# Valuation Of Segregated Funds: Shout Options With Maturity Extensions

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**Abstract**

One of the most popular investments available in the Canadian market today is a mutual fund with the added feature of a long term maturity guarantee. These types of investments are known as segregated funds. They are similar in many respects to variable annuity savings plans in the United States. Segregated funds have very complex option features. These contracts contain multiple embedded shout options which permit the holder to reset the guarantee level and the maturity date for which it applies many times during the life of the contract. These funds also provide mortality benefits if the investor dies prior to the maturity date. This paper explores the valuation of segregated funds using an approach based on the numerical solution of a set of linear complementarity problems. Our results indicate that the option components of many of these contracts seem to be underpriced, especially for riskier funds with relatively high volatilities. This assumes that investors exercise their options optimally. Non-optimal behaviour by investors of course reduces the values of the embedded options, and we provide some illustrative results along these lines. We also show that alternative contract specifications which generate similar present values may require substantially different proportionate fees since the expected durations of the contracts can be quite different.

*Keywords:* Segregated funds, maturity guarantees, option pricing, shout options, linear complementarity

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# 1 Introduction

Segregated funds are Canadian mutual fund investments with the added feature of a maturity guarantee. At a broad level, these contracts can be viewed as insurance policies with embedded financial options. Although these products have been available for a long time, recently the market for them has been increasing rapidly. Net new sales of these products skyrocketed from \$C 700 million in 1996 to \$C 5.9 billion in 1998. The number of segregated funds available increased from about 250 to more than 800 during that time.[1] The current size of the market is estimated to be around \$C 60 billion.[2] Variable annuity savings plans available in the United States are somewhat similar contracts. During 1997, sales of these products were about \$US 85 billion (Milevsky and Posner, 1999).

The academic literature on valuing and hedging option features in insurance products is extensive, dating back at least to Brennan and Schwartz (1976) and Boyle and Schwartz (1977). A very incomplete list of more recent papers on this subject would include those of Bacinello and Ortu (1993), Aase and Persson (1994), Ekern and Persson (1996), Persson and Aase (1997), Boyle and Hardy (1997), Gerber and Shiu (1999), Hardy (1999), and Milevsky and Posner (1999). It is beyond the scope of this paper to provide a summary of this literature; interested readers can find an excellent overview in Ekern and Persson (1996). However, at the risk of doing some injustice to the authors of these papers, it is probably fair to say that in most cases the option features being valued are relatively simple (much less complicated than the options included in segregated funds). In fact, in most of this literature the attached options are of the plain vanilla variety, though when mortality risk is included the maturity dates of the options are uncertain. A prominent exception is Ekern and Persson (1996), which contains analysis of contracts with barrier features, options to switch between two underlying assets, contracts with chooser option components, etc.

It is well-known that a contract with a simple maturity guarantee attached to it can be viewed as an investment in the underlying asset and a European put option. The guarantee level is equal to the strike price of the option involved. In the case of a segregated fund, the guarantee applies for a period of least ten years from the inception of the contract. The holder of the contract is often permitted to reset the level of the guarantee during the term of the contract. There can be many such rights, e.g. two opportunities to reset per year (at any time in the year) for ten years. Typically, upon resetting the strike price is set equal to the prevailing value of the underlying asset, but this is not always the case. For example, multiple resets may involve sequentially reduced guarantee levels. A representative pattern might be as follows: the first reset is to 100% of the value of the underlying asset, the second reset is to 98%, the third to 96%, and so on.

As described thus far, these contracts are a form of shout option, with multiple exercise opportunities. A basic shout option (with a single exercise opportunity) is an option contract which

[1] *The Globe and Mail*, October 23, 1999.

[2] W. Falloon, "Canada's option nightmare", *Risk*, August 1999.

allows the holder to set the strike price equal to some function of the current level of the underlying asset at any time during the life of the contract. If the reset level is simply the prevailing value of the underlying asset, the investor is confronted with the choice of when to take ownership of an at-the-money option. Explicit type (i.e. binomial/trinomial lattice) methods for valuing these types of contracts under the assumption that the underlying asset is described by geometric Brownian motion have been presented in Thomas (1993) and Cheuk and Vorst (1997). Shout option contracts with multiple exercise opportunities have been considered by Cheuk and Vorst (1997) and further analyzed by Windcliff et al. (1999), Windcliff (2000), and Windcliff et al. (2000).

This paper concentrates on the additional complications which arise in the case of segregated funds due to maturity extension features and death benefits. It is frequently the case that when the investor resets the guarantee level (i.e. "shouts"), both the strike price *and* the maturity date of the contract change. For example, the holder of a contract with an original maturity of ten years who shouts might receive a contract with a new strike price and a new maturity date of ten years from the time of exercise (not the original inception of the contract). Financial contracts with extendible maturities have been studied for the cases of bonds (Brennan and Schwartz, 1977; Ananthanarayanan and Schwartz, 1980) and equity options (Longstaff, 1990). The contracts analyzed in these papers are generally simpler than those considered in this work in that there are more opportunities to extend the maturity of the contract in the case of segregated funds. Numerical issues related to the maturity extension aspect of these contracts are described in detail below.

Segregated funds also provide the investor with death benefits: if the investor expires before the contract does, then the contract is paid off immediately. It is traditionally assumed in the literature that mortality risk is completely diversifiable. In papers such as Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Bacinello and Ortu (1993) mortality is modelled in a discrete manner. In other words, death occurs (or at least payments are only made by the insurer) at the end of a period. By contrast, we follow Ekern and Persson (1996) and Milevsky and Posner (1999) and model mortality in a continuous fashion.

The valuation problem for a segregated fund is closely related to the pricing problem for shout options. There are many important numerical and modelling issues which must be considered when valuing shout options which are still pertinent to the valuation of segregated funds. These include:

- While the Black-Scholes setting (i.e. geometric Brownian motion) is obviously a very important benchmark, it is worth considering departures from it for several reasons. First, there is mounting evidence that other option valuation models may outperform the Black-Scholes model, particularly for long term options (see Bakshi et al., 1997, 2000; Hardy, 1999, and references therein). Moreover, alternative specifications such as constant elasticity of variance (Cox, 1975) (CEV) models have been recently shown to imply dramatically different prices for some types of exotic options, particularly barriers and lookbacks (Davydov and Linetsky, 1999; Boyle and Tian, 1999). To handle these types of generalizations of Black-Scholes, a numerical partial differential equation (PDE) method for valuing complex shout options was developed in Windcliff et al. (1999, 2000).[3] Results reported in those papers indicate that, as for barriers and lookbacks, shout option values are quite sensitive to the assumed stochastic process for the underlying asset. This is not surprising, as a shout option can be viewed as a restricted version of a lookback (Cheuk and Vorst, 1997).

- Any numerical scheme will be restricted to utilizing data on a finite mesh. This raises two issues. First, the original pricing problem is posed on an unbounded domain. We need to ensure that the truncation of the solution domain does not cause significant inaccuracy in our computed values. Second, we will be required to interpolate our numerical solution at every timestep in order to model decisions made by the investor. The interpolation method used must be selected with care so as to maximize computational accuracy.

- Since these contracts are typically long term, efficiency of computation is important. The performance of numerical PDE methods can be improved by utilizing high-order timestepping methods and automatic timestep selection, which is not possible with simple lattice methods. Further, the parallel nature of the valuation algorithm developed naturally lends itself to high-performance multiprocessor implementations.

- From a financial modelling perspective, the assumption of constant interest rates and volatilities is highly suspect due to the long term nature of these contracts. Uncertain parameter models (Avellaneda et al., 1995; Wilmott, 1998) are potentially useful in this regard.

These items are dealt with extensively in previous work in the context of shout options (Windcliff et al., 1999, 2000; Windcliff, 2000) and will not be repeated at length here. To briefly summarize, we use PDE methods from the path-dependent options literature (see e.g. Wilmott et al., 1993; Wilmott, 1998; Vetzal and Forsyth, 1999; Zvan et al., 1999). The basic approach involves formulating the pricing problem as a collection of one dimensional PDEs which are embedded in a higher dimensional space and which periodically exchange information with each other. These one dimensional problems are solved using standard techniques. For the application considered in this paper, a Crank-Nicolson method with an automatic time step size selector was used for the individual component PDEs, following the approach described for shout options in Windcliff et al. (1999, 2000) and Windcliff (2000). The reader is referred to those sources for further details regarding these aspects of our computational methods.

[3] As an alternative to PDE methods, Monte Carlo techniques may also be used to handle complex contractual specifications. However, it is not straightforward to incorporate optimal exercise timing decisions. See Boyle et al. (1999) for an application of the stochastic mesh method described in Broadie and Glasserman (1998) to complex shout options (without maturity extensions). Note that this method requires: i) that the continuous early exercise feature of the contracts can be well-approximated by exercise decisions at a set of specified discrete points in time; and ii) that the Green's function for the problem be easily computable.

Several additional complications arise when valuing segregated funds (beyond those encountered for shout options). This paper provides a detailed discussion of the following issues:

- The contract extension feature generates additional numerical complexity because it introduces a moving solution domain. Again, since the numerical solution must be interpolated at every timestep, the choice of interpolation method affects the convergence of the numerical scheme dramatically.

- In order to model the effect of mortality benefits, we assume that the risk is diversifiable over a population demography group (e.g. female, non-smoker, 50 years of age). We can then model the valuation effects of individual deaths as a type of *coupon payment* made out of this aggregate fund occuring at a deterministic rate.

- The value of these contracts is typically quite high. Rather than paying an up front single premium, the investor is usually charged a proportional fee (similar to the management expense fee on a mutual fund) to cover the cost of providing the guarantee. Since the length of the contract is uncertain (due to the maturity extension feature, mortality, and lapsing), the up front value of the guarantee is not easily related to the required proportional expense fee.

- Investors are able to lapse (i.e. withdraw their account). We discuss the effect of two distinct types of lapses. First, investors may lapse because they are in need of liquid assets. Second, they may (optimally) lapse if the value of the guarantee feature of the segregated fund is less than the value of the fees that they are required to pay in order to stay in the fund.

It should be emphasised at this point that, unless otherwise specified, we value these contracts under the assumption that investors will exercise their options in an optimal fashion. Of course, investors do not always do so. As a result, the option values computed assuming optimal behaviour can be viewed as conservative from the perspective of the insurance firms selling these contracts. We also provide some results for a simple model of non-optimal investor behaviour. This of course leads to a reduction in the value of the contract. Unfortunately, we have no way of knowing whether our assumed non-optimal behaviour is realistic because we have no access to data on the behaviour of individual contract holders. Consequently, we are reluctant to draw any firm conclusions here. It is perhaps also worth noting that although the individual investors may not act optimally, there is the chance that an intermediary could do so on their behalf.[4]

The outline of the paper is as follows. Sections 2 and 3 describe the basic model and the numerical solution algorithm. We reiterate that the emphasis here is on those aspects of the valuation problem which differ from the simpler case of shout options without the extendibility and

[4]There have been cases where independent sales representatives, as a sales support feature, have assisted their clients in making decisions regarding resetting their guarantee levels, leading towards optimal investor behaviour.

mortality features of segregated funds. Readers interested in details regarding these aspects of our algorithm should consult Windcliff et al. (1999, 2000). A variety of alternative modelling issues and illustrative examples are presented in Section 4. Section 5 summarizes our main conclusions.

## 2 The Model

In order to be precise we define a shout option with a maturity extension as:

**Definition:** *An **extendible shout option** is a contract defined by the following objects:*

- *An underlying stochastic asset price process $S$ upon which the derivative security is written.*
- *A maturity time $T$ for the contract. Since $T$ is a state variable which depends on the choices made by the holder during the life of the contract, we also define a maximum possible value of the maturity time $T_{max}$. This is consistent with most current contract specifications.*
- *A payoff function $g(S, K)$ that determines the payment made to the holder of the security at maturity $T$ which is a function of the asset level $S$ and the strike $K$, assuming that no premature death benefits are paid out. Note that the strike and maturity can be changed during the life of the contract, depending on the holder's actions.*
- *A maximum number of times $U_{max}$ which the holder of the security can "shout" during a given time period, each time resetting the strike $K$ (and possibly the maturity date $T$). We will use the discrete variable $U = 0, 1, \ldots, U_{max}$ to count the number of shout opportunities used at any point in time.*
- *A function $\mathcal{F}(S, K, T, U, t)$ which determines how the new strike $K^*$ is set upon shouting. In the case of a protective floor where the holder receives a strike setting at the current asset level this function would be defined as*

$$K^* = \mathcal{F}(S, K, T, U, t) = S.$$

- *A function $\mathcal{E}(S, K, T, U, t)$ which determines how the maturity of the contract is to be extended upon shouting. For simplicity, we will assume that the new maturity time $T^*$ of the contract received upon shouting at time $t$ is*

$$T^* = \mathcal{E}(S, K, T, U, t) = \min(t + T_{ext}, T_{max}), \tag{1}$$

  *where $T_{ext}$ is the maturity extension upon shouting.*

In the case of a simple American put option, the value of the contract is constrained to be at least as high as the payoff from immediate exercise. With a shout option, the investor does

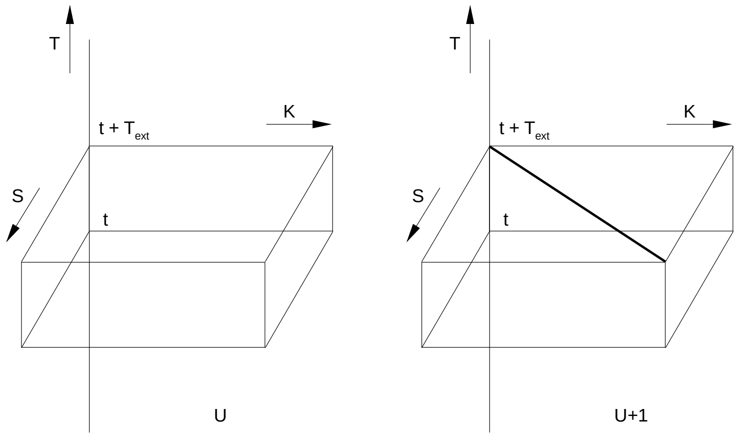



FIGURE 1: *Illustration of how the constraint $V^* = V(S, K = S, T = t+T_{ext}, U+1, t)$ is determined. Note that $V^*$ is obtained for the tuple $(S, K, T, U)$ by interpolation along the line $K = S$, $T = t + T_{ext}$, in the cube $U + 1$. This is shown as the solid line in the right panel.*

not receive a cash payoff from exercise, but instead receives a new contract. In other words, the valuation problem involves a minimum value constraint, $V^*$, which represents the value of the contract the holder receives if they decide to shout,

$$V^*(S,K,T,U,t) = \begin{cases} V(S,K^*,T^*,U+1,t) & \text{if shout permitted} \\ -\infty & \text{otherwise} \end{cases}. \tag{2}$$

The holder cannot shout if, for example, the number of shout opportunities available during this time period has been used up. More precisely, shouts are only allowed if $U + 1 \leq U_{max}$. As well, there are often restrictions on the maximum time allowed for shouting. For example, shouting might not be permitted if $t + T_{ext} > T_{max}$, where $T_{ext}$ is the normal maturity extension upon shouting. Figure 1 shows how the constraint $V^*$ is determined from the data contained in the solution domain.

We assume the existence of a risk free money market account, for which the rate of return is $r(t)$, and that the underlying asset $S$ satisfies the stochastic differential equation

$$dS = (\mu(S,t) - q(t))S dt + \sigma(S,t) S dz, \tag{3}$$

where $dz$ is the increment of a Wiener process, $\mu(S,t)$ is the drift rate, $q(t)$ is a proportional dividend rate, and $\sigma(S,t)$ is the volatility.

Applying standard no-arbitrage arguments for valuing American-type options, we can then model the value of the embedded guarantee, $V = V(S, K, T, U, t)$, as satisfying the linear complementarity problems

$$V_t + (r(t) - q(t))SV_S + \frac{1}{2}\sigma(S,t)^2 S^2 V_{SS} - r(t)V + \mathcal{M}(t)g(S,K) \leq 0 \tag{4}$$

$$V^* \leq V, \tag{5}$$

where at least one of (4-5) holds with equality. The term $\mathcal{M}(t)g(S,K)$ in equation (4) represents the immediate payout made to the estates of deceased holders of this option. The mortality function $\mathcal{M}(t)$ is such that the fraction of the original holders of these contracts who die between $t$ and $t + dt$ is given by $\mathcal{M}(t)dt$. We assume that $\mathcal{M}(t)$ is known with certainty. In essence, we are valuing a large collection of these contracts, so that there is negligible error in assuming that $\mathcal{M}(t)$ is known. In contracts where no death benefits are paid, $\mathcal{M}(t) \equiv 0$.

### 2.1 Maturity Conditions

At contract expiry $t = T$ we impose the terminal condition

$$V(S, K, T, U, T) = \mathcal{R}(T)g(S, K), \tag{6}$$

where $\mathcal{R}(t)$ denotes the fraction of the original investors who are *remaining* in the contract at time $t$. For example, if death benefits are paid, then only the proportion

$$\mathcal{R}(t) = 1 - \int_0^t \mathcal{M}(u)du \tag{7}$$

of the original holders of the contract are still living to take the payout at $t = T$. Of course, those holders who die before the maturity of the option receive an immediate payout, which is taken into account in equation (4). We will later extend the definition of $\mathcal{R}(t)$ to incorporate a deterministic lapse rate.

Although all of the methods discussed in this paper can be used with arbitrary $g(S, K)$, for simplicity we will restrict attention to the case of put options, where the payoff is given by

$$g(S, K) = \max(K - S, 0). \tag{8}$$

### 2.2 No-Arbitrage Jump Conditions

As noted earlier, in many cases segregated funds contracts are sold with the feature that the holder is allowed to shout up to a maximum of $U_{max}$ times during a given time interval; normally one year. At the end of this time interval the discrete shout counter variable $U$ is reset to zero. If $t_i$ is a shout counter reset time, then no-arbitrage considerations require that the condition

$$V(S, K, T, U, t_i^-) = V(S, K, T, 0, t_i^+) \tag{9}$$

be satisfied for all $U$ where $t_i^-$ and $t_i^+$ are the times immediately before and after $t_i$.

### 2.3 Asymptotic Conditions

Since we will be using an implicit numerical scheme we must also specifiy the behaviour of the solution on the boundary $S \to \infty$. We can develop asymptotic conditions (see Windcliff, 2000) which apply for $S \gg K$. If there are no shout opportunities left, then the contract is simply a standard European put option which satisfies the asymptotic condition $V\big|_{S \gg K} = 0$. The boundary condition which is applied on planes with shout opportunities remaining ($U \neq U_{max}$) is given by

$$V(S \to \infty, K, T, U, t) = \begin{cases} V^* & \text{if it is optimal to shout} \\ V_t = q(t)V - \mathcal{M}(t)g(S, K) & \text{otherwise} \end{cases}. \tag{10}$$

Windcliff (2000) demonstrates that if the volatility $\sigma$ is bounded within some range, then $V$ becomes linear in $S$ as $S \to \infty$. The condition involving $V_t$ in equation (10) follows from substitution into equation (4). Note that these asymptotic conditions are superseded by the jump condition (9) at shout counter reset times.

## 3 An Algorithm For Pricing Segregated Funds

The equation to be solved (4-5) is four dimensional, time dependent, non-linear, and constrained by the solutions of other similarly complex equations. However, as described below, this PDE can be simplified by exploiting the structure of the dependence on the variables $K$, $T$ and $U$.

Although the value of the extendible shout option is in general a function of the variables $(S, K, T, U, t)$, we can see from equation (4) that the dependence on $(K, T, U)$ is only through the terminal condition (6) and the minimum value constraint, $V^*$. Once $V^*$ is known, the equations decouple and can be solved efficiently.[5] Since the minimum value constraint (2) for a particular $U$ depends only on the solution for $U + 1$, this indicates that we must solve the equations recursively in the order $U = U_{max}, \dots, 0$.

Since equations (4-5) are solved backwards in time, solutions for $V$ are undefined (and unneeded) for $T < t$. Moreover, when the contract is first purchased ($t = 0$), the initial maturity is $T_{ext}$. Also, from equation (1) we can see that data with $T > \min(t + T_{ext}, T_{max})$ is also not required. As a result, the solution is only *activated* for a given $T$ for a subset of the lifetime of the option.

[5] Windcliff et al. (2000) show how this can be done on parallel, high-performance computer architectures. This may be useful for valuing a large collection of segregated fund contracts.

Specifically, the solution with maturity $T$ is active if

$$\max(t, T_{ext}) \leq T \leq \min(t + T_{ext}, T_{max}). \tag{11}$$

The left inequality of equation (11) simply says that we need not be concerned with contracts which have already matured. The right inequality says that we do not need to be valuing contracts with maturity dates that could not possibly be reached at present time $t$. This offers a couple of benefits. First, we obtain a computational performance advantage since only a subset of the equations are active at any given time. Second, and more importantly, it is unclear how to define the minimum value constraint during the interval $t + T_{ext} < T$. This raises some important issues which are discussed in detail below.

At this point, we will provide some pseudo code to outline the solution process. To simplify the discussion, we will consider the discretization of the $U$ and $T$ variables only. We denote the discrete values of $U = \{0, 1, \ldots, U_{max}\}$ by $U_p$; $p = 0, \ldots, P$. Similarly, the discrete values of $T$ are denoted by $T_m$; $m = 0, \ldots, M$. To cover the complete range of possible maturity dates, $T_0 = T_{ext}$ and $T_M = T_{max}$. The values of the solution at discrete times $t = t_N, t_{N-1}, \ldots, t_0$ are obtained using the following algorithm:

$active[m] :=$ *false* ; `for each` $m = 0, \ldots, M-1$
$active[M] :=$ *true*
$t := T_{max} := T_M$
$V(S, K, T_M, U_p, T_M) := \mathcal{R}(T_M) g(S, K)$ ; `for each` $p = 0, \ldots, P$
$stop := M - 1$
$t := \max(t - \Delta t, T_{stop})$
`while` $(t \geq 0)$
    `for` $p = P, \ldots, 0$
        `for` $m = 0, \ldots, M$
            `if` $(active[m] = true)$ `then`
                `if` $(p < P)$ `then`
                    Constraint $V^*(S, K, T_m, U_p, t)$ obtained from equation (2)
                `endif`
                Solve equation (4-5) for $V(S, K, T_m, U_p, t)$
            `endif`
        `endfor` $m$
    `endfor` $p$
    `if` $(t = T_{stop})$ `then`
        $active[stop] := true$
        $V(S, K, T_{stop}, U_p, T_{stop}) := \mathcal{R}(T_{stop}) g(S, K)$ ; `for each` $p = 0, \ldots, P$

```
        stop := stop − 1
    endif
    if (stop ≥ 0 and t − Δt < T_stop) then
        t := T_stop
    else
        t := t − Δt
    endif
    m* = min_m{T_m ≥ t + T_ext}
    active[m] := false if (m > m*) ; for each m = 0, ... , M
endwhile
```
(12)

Note that in pseudo code (12), if $m^*$ is the smallest $m$ such that $T_m \geq t + T_{ext}$, then solutions are only deactivated for $T_m$ ; $m > m^*$. At first glance this would seem to be one more value of $T_m$ than necessary. However, since solutions are only known at discrete locations $T_m$, then some type of interpolation must be used to determine $V^*$ (see equation (2)). Depending on the form of interpolation used, this may require the extra discrete $T_m$ value. The interpolation method will be discussed below.

### 3.1 The Similarity Reduction

Solution of the full four dimensional problem is clearly a computationally intensive task. In some cases, we can reduce the dimensionality of the problem by one through use of a similarity transformation. Assuming that i) the payoff is given by $g(S, K) = \max(K - S, 0)$; ii) the volatility $\sigma(S, t) = const.$ independent of $S$; and iii) the function $\mathcal{F}(S, K, T, U, t)$ which determines the strike setting upon shouting is given by $\mathcal{F}(S, K, T, U, t) = \alpha(t)S$, we can determine the value of $V(S, K, T, U, t)$ by solving only for a single reference strike $K_0$:

$$V(S, K, T, U, t) = \frac{K}{K_0} V\left(S\frac{K_0}{K}, K_0, T, U, t\right). \tag{13}$$

Note that as long as the node $S = K_0/\alpha$ is in the $S$ direction mesh, then no interpolation is required in the $S$ direction to determine $V^*$ (from equation (2)).

The similarity reduction can also be shown to be valid (see Windcliff, 2000) for the uncertain parameter models described in Wilmott (1998) and many stochastic volatility/interest rate models. All of the numerical results presented in this paper utilize this similarity reduction. For some more general modelling specifications such as implied volatility surfaces, we would be faced with the task of solving the full four dimensional, time dependent problem.

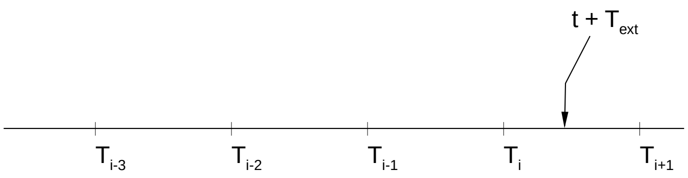



FIGURE 2: *Discrete data available for interpolating/extrapolating to find the constraint $V^*$ which requires the solution at $T^* = t + T_{ext}$.*

### 3.2 Details Concerning the Discretization of $T$

One of the numerical difficulties which must be addressed in the valuation of segregated funds is the discretization of the variable $T$, which keeps track of the current expiry of the contract. Recall that this additional variable is necessary because the holder can modify the expiry date of the contract by shouting. As we will see, the discretization of this variable requires careful attention.

The solution domain in the $T$ variable is characteristically different from that for the other variables. As noted above, in this dimension the solution domain *moves*, corresponding to the window of contracts which are active at a given time. We can improve the computational efficiency of the pricing algorithm by only solving for contracts which are active at any particular time $t$ (see equation (11)). However, we will find that the handling of contracts which have expiry dates $T > t + T_{ext}$ has subtle implications.

Taking into account the nature of the solution domain described previously, we choose a fixed discretization of the $T$ direction, and dynamically activate the solutions during the valuation procedure (this is presented in detail in the pseudo code (12) above). The nodes in the $T$ discretization are activated when $t = T$ and are deactivated when $t + T_{ext} < T$. The contracts within this *window* are the only contracts which have concrete financial meaning at time $t$. Due to the long term nature of these contracts, the active window is often only a small portion of the $T$ discretization. By only solving the PDEs associated with contracts in this window, we obtain a significant computational performance gain.

This dynamic procedure has subtle yet important interpolation implications. Notice that since we will be interpolating the numerical solution at each timestep in order to determine the constraint $V^*$, we must be careful to ensure that we control the cumulative interpolation error (Windcliff et al., 1999). Consider the sequential nodes $T_{i-1}$, $T_i$, and $T_{i+1}$ in the $T$ discretization (see Figure 2). Assume that $T_i < t + T_{ext} < T_{i+1}$. In order to determine the constraint $V^*$ we are required to interpolate the solution between $T_i$ and $T_{i+1}$ (see equation (2)). It is straightforward to define linear interpolation using data at $T_i, T_{i+1}$ and quadratic interpolation obtained from the solutions for $T_{i-1}, T_i, T_{i+1}$ using the usual Lagrange interpolants. Unfortunately, the data at $T_{i+1}$ is no longer relevant since it is unclear how one should define the minimum value constraint for these contracts

| | | Nodes in T discretization | | | |
|---|---|---|---|---|---|
| | | 60 | 120 | 240 | 480 |
| Method | Order | Value of guarantee at $S = \$100$ | | | |
| Interpolation | linear | 38.742 | 38.883 | 38.931 | 38.950 |
| (**inactive** info) | quadratic | 38.781 | 38.895 | 38.936 | 38.952 |
| Extrapolation | linear | 38.901 | 38.946 | 38.957 | 38.961 |
| (**active** info) | quadratic | 38.962 | 38.963 | 38.962 | 38.962 |

TABLE 1: *Comparison of interpolation and extrapolation methods for determining the constraint $V^*$ when $T_i < t + T_{ext} < T_{i+1}$. Linear interpolation (inactive info) uses data from the solutions for $T_i, T_{i+1}$. Quadratic interpolation (inactive info) includes data from the solutions for $T_{i-1}$. Linear extrapolation (active info) uses data from the solutions for $T_i, T_{i-1}$, and quadratic extrapolation (active info) includes data from the solutions for $T_{i-2}$. A similarity solution with* 200 *nodes in the S discretization using automatic Crank-Nicholson timestepping was employed. The contract allows* 2 *shouts per year with no shouting allowed during the final* 10 *years and no mortality provisions.* $\sigma = .25$, $r = .06$, $T_{initial} = 10$ *years*, $T_{ext} = 10$ *years*, $T_{max} = 30$ *years.*

during the time interval $t + T_{ext} < T_{i+1}$.

In order to avoid the use of information from contracts which are formally inactive, we extrapolate using the data from active solutions in our valuation algorithm. Linear extrapolation uses the solutions from $T_{i-1}$ and $T_i$. Quadratic extrapolation methods also include information obtained from the solution for $T_{i-2}$ when this data exists and reduce to linear extrapolation otherwise. As shown in Table 1, extrapolation provides substantial improvement over the interpolation techniques which use inappropriate data. In particular, quadratic extrapolation using active data can be seen to be accurate to within the precision of our numerical approximation for this problem using even the coarsest $T$ discretization. The importance of using an effective interpolation method is consistent with findings presented in Windcliff et al. (1999). For the remainder of the cases considered in this paper we use quadratic extrapolation from active data.

### 3.3 Convergence Tests

For the purpose of performing convergence tests we define three refinement levels. At each successive level, the number of nodes in the discretization is doubled in each dimension. Further, the parameters controlling the automatic timestep selector are modified, approximately doubling the number of timesteps.[6] Thus, increasing the refinement level represents refinement in both the spatial and temporal variables.

In Table 2 we observe quadratic convergence as the discretization is refined. Furthermore, we note that even the first refinement level provides results which are numerically accurate to within 0.25%. For the balance of this paper, we use the *Level 1* discretization.

[6] Interested readers are referred to Windcliff et al. (1999) for more details regarding the timestep selection mechanism used.

| | Refinement Level | | |
|---|---|---|---|
| | *Level 1* | *Level 2* | *Level 3* |
| Asset ($S$) Nodes | 100 | 200 | 400 |
| Expiry Grid ($T$) Nodes | 60 | 120 | 240 |
| Value of guarantee at $S = \$100$ | 38.910 | 38.963 | 38.973 |

TABLE 2: *Convergence tests for the guarantee component of a segregated fund which allows* 2 *shouts per year with no shouting allowed during the final* 10 *years;* $T_{initial} = 10$ *years,* $T_{ext} = 10$ *years,* $T_{max} = 30$ *years,* $\sigma = .25$, $r = .06$, $K_0 = \$100$. *No mortality benefits.*

# 4 Further Modelling Issues With Segregated Funds

## 4.1 Mortality Benefits

As noted above, segregated fund contracts provide for immediate payout of the contract should the investor die. Estimating the value of this feature is of interest to both investors and writers of these contracts.

It is not possible for the writer to hedge against individual deaths. An unrealistic exception to this occurs if we assume that individual deaths occur in such a way as to maximize the value of the contract. This would be an "optimal death" strategy from the view of the holder's estate. This simply involves valuing the contract with an additional American minimum floor constraint. Although unrealistic, the values calculated under this scenario represent an upper bound for the value of the mortality provision in segregated fund contracts. Of course, the investor has a vested interest in not employing an optimal death exercise strategy. More realistically, we can value these securities in an *expected* sense, by averaging over a particular population type. As described above, this is handled by the term $\mathcal{M}(t)g(S,K)$ in equation (4).

In Figure 3 we show results for the case of a fifty year old Canadian female. It is interesting to note that the immediate mortality payout provided by these contracts only nominally increases their worth to the investor. From the writer's perspective, we must consider the following competing factors. On the negative side, the writer i) must pay out the guarantee immediately upon the death of the investor; and ii) will receive less proportional payments, in the case where the payment of the guarantee feature is amortized over the life of the contract. On the positive side (again to the writer), i) it is possible that the fund value at the time of death is above (or at least not far below) the current guarantee level, so that the investor derives little or no benefit from the guarantee feature; and ii) the writer does not have to hedge against future lock in levels since the investor no longer has the ability to reset the guarantee.

Figure 3 also shows that, as expected, the effect of the mortality benefits is more pronounced for the seventy year old Canadian female scenario. In this case, the investor is much more likely to utilize the mortality provisions and we find an increase in the value of the contract when the asset price is below the current guarantee level of \$100. On the other hand, the value of the contract is

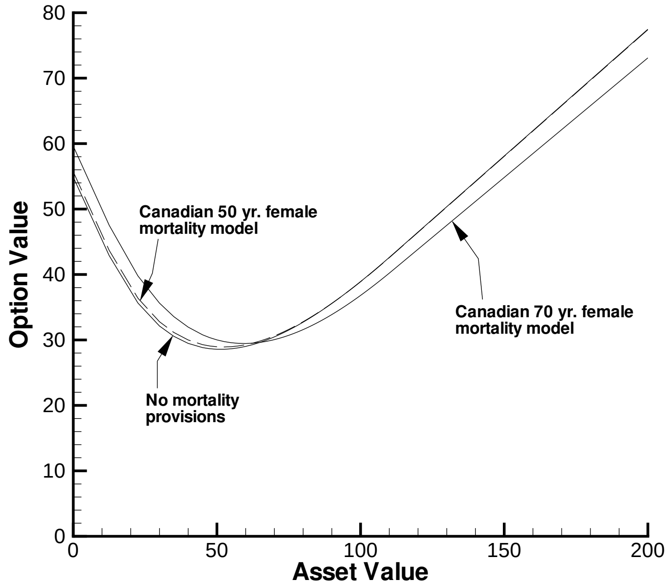



FIGURE 3: *The value of a segregated fund guarantee under various mortality models. The contract specifies 2 shouts per year with no shouting allowed during the final* 10 *years;* $T_{initial} = 10$ *years,* $T_{ext} = 10$ *years,* $T_{max} = 30$ *years,* $\sigma = .25$, $r = .06$, $K_0 = \$100$. *The female mortality model uses data for a Canadian female.*

diminished for high asset levels. This is because the major contributing factor to the value of the contract in this region is the holder's ability to reset the guarantee to a higher level. With a higher chance of mortality, the investor is less likely to be able to use all of her shout optionality to lock in higher guarantee levels.

As will be discussed shortly, segregated funds are often purchased by paying a proportional rate to the insurance provider throughout the life of the contract. Although the present value of the guarantee is similar regardless of mortality provisions in the contract, the required proportional payment rates may be dramatically different. This is because the incoming payments received by the writer of the contract will be over time intervals of different length, due to the variable maturity of the contract. As a result, knowing the present value of a contract outside of the context of the maturity strategy employed by the investor may not be very useful.

## 4.2 Deferred Payment Schemes

The investor who purchases a segregated fund is not charged an up front fee for the embedded option guarantee. Instead, the investor pays a higher management expense fee for the contract (over and above the standard fees for a mutual fund). This fee is proportional to the value of the underlying asset, and is often deducted daily. This is basically the same procedure used to charge for mortality benefits in variable annuity savings plans in the United States (Milevsky and

Posner, 1999). Note that this effectively means that the value of the guarantee depends on the amount charged to finance the guarantee. This is because the guarantee applies to the value of the investor's account, which is reduced by the fees being extracted. This shows that the guarantee level is endogenous, in the sense described in Bacinello and Ortu (1993).

If the proportional expense rate charged to finance the embedded guarantee is $r_e$, and the underlying mutual fund follows (3), then the units of account held by an investor holding the corresponding segregated fund will follow the stochastic differential equation

$$dS = (\mu(S,t) - q(t) - r_e)S\,dt + \sigma(S,t)S\,dz. \tag{14}$$

A relevant task in the valuation of these contracts is to determine the expense rate $r_e$ which makes the investor indifferent to paying the premium up front. Previous research (Milevsky and Posner, 1999; Windcliff et al., 2000) has focused on dividing this task into the following two problems. First, the premium or up-front value of the embedded guarantee is determined, then the present expected value of the incoming payments is modelled. The rate $r_e$ is chosen such that these two quantities are equal. As pointed out by Milevsky and Posner (1999), the up-front option value is a non-linear function of $r_e$, so that iteration is necessary to determine the fair market value of $r_e$.

Modelling the present value of the incoming payment stream separately presents some difficulties. First, the actual expiry date of the contract is uncertain. This occurs because the expiry date of the contract can be modified by the investor for optimization reasons via shouting and because of mortality benefits and lapses.[7] Moreover, the stochastic process followed by the underlying mutual fund may be quite complex, rendering this approach infeasible.

As a more flexible alternative, we introduce an additional term into equation (4) which represents the amortized payments received by the writer. During the time interval $dt$, the holders make the proportional payment $\mathcal{R}(t)r_eS$, where $\mathcal{R}(t)$ represents the proportion of the original investors remaining in the contract at time $t$. Thus equation (4) modelling the value of the segregated fund guarantee is replaced with

$$V_t + (r(t) - q(t) - r_e)SV_S + \frac{1}{2}\sigma(S,t)^2S^2V_{SS} - r(t)V - \mathcal{R}(t)r_eS + \mathcal{M}(t)g(S,K) \leq 0. \tag{15}$$

The boundary condition which is applied as $S \to \infty$ if it is not optimal to shout now becomes

$$V_t = (q(t) + r_e)V + \mathcal{R}(t)r_eS - \mathcal{M}(t)g(S,K). \tag{16}$$

Since there is no initial premium for entering into a segregated fund contract, we determine the expense rate $r_e$ which makes the segregated fund guarantee value initially zero for the current asset price $S_0$. In Figure 4 we present the initial value of a segregated fund contract with various

[7] See Milevsky and Posner (1999) for a description of an integral equation approach for valuing mortality benefits (in the absence of shouting).

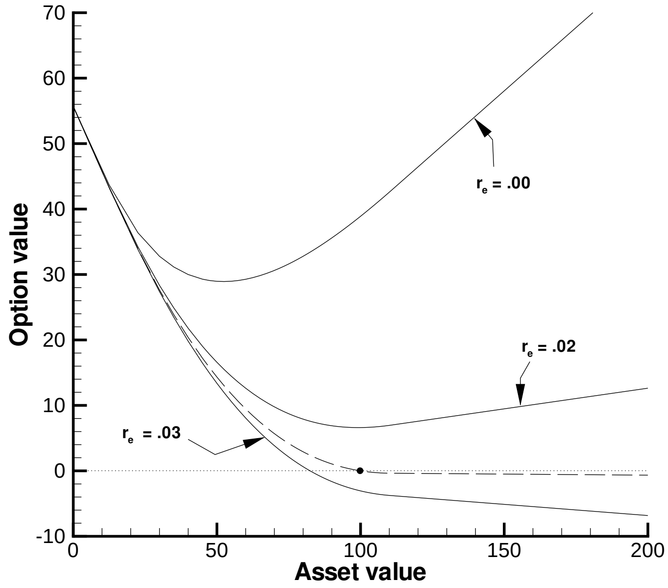



FIGURE 4: *The value of a segregated fund guarantee which specifies 2 shouts per year with no shouting allowed during the final* 10 *years for different deferred proportional fees. For this case, the break even expense rate is* $r_e = .0265$*, shown by the dashed line, which makes the initial value of the contract zero at* $S_0 = \$100$*.* $T_{initial} = 10$ *years,* $T_{ext} = 10$ *years,* $T_{max} = 30$ *years,* $\sigma = .25$*,* $r = .06$*,* $K_0 = \$100$*. The female mortality model uses data for a Canadian female aged 50 years.*

proportional expense fees charged. Notice that for large $r_e$, there are regions where the value of the contract is negative. As discussed below, this can have important implications if investors have the ability to lapse without penalty.

In practice, we determine the value of the segregated fund guarantee for several expense rates and linearly interpolate to approximate the $r_e$ which makes the initial value of the contract zero. Table 3 presents some illustrative calculations for various levels of $r$ and $\sigma$. The table shows that the fair value of $r_e$ varies widely with these parameters. Depending on the level of $r$, higher values of $\sigma$ may imply that the fee should be more on the order of 2%-5%. We note that although the extra amount charged for the guarantee in practice varies both with the underlying fund and the particular contract specification, 1% appears to be an upper bound on the incremental fee.[8] This suggests that these contracts are undervalued by the writers for riskier funds with higher volatilities.

Previously we found that the inclusion of mortality benefits had little effect on the initial value of the contract (see Figure 3). However, it was noted that since the duration of the contracts is altered by mortality, the initial value of the contract is not directly related to the required proportional expense rate.

[8]According to The Investment Funds Institute of Canada, the incremental fees charged are typically in the range of 40 to 80 basis points.

| | | Mortality provision | | | |
|---|---|---|---|---|---|
| | | *None* | | *Female aged 50 years* | |
| $\sigma$ | $r$ | Initial value | Break even $r_e$ | Initial value | Break even $r_e$ |
| .15 | .06 | 10.41 | .006 | 11.17 | .007 |
| | .10 | 1.57 | .001 | 2.58 | .002 |
| .25 | .06 | 38.92 | .025 | 38.90 | .027 |
| | .10 | 15.08 | .008 | 16.67 | .009 |
| .35 | .06 | 76.60 | .055 | 75.21 | .058 |
| | .10 | 38.86 | .020 | 40.33 | .023 |

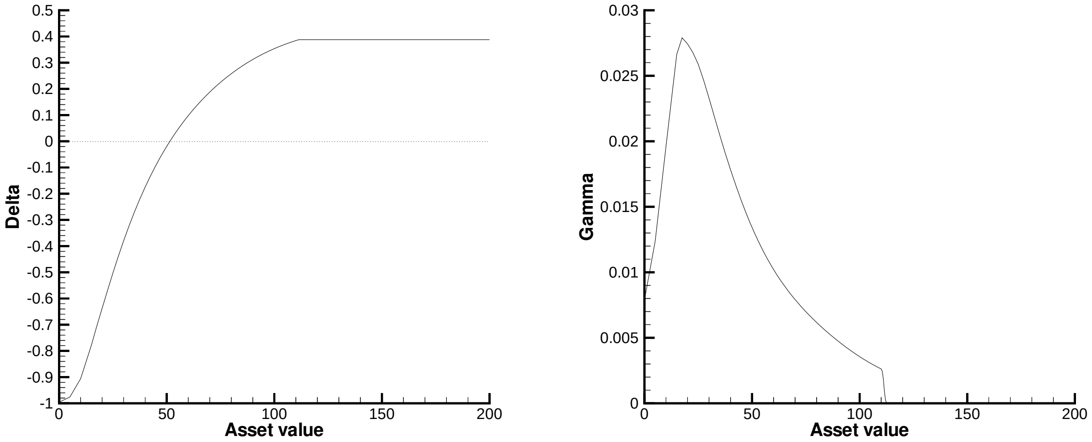
TABLE 3: *The initial value (computed at $S_0 = \$100$) and the proportional expense rate required to finance the guarantee for a segregated fund with 2 shouts per year with no shouting allowed during the final* 10 *years; $T_{initial} = 10$ years, $T_{ext} = 10$ years, $T_{max} = 30$ years, $K_0 = \$100$. Mortality benefits providing immediate payout upon death modelled using data for a Canadian female aged 50 years (from* `www.soa.org`*). Refinement Level 1 was used.*

In Table 3 we find that the inclusion of mortality benefits only marginally increases the required guarantee expense rate. For the market parameters, $\sigma = .25$, $r = .06$, the fee increases from $r_e = .025$ to $r_e = .027$ for the case of a 50 year old female demography type. As an extreme example, we can consider the case of a 70 year old female. For this demography type, over a 30 year maximum contract duration, there is a much greater probability that the contract will be terminated before the maturity date because of mortality. As a result, we expect the required expense rate to be larger because the duration of payments received is likely to reduced. For this contract we find that the required expense rate for providing the guarantee is significantly larger at $r_e = .037$. Notice that most contracts avoid distinguishing between demography types by restricting the ability to shout when mortality becomes more likely. Many funds, for example, limit shouting after the investor's seventieth birthday. In summary, we find that much of the value of the guarantee feature of a segregated fund comes from the reset feature, not the inclusion of mortality benefits.

### 4.3 Some Hedging Considerations

Valuation is not the only problem of practical importance in the context of segregated funds. It is also critical for writers of these complex contracts to have strategies available to hedge the risks involved in selling them. As a first attempt we can consider a dynamic delta hedging strategy. Since these contracts are extremely long term, such an approach must be considered with care due to the sensitivity to small variations in parameters which are assumed to be constant (i.e. $r$ and $\sigma$).

Figure 5 shows the delta ($\Delta = \frac{\partial V}{\partial S}$) for the segregated fund guarantee. It is very similar to that for a standard European put option with the exception that the delta for the guarantee can change sign. This can have some important financial consequences in the context of models with uncertain




(a) *Delta* ($\Delta = \frac{\partial V}{\partial S}$). (b) *Gamma* ($\Gamma = \frac{\partial^2 V}{\partial S^2}$).

FIGURE 5: *Delta and gamma for the guarantee component of a segregated fund. The contract specifies 2 shouts per year with no shouting allowed during the final* 10 *years;* $T_{initial} = 10$ *years,* $T_{ext} = 10$ *years ,* $T_{max} = 30$ *years,* $\sigma = .25$, $r = .06$, $K_0 = \$100$. *No mortality benefits. The value of the guarantee is plotted in Figure 3.*

parameters (Avellaneda et al., 1995; Wilmott, 1998). In particular, in these models we specify an assumed range for a parameter (e.g. the interest rate), rather than a unique value. Given this range, we calculate a range of possible values for the option. It might be expected that the highest and lowest values for the option correspond to the end-points of the specified range, but this is not true if the terms involving the parameters in the PDE can change sign. As a result, the worst-case scenario for the option writer cannot be found by simply solving the PDE with an extreme value of the parameter. Exactly the same issues arise with shout options if there are no maturity extension features, and a complete discussion can be found in Windcliff (2000) and Windcliff et al. (2000).

Figure 5 also shows the gamma $\Gamma = \frac{\partial^2 V}{\partial S^2}$ for the guarantee component. Again, the profile is roughly similar to that of a vanilla European put option, though the value of gamma does fall very rapidly to zero when $S$ rises above a level of around \$110, corresponding to the exercise region where the investor should lock in at a new guarantee level. Taken together, the two panels of this figure suggest that these contracts may not be much more difficult to hedge than standard options. However, as noted above, given the long term nature of these contracts, there are reasons to question the applicability of a dynamic hedging strategy in the first place.

### 4.4 Alternative Contract Specifications

The competitive market for segregated funds has led to a very diverse set of contract specifications. In this section we study some common contract variations. These examples emphasize the fact that the initial value of the contract is not directly related to the required expense ratio which must be

| $T_{max}$ | Last 10 years | Initial value at $S = \$100$ | Break even $r_e$ |
|---|---|---|---|
| | *No mortality provision* | | |
| 30 yrs. | No shouting | 38.92 | .025 |
| | 2% penalty/yr. | 48.39 | .027 |
| 40 yrs. | No shouting | 50.59 | .026 |
| | 2% penalty/yr. | 58.82 | .027 |
| | *Mortality benefits: Female aged 50 years* | | |
| 30 yrs. | No shouting | 38.90 | .027 |
| | 2% penalty/yr. | 46.31 | .028 |
| 40 yrs. | No shouting | 49.40 | .028 |
| | 2% penalty/yr. | 53.51 | .029 |

TABLE 4: *The value of a segregated fund guarantee with 2 shouts per year with different shouting provisions during the final* 10 *years and different maximum extensions, $T_{max}$ allowed; $T_{initial} = 10$ years, $T_{ext} = 10$ years, $K_0 = \$100$, $\sigma = .25$, $r = .06$. Notice that the required expense rate, $r_e$, is similar for all of these various contract provisions. Refinement Level 1 was used to obtain the numerical results.*

charged to cover the cost of providing the guarantee. Contrast this with a contract with a fixed expiry date for which an increase in the present value of the contract implies an increase in the required expense rate.

Often the maximum life of the maturity is set so that the funds must be redeemable when the holder reaches a certain age. In Table 4 we see that the contracts with longer maximum maturities are indeed worth more in a *present value* sense, since they include additional flexibility. The additional value in the contract with larger $T_{max}$ comes from exercise strategies with longer realized actual maturities. The fair value of the expense rate, $r_e$, is relatively unaffected, lying between 2.5% and 3% for these contracts. This occurs because there will be more incoming payments received by the writer if these longer maturity exercise strategies are actually employed.

Another variation in the contract design allows the investor to shout during the last 10 years of the contract life, albeit at a reduced percentage guarantee. It is clear that the present value of this contract must be worth more since the investor can always elect not to use these shout opportunities. This is also illustrated in Table 4. However, due to the amortization, and because making use of this additional provision increases the duration of the payments, the inclusion of this feature does not change the required expense rate as much as might be expected.

To emphasize the difference between the present value of a contract and the required payment rate we consider the following two contracts, both with maximum expiry of 20 years. The first contract provides an initial guarantee in $T_{initial} = 10$ years. The maturity of the contract is extended by $T_{ext} = 10$ years upon shouting where shouting is without penalty during the final 10 years and does not extend the maturity past $T_{max} = 20$ years. This contract gains much of its value near the maximum maturity, $T_{max}$. The second contract specifies $T_{initial} = T_{ext} = 5$ years and does not allow shouting after 15 years from the date of sale of the contract. More of the

| | | Expense rate $r_e$ | | | | |
|---|---|---|---|---|---|---|
| | | .00 | .02 | .04 | .06 | |
| $T_{ext} = T_{initial}$ | Last $T_{ext}$ years | Value at $S = \$100$ | | | | Break even $r_e$ |
| 10 yrs. | No penalty | 46.21 | 13.60 | −6.69 | −18.86 | .033 |
| 5 yrs. | No shouting | 46.96 | 23.73 | 9.68 | 0.87 | .063 |

TABLE 5: *The value of a segregated fund guarantee with 2 shouts per year with different shouting provisions during the final $T_{ext}$ years and different expiry extensions $T_{ext}$; $T_{max} = 20$ years, $K_0 = \$100$, $\sigma = .25$, $r = .06$. No mortality benefits. Notice that the present values of these contracts are nominally the same, yet the proportional fees required are dramatically different. Refinement Level 1 was used to obtain the numerical results.*

| | Initial Value at $S = \$100$ | Break even $r_e$ |
|---|---|---|
| No lapsing | 38.92 | .025 |
| 5% lapse rate | 15.61 | .015 |
| Optimal lapsing | 38.92 | .030 |

TABLE 6: *The value of a segregated fund guarantee with different lapse models. The contract specifies 2 shouts per years with no shouting during the final 10 years. $T_{max} = 30$ years, $T_{initial} = 10$ years, $T_{ext} = 10$ years, $K_0 = \$100$, $\sigma = .25$, $r = .06$. No mortality provisions. Refinement Level 1 was used to obtain the numerical results.*

value of this contract is a result of the initial guarantee, when compared with the first contract. Although they have nominally similar present values (i.e. values with $r_e = 0$), these contracts require substantially different proportional payment rates. This is demonstrated in Table 5, which illustrates the distinction between these two notions of value.

### 4.5 Modelling Investor Lapsing

As a first approximation, we can model non-optimal investor lapsing using a deterministic lapse rate. With this simple model, a fraction of the remaining investors in the contract lapse each year. We denote the fraction of the original holders who lapse during the time interval $t \to t + dt$ by $\mathcal{L}(t)$. Since lapsing affects the number of investors remaining in the contract we must change $\mathcal{R}(t)$, the proportion of the original investors remaining in the contract to

$$\mathcal{R}(t) = 1 - \int_0^t \big(\mathcal{M}(u) + \mathcal{L}(u)\big) du. \tag{17}$$

This will affect both the terminal condition (6) and the proportional payment received by the writer. The effect of incorporating a simple deterministic lapse rate is shown in Table 6. A 5% lapse rate reduces the up front value from about \$39 to less than \$16, and the required $r_e$ from 2.5% to 1.5%.

By lapsing the investor forfeits his right to the guarantee held, and the writer is not responsible for any payments to the investor. However, it should be noted that it can sometimes be detrimental to the writer if the investor lapses. As a simple example, consider a case near the maximum maturity date $T_{max}$, where the investor has no more opportunities to shout. If $S \gg K$, the option is out-of-the-money and is not likely to have any value at the maturity date. In a deferred payment plan, the investor would be better off lapsing and avoiding the additional proportional expense costs. This type of optimal behaviour can be modelled by ensuring that the value of the guarantee satisfies an additional constraint $V(S, K, U, T, t) \geq 0$, where the investor will lapse if the value maintaining the guarantee becomes negative. Of course, this type of investor behaviour will increase the cost of hedging the guarantee feature, as demonstrated in Table 6. Note that the initial values of the contracts are the same under optimal lapsing and no lapsing. This is because the investor derives no benefit from lapsing if the contract has already been paid for at inception.

# 5 Conclusions

In this work, we have described a general numerical approach for valuing segregated fund guarantees (extendible shout options) with mortality benefits. The examples presented here use a similarity reduction, resulting in a three dimensional problem. Under alternative modelling assumptions (e.g. a CEV process for the value of the underlying asset), then a full four dimensional problem must be solved. However, the nature of the problem allows fast computation on modern multiprocessor architectures (Windcliff et al., 2000).

As in the simpler case of shout options without extendibility (Windcliff et al., 1999), it is necessary to interpolate the discrete solution data in order to determine the value obtained upon resetting the strike. In the case of segregated fund guarantees, we have the additional complication of requiring data (in general) at points between active and inactive $T$ planes. Since inactive planes are financially meaningless, we are in the awkward position of having to interpolate in the $T$ direction, using highly suspect data. However, this situation can be avoided if we extrapolate using valid (i.e. financially meaningful) data. Numerical tests show, rather surprisingly, that quadratic extrapolation (with valid data) in the $T$ direction converges more rapidly than interpolation in the $T$ direction. This is consistent with the observation in Forsyth et al. (1999) that the choice of interpolation method used when valuing path-dependent options can have a very large effect on convergence as the mesh is refined.

An interesting feature of our results is that contract specifications which result in quite similar present values may imply substantially different deferred payment expense rates. Basically, this is due to the maturity extension aspect: contracts which can be expected to last longer will receive payments over a longer period of time.

Our results also indicate that mortality benefits (at least for a 50 year old female) are not a significant contributor to the value of segregated funds. This is consistent with results reported in Milevsky and Posner (1999) for death benefits in variable annuity savings plans in the United States. However, a significant difference is that it is concluded in Milevsky and Posner (1999) that investors are apparently being significantly overcharged for the death benefit component in these plans. To reiterate, our results suggest that this is not true in Canada for segregated fund guarantees, including both death benefits—which have relatively little value, and very valuable reset provisions. Indeed, in many cases insurance companies appear to be undercharging for these contracts, particularly in cases of riskier funds with higher volatilities. We have also shown that lapsing cannot simply be assumed to be beneficial to the writer because in some cases it may be optimal for investors to lapse. These conclusions presume optimal behaviour on the part of investors in these contracts. Non-optimal behaviour does reduce the estimated contract values, but it is not possible for us to draw firm conclusions about this due to an absence of data on the behaviour of individual contract holders. It is worth noting that our modelling framework does allow for some investigation along these lines, should this type of data become available at some future time.

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