 11 *The problem* #SAT(clu) *is fixed-parameter tractable.*

Note that the algorithm outlined above also checks whether the promise $\mathrm{clu}(F) \leq k$ is true. Furthermore, from (2) it follows that every instance (F, k) of #SAT($\mathrm{del}_{\mathrm{CLU}}$) is also an instance of #SAT(clu). Whence Corollary 11 also implies fixed-parameter tractability of #SAT($\mathrm{del}_{\mathrm{CLU}}$).

Corollary 12 *The problem* $\#\mathrm{SAT}(\mathrm{del}_{\mathrm{CLU}})$ *is fixed-parameter tractable.*

Although we do not know whether DELETION $\mathcal{C}$-BACKDOOR is fixed-parameter tractable, we emphasize that the algorithm for Corollary 12 will not produce an incorrect solution, even if the promise $\mathrm{del}_{\mathrm{CLU}}(F) \leq k$ does not hold. Consider F and k with $\mathrm{del}_{\mathrm{CLU}}(F) > k$. The algorithm checks whether $\mathrm{clu}(F) \leq k$. If $\mathrm{clu}(F) \leq k$, then the algorithm outputs the correct solution $\#\mathrm{SAT}(F)$. If, however, $\mathrm{clu}(F) > k$, then we know by (2) that also $\mathrm{del}_{\mathrm{CLU}}(F) > k$, hence the algorithm can reject the input.

4 Comparison with Other Parameters

4.1 Other Width Parameters

Several parameters are defined in terms of the following directed and undirected graphs associated with a formula F. The *primal graph* $P(F)$ is the graph whose vertices are the variables of F, and where two variables x and y are joined by an edge if and only if F contains a clause C with $x, y \in \mathrm{var}(C)$. The *formula hypergraph* $\mathcal{H}(F)$ is the hypergraph whose vertices are the variables of F and whose hyperedges are the sets $\mathrm{var}(C)$ for clauses C of F. The *incidence graph* $I(F)$ is the bipartite graph where one vertex class consists of the variables of F, the other vertex class consists of the clauses of F; a variable x and a clause C are joined by an edge if and only if $x \in \mathrm{var}(C)$. The *directed* or *signed* incidence graph $I_d(F)$ arises from $I(F)$ by orienting edges from C to x if $x \in C$, and from x to C if $\overline{x} \in C$. The *underlying graph* G_D of a directed graph D is the undirected graph obtained from D by "forgetting" the orientation of edges and by replacing parallel edges by a single representative. Thus $I(F)$ is the underlying graph of $I_d(F)$. For an undirected graph G we consider its treewidth $\mathrm{tw}(G)$, its branchwidth $\mathrm{bw}(G)$, and its clique-width $\mathrm{cwd}(G)$; clique-width is also defined for directed graphs. These graph parameters are defined below; for more detailed discussion we refer the reader to related work [1,2,5,6,13,26].

The notion of tree-width is derived from the decomposition of a graph as a tree. Let G be a graph, $T = (V, E)$ a tree, and χ a labeling of the vertices of T by sets vertices of G. Then (T, χ) is a *tree decomposition* of G if the following conditions hold:

(T1) Every vertex of G belongs to $\chi(t)$ for some vertex t of T;
(T2) for every edge (v, w) of G there is some vertex t of T such that $v, w \in \chi(t)$;
(T3) for any vertices t_1, t_2, t_3 of T, if t_2 lies on a path from t_1 to t_3, then $\chi(t_1) \cap \chi(t_3) \subseteq \chi(t_2)$.

The *width* of a tree decomposition (T, χ) is the maximum $|\chi(t)| - 1$ over all vertices t of T. The tree-width $tw(G)$ of G is the minimum width over all its tree-decompositions.

A branch decomposition is defined in terms of a hypergraph. Let $\mathcal{H}$ be a hypergraph, $T = (V, E)$ a ternary tree (i.e., all vertices of T have either degree 0 or 3), and τ a bijection from the set of leaves of T to the set of

hyperedges of $\mathcal{H}$; (T, τ) is called a *branch decomposition* of $\mathcal{H}$. The *order* of an edge e of T is the number of vertices of $\mathcal{H}$ which are incident to hyperedges $\tau(t_1), \tau(t_2)$ such that t_1 and t_2 belong to different components of $T - e$. The *width* of a branch decomposition (T, τ) is the maximum order of all edges of T; the *branch-width* $bw(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the smallest width over all its branch decompositions.

Finally, clique-width is defined as follows. For k a positive integer, a *k-graph* is a graph whose vertices are labeled by integers from $\{1, \ldots, k\}$. We consider an arbitrary graph as a k-graph with all vertices labeled by 1. We call the k-graph consisting of exactly one vertex v (say, labeled by $i \in \{1, \ldots, k\}$) an *initial k-graph* and denote it by $i(v)$. Let $\mathcal{C}(k)$ denote the class of k-graphs which can be constructed from initial k-graphs by means of the following three operations.

(C1) If $G, H \in \mathcal{C}(k)$ and $V(G) \cap V(H) = \emptyset$, then the union of G and H, denoted by $G \oplus H$, belongs to $\mathcal{C}(k)$.
(C2) If $G \in \mathcal{C}(k)$ and $i, j \in \{1, \ldots, k\}$, then the k-graph $\rho_{i \to j}(G)$ obtained from G by changing the labels of all vertices which are labeled by i to j belongs to $\mathcal{C}(k)$.
(C3) If $G \in \mathcal{C}(k)$, $i, j \in \{1, \ldots, k\}$, and $i \neq j$, then the k-graph $\eta_{i,j}(G)$ obtained from G by connecting all vertices labeled by i with all vertices labeled by j belongs to $\mathcal{C}(k)$.

The *clique-width* $\mathrm{cwd}(G)$ of a graph G is the smallest integer k such that $G \in \mathcal{C}(k)$.

By means of primal, incidence and directed incidence graphs, these graph parameters apply to formulas as follows: For a formula F we call $\mathrm{tw}(F) = \mathrm{tw}(P(F))$ the *primal treewidth* of F, $\mathrm{tw}^*(F) = \mathrm{tw}(I(F))$ the *incidence treewidth* of F, $\mathrm{bw}(F) = \mathrm{bw}(\mathcal{H}(F))$ the *branchwidth* of F, $\mathrm{cwd}(F) = \mathrm{cwd}(I_d(F))$ the *clique-width* of F.

4.2 Comparisons

In this section we introduce a general framework for comparing parameters that allow fixed-parameter algorithms for #SAT.

We introduce the notion of dominance for the purpose of relating parameters with respect to fixed-parameter tractability, so that if p dominates q, then #SAT(p) being fixed-parameter tractable implies that #SAT(q) is fixed-parameter tractable and #SAT(q) being W-hard implies #SAT(p) is W-hard. For two formula parameters p and q we say that p *dominates* q if there is a computable function f such that $p(F) \leq f(q(F))$ holds for all formulas F, and that p *strictly dominates* q if p dominates q but q does not dominate p. We say that p and q are *incomparable* if neither p dominates q nor q dominates p. From known results it follows that clique-width strictly dominates incidence treewidth, and that, in turn, incidence treewidth dominates primal treewidth and branchwidth [26]. Whence, clique-width can be considered as the most general parameter considered so far. The fixed-parameter tractability of #SAT(cwd) follows from work of Courcelle, Makowsky, and Rotics [5]

and that of Oum and Seymour [22], improved by Fischer, Makowsky, and Ravve [8]. Samer and Szeider [25] present dynamic programming algorithms for #SAT(tw) and #SAT(tw*). By the above relationships among the various parameters, this result also implies the fixed-parameter tractability of #SAT(tw*), #SAT(tw), and #SAT(bw):

Theorem 13 *[1, 5, 8, 22, 26] The problems* #SAT(cwd)*,* #SAT(tw*)*,* #SAT(tw)*, and* #SAT(bw)*, are fixed-parameter tractable.*

The question arises how our new parameter, the clustering-width, is related to the other parameters. Does any of the above parameters dominate clustering-width, or does clustering-width dominate any of the other parameters? We will show that the answer to both questions is 'no': clustering-width is *incomparable* with any of the other parameters.

Lemma 14 *The class* HIT *has unbounded clique-width.*

Proof Let $n \geq 3$ be an integer and let G denote an $n \times n$ grid. That is, G is a bipartite graph with n^2 vertices $v_{i,j}$, $i, j \in \{1, \ldots, n\}$, where two vertices $v_{i,j}$ and $v_{i',j'}$ are joined by an edge if and only if either $i = i'$ and $|j - j'| = 1$, or $|i - i'| = 1$ and $j = j'$. Let V_1, V_2 be a bipartition of the vertex set of G. We obtain a formula F with $I(F) = G$ by considering vertices in V_1 as variables and setting $F = \{ N(v_{i,j}) : v_{i,j} \in V_2 \}$; here $N(v_{i,j})$ denotes the set of neighbors of $v_{i,j}$ in G.

Consider a directed graph D whose underlying graph is the complete graph K_m for $m = |V_2|$. We construct the hitting formula F_D as described at the beginning of Section 3.1; we assume that F and F_D do not share variables. Observe that $|F_D| = m$; thus we can write $F = \{C_1, \ldots, C_m\}$ and $F_D = \{C_{1,D}, \ldots, C_{m,D}\}$, ordering the clauses arbitrarily.

Let H be the formula $\{C_1 \cup C_{1,D}, \ldots, C_m \cup C_{m,D}\}$. Clearly H is a hitting formula since F_D is a hitting formula. Golumbic and Rotics [11] show that the clique-width of $n \times n$ grids, $n \geq 3$, is exactly $n + 1$, hence $\mathrm{cwd}(G) = n + 1$. Note that $I(F) = G$ is isomorphic to a vertex-induced subgraph of $I(H)$; this implies that $\mathrm{cwd}(H) \geq \mathrm{cwd}(G) = n + 1$ (see Courcelle and Olariu [6]). Moreover, also noted by Courcelle and Olariu, the clique-width of a directed graph is at least as large as the clique-width of its underlying graph; hence we have $\mathrm{cwd}(I_d(H)) \geq \mathrm{cwd}(I(H)) \geq \mathrm{cwd}(I(F)) = \mathrm{cwd}(G) = n + 1$. We conclude that for every positive integer n there exists a hitting formula H with $\mathrm{cwd}(H) > n$. □

Lemma 15 *The class of formulas with primal treewidth* 1 *has unbounded clustering-width.*

Proof Let $\mathcal{C}$ denote the class of formulas with primal treewidth 1. Let n be an even positive integer and consider the formula

$$F = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}\}.$$

The primal graph of F is a path. Since paths have treewidth 1, $F \in \mathcal{C}$ follows.

For every $i = 1, \ldots, n - 1$, the formula F contains the overlap obstruction $\{\{x_{i-1}, x_i\}, \{x_i, x_{i+1}\}\}$ with the corresponding deletion pair

$\{\{x_i\}, \{x_{i-1}, x_{i+1}\}\}$. There are no clash obstructions. The obstruction graph is therefore a path P on the vertices $x_1, \ldots, x_n$. Any vertex cover of P contains at least $n/2$ vertices, hence $\mathrm{clu}(F) \geq n/2$ follows.

As we can choose arbitrarily large n, $\mathcal{C}$ has unbounded clustering-width. □

In view of the relationships among the parameters cwd, tw*, tw, and bw stated above, the last two lemmas imply the following result.

Theorem 16 *The parameters* cwd*,* tw**, tw, and* bw*, are all incomparable with clustering-width.*

5 Conclusion

Our main contributions include an exact algorithm for counting the models of formulas in CNF as well as a resulting hardness index (clustering-width). The approach used in our algorithm is that of detecting strong CLU-backdoor sets of bounded size, applying a vertex cover algorithm to an obstruction graph associated with the formula. Given a formula and its obstruction graph, for a parameter k we can in $O(1.273^k + nk)$ time either find a strong CLU-backdoor set of size at most k or decide that every deletion CLU-backdoor set of F exceeds k, where n is the number of variables in the formula.

The clustering width of a formula is the size of a smallest vertex cover of the obstruction graph of the formula. We demonstrate that this parameter is incomparable to the parameters clustering-width, primal treewidth, treewidth, and branchwidth.

It would be interesting to complement our theoretical results with empirical evidence on the significance of our new parameter. In particular, it would be interesting to know the clustering-width of CNF formulas that encode real-world instances from different domains. However, one must choose the encoding carefully in order to avoid a large clustering-width caused by the gadgets of the encoding itself.

Finally, we observe that as a consequence of our results there is an efficient algorithm to determine whether or not a CNF formula has small clustering-width. For any #SAT algorithm on any type of inputs, we can use that algorithm as subroutine in a variant of the algorithm, where the subroutine is used to determine whether the clustering-width is small, and if so, to run our #SAT algorithm.

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