

# Comparison of Mean Variance Like Strategies for Optimal Asset Allocation Problems \*

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## Abstract

We determine the optimal dynamic investment policy for a mean quadratic variation objective function by numerical solution of a nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). We compare the efficient frontiers and optimal investment policies for three mean variance like strategies: pre-commitment mean variance, time-consistent mean variance, and mean quadratic variation, assuming realistic investment constraints (e.g. no bankruptcy, finite shorting, borrowing). When the investment policy is constrained, the efficient frontiers for all three objective functions are similar, but the optimal policies are quite different.

**Keywords:** Mean quadratic variation investment policy, mean variance asset allocation, HJB equation, optimal control

**JEL Classification:** C63, G11

**AMS Classification** 65N06, 93C20

## 1 Introduction

In this paper, we consider optimal continuous time asset allocation using mean variance like strategies. This contrasts with the classic power law or exponential utility function approach [24].

Mean variance strategies have a simple intuitive interpretation, which is appealing to both individual investors and institutions. There has been considerable recent interest in continuous time mean variance asset allocation [32, 21, 25, 20, 6, 11, 31, 18, 19, 29]. However, the optimal strategy in these papers was based on a *pre-commitment* strategy which is not *time-consistent* [7, 5].

Although the pre-commitment strategy is optimal in the sense of maximizing the expected return for a given standard deviation, this may not always be economically sensible. A real-world investor experiences only one of many possible stochastic paths [22], hence it is not clear that a strategy which is optimal in an average sense over many stochastic paths is appropriate. In

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29 addition, the optimal strategy computed from the pre-commitment objective function assumes that  
30 the stochastic parameters are known at the beginning of the investment horizon, and do not change  
31 over the investment period. In practice, of course, one would normally recompute the investment  
32 strategy based on the most recent available data.

33 For these reasons, a *time-consistent* form of mean variance asset allocation has been suggested  
34 recently [7, 5, 30]. We may view the time-consistent strategy as a pre-commitment policy with a  
35 time-consistent constraint [30].

36 Another criticism of both time-consistent and pre-commitment strategies is that the risk is only  
37 measured in terms of the standard deviation at the end of the investment period. In an effort to  
38 provide a more direct control over risk during the investment period, a mean quadratic variation  
39 objective function has been proposed in [9, 16].

40 This article is the third in a series. In [29], we developed numerical techniques for determining  
41 the optimal controls for pre-commitment mean variance strategies. The methods in [29] allowed us  
42 to apply realistic constraints to the control policies. In [30], we developed numerical methods for  
43 solution of the time-consistent formulation of the mean-variance strategy [5]. The methods in [30]  
44 also allowed us to apply constraints to the control policies.

45 In this article, we develop numerical methods for solution of the mean quadratic variation policy,  
46 again for the case of constrained controls. We also present a comparison of pre-commitment, time  
47 consistent and mean quadratic variation strategies, for two typical asset allocation problems. We  
48 emphasize here that we use numerical techniques which allow us to apply realistic constraints (e.g.  
49 no bankruptcy, finite borrowing and shorting), on the investment policies. This is in contrast to  
50 the analytic approaches used previously [32, 21, 6, 7].

51 We first consider the optimal investment policy for the holder of a pension plan, who can  
52 dynamically allocate his wealth between a risk-free asset and a risky asset. We will also consider  
53 the case where the pension plan holder desires to maximize the wealth-to-income ratio, in the case  
54 where the plan holder's salary is stochastic [10].

55 The main results in this paper are

- 56 • We formulate the optimal investment policy for the mean quadratic variation problem as  
57 a nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). We extend  
58 the numerical methods in [29, 30] to handle this case.
- 59 • We give numerical results comparing all three investment policies: pre-commitment mean  
60 variance, time-consistent mean variance, and mean quadratic variation. In the case where  
61 analytic solutions are available, our numerical results agree with the analytic solutions. In the  
62 case where typical constraints are applied to the investment strategy, the efficient frontiers  
63 for all three objective functions are very similar. However, the investment policies are quite  
64 different.

65 These results show that, in deciding which objective function is appropriate for a given economic  
66 problem, it is not sufficient to simply examine the efficient frontiers. Instead, the actual investment  
67 policies need to be studied in order to determine if a particular strategy is applicable to specific  
68 investment objectives.

## 69 2 Dynamic Strategies

70 In this paper, we first consider the problem of determining the mean variance like strategies for a  
 71 pension plan. It is common to write the efficient frontier in terms of the investor's final wealth. We  
 72 will refer to this problem in the following as the *wealth* case.

73 Suppose there are two assets in the market: one is risk free (e.g. a government bond) and the  
 74 other is risky (e.g. a stock index). The risky asset  $S$  follows the stochastic process

$$dS = (r + \xi\sigma)S dt + \sigma S dZ_1, \quad (2.1)$$

75 where  $dZ_1$  is the increment of a Wiener process,  $\sigma$  is volatility,  $r$  is the interest rate, and  $\xi$  is the  
 76 market price of risk (or Sharpe ratio). The stock drift rate can then be defined as  $\mu_S = r + \xi\sigma$ . We  
 77 specify the drift rate of the stock in terms of the market price of risk  $\xi$ , to be consistent with [10].  
 78 This also allows us to compare results obtained by varying  $\sigma$ , while keeping  $\xi$  constant, in addition  
 79 to varying  $\sigma$ , while keeping  $\mu_S$  constant.

80 Suppose that the plan member continuously pays into the pension plan at a constant contri-  
 81 bution rate  $\pi \geq 0$  in the unit time. Let  $W(t)$  denote the wealth accumulated in the pension plan  
 82 at time  $t$ , let  $p$  denote the proportion of this wealth invested in the risky asset  $S$ , and let  $(1 - p)$   
 83 denote the fraction of wealth invested in the risk free asset. Then,

$$\begin{aligned} dW &= [(r + p\xi\sigma)W + \pi]dt + p\sigma W dZ_1, \\ W(t=0) &= \hat{w}_0 \geq 0. \end{aligned} \quad (2.2)$$

84 Define,

$$\begin{aligned} E[\cdot] &: \text{expectation operator,} \\ \text{Var}[\cdot] &: \text{variance operator,} \\ \text{Std}[\cdot] &: \text{standard deviation operator,} \\ E_{t,w}[\cdot], \text{Var}_{t,w}[\cdot] \text{ or } \text{Std}_{t,w}[\cdot] &: E[\cdot|W(t) = w], \text{Var}[\cdot|W(t) = w] \text{ or } \text{Std}[\cdot|W(t) = w] \\ &\quad \text{when sitting at time } t, \\ E_{t,w}^p[\cdot], \text{Var}_{t,w}^p[\cdot] \text{ or } \text{Std}_{t,w}^p[\cdot] &: E_{t,w}[\cdot], \text{Var}_{t,w}[\cdot] \text{ or } \text{Std}_{t,w}[\cdot], \text{ where } p(s, W(s)), s \geq t, \\ &\quad \text{is the policy along path } W(t) \text{ from stochastic process (2.2).} \end{aligned} \quad (2.3)$$

85 For the convenience of the reader, we will first give a brief summary of the pre-commitment  
 86 and time consistent policies.

### 87 2.1 Pre-commitment Policy

88 We review here the pre-commitment policy, as discussed in [29]. In this case, the optimal policy  
 89 solves the following optimization problem,

$$\mathcal{V}(w, t) = \sup_{p(s \geq t, W(s))} \left\{ E_{t,w}^p[W(T)] - \lambda \text{Var}_{t,w}^p[W(T)] \mid W(t) = w \right\}, \quad (2.4)$$

90 where  $W(T), t < T$  is the investor's terminal wealth, subject to stochastic process (2.2), and where  
 91  $\lambda > 0$  is a given Lagrange multiplier. The multiplier  $\lambda$  can be interpreted as a coefficient of risk  
 92 aversion. The optimal policy for (2.4) is called a *pre-commitment* policy [5].

93 Let  $p_t^*(s, W(s))$ ,  $s \geq t$ , be the optimal policy for problem (2.4). Then,  $p_{t+\Delta t}^*(s, W(s))$ ,  $s \geq t+\Delta t$ ,  
 94 is the optimal policy for

$$\mathcal{V}(w, t + \Delta t) = \sup_{p(s \geq t + \Delta t, W(s))} \left\{ E_{t+\Delta t, w}^p [W(T)] - \lambda \text{Var}_{t+\Delta t, w}^p [W(T)] \mid W(t + \Delta t) = w \right\}. \quad (2.5)$$

95 However, in general

$$p_t^*(s, W(s)) \neq p_{t+\Delta t}^*(s, W(s)) ; s \geq t + \Delta t, \quad (2.6)$$

96 i.e. solution of problem (2.4) is not time-consistent. Therefore, a dynamic programming princi-  
 97 ple cannot be directly applied to solve this problem. However, problem (2.4) can be embedded  
 98 into a class of auxiliary stochastic Linear-Quadratic (LQ) problems using the method in [32, 21].  
 99 Alternatively, equation (2.4) can be posed as a convex optimization problem [22, 6, 1, 17]. More  
 100 precisely, if  $p_t^*(s, W(s))$  is the optimal control of equation (2.4), then there exists a  $\gamma(t, w)$ , such  
 101 that  $p_t^*(s, W(s))$  is also the optimal control of

$$\inf_{p(s \geq t, W(s))} \left\{ E_{t, w}^p \left[ \left( W(T) - \frac{\gamma(t, w)}{2} \right)^2 \right] \mid W(t) = w \right\}. \quad (2.7)$$

102 Hence we can solve for  $p_t^*(s, W(s))$  using dynamic programming. Note that this does not contradict  
 103 our assertion that the optimal policy for equation (2.4) is not time consistent, since in general  $\gamma(t, w)$   
 104 depends on the initial point  $(t, w)$  (see Remark 1.1 in [7], and [19]). Problem 2.7 can be determined  
 105 from the solution of an Hamilton Jacobi Bellman (HJB) equation. We have discussed the numerical  
 106 solution of the resulting HJB equation in detail in [29].

## 107 2.2 Time-consistent Policy

108 In [30], we focused on the so called *time-consistent* policy. We can determine the time-consistent  
 109 policy by solving problem (2.4) with an additional constraint,

$$p_t^*(s, W(s)) = p_{t'}^*(s, W(s)) ; s \geq t', t' \in [t, T]. \quad (2.8)$$

110 In other words, we optimize problem (2.4), given that we follow the optimal policy in the future,  
 111 which is determined by solving (2.4) at each future instant. Obviously, dynamic programming  
 112 can be applied to the time-consistent problem. We have discussed the numerical algorithm for  
 113 determining the optimal time-consistent policy in [30].

114 **Remark 2.1** *We follow the definition of a time consistent policy as given in [5], with a constant*  
 115 *risk aversion parameter. Note that in [8], it is suggested that a wealth dependent risk aversion*  
 116 *parameter is more meaningful. Some computations with a wealth dependent risk aversion parameter*  
 117 *are given in [30]. However, we will use the original form with a constant risk aversion, in the*  
 118 *following. See Remark 2.2.*

## 119 2.3 Mean Quadratic Variation

120 Instead of using the variance/standard deviation as the risk measure, we can use the quadratic  
 121 variation [9],  $\int_t^T (dW_s)^2$ . From equation (2.2) we have

$$(dW_t)^2 = (p(t, W(t))\sigma W(t))^2 dt, \quad (2.9)$$

122 and consequently, we obtain,

$$\int_t^T (dW_s)^2 = \int_t^T (\sigma p(s, W(s))W(s))^2 ds . \quad (2.10)$$

123 **Remark 2.2 (Relation to Time Consistent Mean Variance)** In [7], the following rather sur-  
 124 prising result is obtained. Without constraints (the allowing bankruptcy case, discussed in later  
 125 sections), if  $\int_t^T (e^{r(T-s)}dW_s)^2$  is used as the risk measure, the mean quadratic variation strategy  
 126 has the same solution as the time-consistent strategy. The term  $(e^{r(T-s)}dW_s)^2$  represents the future  
 127 value of the instantaneous risk due to investing  $pW$  (in monetary amount) in the risky asset. In  
 128 order to facilitate comparison with various alternative strategies, we will use the risk measure

$$\int_t^T (e^{r(T-s)}dW_s)^2 = \int_t^T (e^{r(T-s)}\sigma p(s, W(s))W(s))^2 ds , \quad (2.11)$$

129 in the following. Note that as discussed in [16], risk measure (2.11) is commonly used in optimal  
 130 trade execution (with  $r = 0$ ).

131 Using equation (2.11) as a risk measure, we seek the optimal policy which solves the following  
 132 optimization problem,

$$\mathcal{V}(w, t) = \sup_{p(s \geq t, W(s))} E_{t,w}^p \left\{ W(T) - \lambda \int_t^T (e^{r(T-s)}\sigma p(s, W(s))W(s))^2 ds \mid W(t) = w \right\}, \quad (2.12)$$

133 where  $\lambda$  is a given Lagrange multiplier, subject to stochastic process (2.2). Let  $p_t^*(s, W(s))$ ,  $s \geq t$ ,  
 134 be the optimal policy for problem (2.12). Then clearly,

$$p_t^*(s, W(s)) = p_{t'}^*(s, W(s)) ; s \geq t', t' \in [t, T] . \quad (2.13)$$

135 Hence, dynamic programming can be directly applied to this problem.

### 136 3 Mean Quadratic Variation Wealth Case

137 In this section, we formulate the mathematical model for the optimal mean quadratic variation  
 138 investment strategy. Let,

$$\begin{aligned} \mathbb{D} &:= \text{the set of all admissible wealth } W(t), \text{ for } 0 \leq t \leq T; \\ \mathbb{P} &:= \text{the set of all admissible controls } p(t, W(t)), \text{ for } 0 \leq t \leq T \text{ and } W(t) \in \mathbb{D}. \end{aligned} \quad (3.1)$$

139

140 **Remark 3.1** Strictly speaking, we choose  $\mathbb{P}$  so as to enforce  $W \in \mathbb{D}$ . However, when the control  
 141 problem is formulated as the solution to an HJB PDE, then we choose  $\mathbb{D}$ , and select a boundary  
 142 condition which specifies a control which ensures that  $W \in \mathbb{D}$ .

143 We seek the solution of the optimization problem (2.12). Define

$$\mathcal{V}(w, t) = \sup_{p \in \mathbb{P}} E_{t,w}^p \left\{ W(T) - \lambda \int_t^T (e^{r(T-s)}\sigma p(s, W(s))W(s))^2 ds \mid W(t) = w \right\}. \quad (3.2)$$

144 Let  $\tau = T - t$ . Then using equation (2.2) and Ito's Lemma, we have that  $V(w, \tau) = \mathcal{V}(w, t)$  satisfies  
 145 the HJB equation

$$V_\tau = \sup_{p \in \mathbb{P}} \left\{ \mu_w^p V_w + \frac{1}{2} (\sigma_w^p)^2 V_{ww} - \lambda (e^{r\tau} \sigma p w)^2 \right\} ; \quad w \in \mathbb{D}, \quad (3.3)$$

146 with terminal condition

$$V(w, 0) = w, \quad (3.4)$$

147 and where

$$\begin{aligned} \mu_w^p &= \pi + w(r + p\sigma\xi) \\ (\sigma_w^p)^2 &= (p\sigma w)^2. \end{aligned} \quad (3.5)$$

148 Since PDE (3.3) can be degenerate depending on the control, we have no reason to believe that  
 149 the solution is smooth. For this reason, we seek the viscosity solution to equation (3.3). This is  
 150 discussed further in Section 5.1.

151 In order to trace out the efficient frontier solution (in terms of mean and quadratic variation of  
 152 the wealth) of problem (2.12), we proceed in the following way. Pick an arbitrary value of  $\lambda$  and  
 153 solve problem (2.12), which determines the optimal control  $p^*(t, w)$ . We also need to determine  
 154  $E_{t=0, w}^{p^*}[W(T)]$ .

155 Let  $G = G(w, \tau) = E[W(T) | W(t) = w, p(s \geq t, w) = p^*(s \geq t, w)]$ . Then  $G$  is given from the  
 156 solution to

$$G_\tau = \left\{ \mu_w^p G_w + \frac{1}{2} (\sigma_w^p)^2 G_{ww} \right\}_{p(T-\tau, w) = p^*(T-\tau, w)} ; \quad w \in \mathbb{D}, \quad (3.6)$$

157 with the terminal condition

$$G(w, 0) = w. \quad (3.7)$$

158 Since the most costly part of the solution of equation (3.3) is the determination of the optimal  
 159 control  $p^*$ , solution of equation (3.6) is very inexpensive, once  $p^*$  is known.

160 Then, if

$$\begin{aligned} V(\hat{w}_0, T) &= E_{t=0, \hat{w}_0}^{p^*} \left\{ W(T) - \lambda \int_0^T (e^{r(T-s)} \sigma p^*(s, W(s)) W(s))^2 ds \mid W(0) = \hat{w}_0 \right\}, \\ G(\hat{w}_0, T) &= E_{t=0, \hat{w}_0}^{p^*} \left\{ W(T) \mid W(0) = \hat{w}_0 \right\}, \end{aligned} \quad (3.8)$$

161 we have that

$$\begin{aligned} &E_{t=0, \hat{w}_0}^{p^*} \left\{ \int_0^T (e^{r(T-s)} \sigma p^*(s, W(s)) W(s))^2 ds \mid W(0) = \hat{w}_0 \right\} \\ &= (G(\hat{w}_0, T) - V(\hat{w}_0, T)) / \lambda. \end{aligned} \quad (3.9)$$

162 It is useful also to determine the variance of the terminal wealth,  $Var_{t=0, w}^{p^*}[W(T)]$ , under the  
 163 optimal strategy in terms of mean quadratic variation. Let  $F = F(w, \tau) = E[W(T)^2 | W(t) =$   
 164  $w, p(s \geq t, w) = p^*(s \geq t, w)]$ . Then  $F$  is given from the solution to

$$F_\tau = \left\{ \mu_w^p F_w + \frac{1}{2} (\sigma_w^p)^2 F_{ww} \right\}_{p(T-\tau, w) = p^*(T-\tau, w)} ; \quad w \in \mathbb{D}, \quad (3.10)$$

165 with the terminal condition

$$F(w, 0) = w^2 . \quad (3.11)$$

166 Assuming  $F(\hat{w}_0, T), G(\hat{w}_0, T)$  are known, for a given  $\lambda$ , we can then compute the pair

$$(Var_{t=0, \hat{w}_0}^{p^*}[W(T)], E_{t=0, \hat{w}_0}^{p^*}[W(T)])$$

167 from  $Var_{t=0, \hat{w}_0}^{p^*}[W(T)] = F(\hat{w}_0, T) - [G(\hat{w}_0, T)]^2$  and  $E_{t=0, \hat{w}_0}^{p^*}[W(T)] = G(\hat{w}_0, T)$ .

168 **Remark 3.2** *If we allow an unbounded control set  $\mathbb{P} = (-\infty, +\infty)$ , then the total wealth can*  
 169 *become negative (i.e. bankruptcy is allowed). In this case  $\mathbb{D} = (-\infty, +\infty)$ . If the control set  $\mathbb{P}$  is*  
 170 *bounded, i.e.  $\mathbb{P} = [p_{\min}, p_{\max}]$ , then negative wealth is not possible, in which case  $\mathbb{D} = [0, +\infty)$ . We*  
 171 *can also have  $p_{\max} \rightarrow +\infty$ , but prohibit negative wealth, in which case  $\mathbb{D} = [0, +\infty)$  as well.*

### 172 3.1 Localization

173 Let,

$$\hat{\mathbb{D}} := \text{a finite computational domain which approximates the set } \mathbb{D}. \quad (3.12)$$

174 In order to solve the PDEs (3.3), (3.6) and (3.10), we need to use a finite computational domain,  
 175  $\hat{\mathbb{D}} = [w_{\min}, w_{\max}]$ . When  $w \rightarrow \pm\infty$ , we assume that

$$\begin{aligned} V(w \rightarrow \pm\infty, \tau) &\simeq H_1(\tau)w^2 , \\ G(w \rightarrow \pm\infty, \tau) &\simeq J_1(\tau)w , \\ F(w \rightarrow \pm\infty, \tau) &\simeq I_1(\tau)w^2 , \end{aligned} \quad (3.13)$$

176 then, ignoring lower order terms and taking into account the initial conditions (3.4), (3.7), (3.11),

$$\begin{aligned} V(w \rightarrow \pm\infty, \tau) &\simeq \frac{\lambda e^{r\tau} k_2}{2k_1 + k_2} (1 - e^{(2k_1 + k_2)\tau}) w^2 , \\ G(w \rightarrow \pm\infty, \tau) &\simeq e^{k_1\tau} w , \\ F(w \rightarrow \pm\infty, \tau) &\simeq e^{(2k_1 + k_2)\tau} w^2 , \end{aligned} \quad (3.14)$$

177 where  $k_1 = r + p\sigma\xi$  and  $k_2 = (p\sigma)^2$ . We consider three cases.

#### 178 3.1.1 Allowing Bankruptcy, Unbounded Controls

179 In this case, we assume there are no constraints on  $W(t)$  or on the control  $p$ , i.e.,  $\mathbb{D} = (-\infty, +\infty)$   
 180 and  $\mathbb{P} = (-\infty, +\infty)$ . Since  $W(t) = w$  can be negative, bankruptcy is allowed. We call this case the  
 181 *allowing bankruptcy* case.

182 Our numerical problem uses

$$\hat{\mathbb{D}} = [w_{\min}, w_{\max}] , \quad (3.15)$$

183 where  $\hat{\mathbb{D}} = [w_{\min}, w_{\max}]$  is an approximation to the original set  $\mathbb{D} = (-\infty, +\infty)$ .

184 Applying equation (3.13) at finite  $[w_{\min}, w_{\max}]$  will cause some error. However, we can make  
 185 these errors small by choosing large values for  $(|w_{\min}|, w_{\max})$ . We have verified this in [29, 30], and  
 186 numerical tests show that this property holds for the mean quadratic variation strategy as well. If  
 187 asymptotic forms of the solution are unavailable, we can use any reasonable estimate for  $p^*$  for  $|w|$   
 188 large, and the error will be small if  $(|w_{\min}|, w_{\max})$  are sufficiently large [3].

189 **3.1.2 No Bankruptcy, No Short Sales**

190 In this case, we assume that bankruptcy is prohibited and the investor cannot short the stock index,  
 191 i.e.,  $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, +\infty)$ . We call this case the *no bankruptcy* (or *bankruptcy prohibition*)  
 192 case.

193 Our numerical problem uses,

$$\hat{\mathbb{D}} = [0, w_{\max}] . \quad (3.16)$$

194 The boundary conditions for  $V, G, F$  at  $w = w_{\max}$  are given by equations (3.14). We prohibit the  
 195 possibility of bankruptcy ( $W(t) < 0$ ) by requiring that (see Remark 3.3 )  $\lim_{w \rightarrow 0}(pw) = 0$ , so that  
 196 equations (3.3), (3.6) and (3.10) reduce to (at  $w = 0$ )

$$\begin{aligned} V_{\tau}(0, \tau) &= \pi V_w , \\ G_{\tau}(0, \tau) &= \pi G_w , \\ F_{\tau}(0, \tau) &= \pi F_w . \end{aligned} \quad (3.17)$$

197 Another way of deriving this boundary condition is to note that we can rewrite equation (3.3)  
 198 using the control  $q = pw$ . In this case  $q$  is the dollar amount invested in the risky asset. We can  
 199 prohibit negative wealth by requiring that the amount invested in the risky asset  $q = 0$  at  $w = 0$ .

200 **Remark 3.3** *It is important to know the behavior of  $p^*w$  as  $w \rightarrow 0$ , since it helps us determine*  
 201 *whether negative wealth is admissible or not. As shown above, negative wealth is admissible for the*  
 202 *case of allowing bankruptcy. In the case of no bankruptcy, although  $p \in \mathbb{P} = [0, +\infty)$ , we must*  
 203 *have  $\lim_{w \rightarrow 0}(pw) = 0$  so that  $W(t) \geq 0$  for all  $0 \leq t \leq T$ . In particular, we need to make sure*  
 204 *that the optimal strategy never generates negative wealth, i.e.,  $\text{Probability}(W(t) < 0 | p^*) = 0$  for all*  
 205  *$0 \leq t \leq T$ . We will see from the numerical solutions that boundary condition (3.17) does in fact*  
 206 *result in  $\lim_{w \rightarrow 0}(p^*w) = 0$ . Hence, negative wealth is not admissible under the optimal strategy.*  
 207 *More discussion of this issue are given in Section 6.*

208 **Remark 3.4 (Behavior of  $W(t), W \rightarrow 0$ )** *The precise behavior of the controlled stochastic pro-*  
 209 *cess will depend on the behavior of  $pw$  as  $w \rightarrow 0$ , from the solution of the HJB equation (3.3). If*  
 210  *$pw \rightarrow C_1 w^{\gamma}, w \rightarrow 0$ ,  $C_1$  independent of  $w$ , then the behaviour of  $W(t)$  near  $W = 0$  can be deter-*  
 211 *mined from the usual Feller conditions. Equivalently, and more germane for PDE analysis, we can*  
 212 *examine the Fichera function [15] to determine if boundary conditions are required at  $w = 0$ . We*  
 213 *have the following possibilities*

- 214 •  $\gamma \geq 1$ ,  $w = 0$  is unattainable, and no boundary condition is required;
- 215 •  $1/2 < \gamma < 1$ , which implies that  $w = 0$  is attainable but no boundary condition is required;
- 216 •  $\gamma = 1/2$ ,  $w = 0$  is attainable, and no boundary condition is required if  $\pi \geq C_1^2 \sigma^2 / 2$ , otherwise  
 217 a boundary condition is required;
- 218 •  $0 < \gamma < 1/2$ ,  $w = 0$  is attainable, and we need to supply a boundary condition. In this case  
 219 we apply a reflecting condition (from equation (3.17)) if  $\pi > 0$  and an absorbing condition if  
 220  $\pi = 0$ .

221 Note that for numerical purposes, we always apply conditions (3.17). In some cases, as noted above,  
 222 boundary conditions at  $w = 0$  are not actually required, but this does not cause any problems if the  
 223 boundary condition is superfluous [26, 15]. This is convenient, since of course we do not know the  
 224 precise behavior of  $(pw)$ ,  $w \rightarrow 0$  until we solve the HJB equation (3.3) and determine the control  $p$ .

225 **Remark 3.5 (Economic Interpretation of the Conditions on  $pw$ )** The quantity  $pw$  is the  
 226 dollar amount invested in the risky asset. Consequently, the above conditions on  $pw$  can be in-  
 227 terpreted in economic terms. For example, we prohibit negative wealth by requiring that the dollar  
 228 amount invested in the risky asset must tend to zero as the the investor's wealth tends to zero. Note  
 229 that this economically reasonable condition permits  $p$  to be finite or infinite at  $w = 0$ .

### 230 3.1.3 No Bankruptcy, Bounded Control

231 This is a realistic case, in which we assume that bankruptcy is prohibited and infinite borrowing  
 232 is not allowed. As a result,  $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, p_{\max}]$ . We call this case the *bounded control*  
 233 case. For example, typical margin requirements on a brokerage account limit borrowing to 50% of  
 234 the market value of the assets in the account. This would translate into a value of  $p_{\max} = 1.5$ .

235 Our numerical problem uses,

$$\hat{\mathbb{D}} = [0, w_{\max}] , \quad (3.18)$$

236 where  $w_{\max}$  is an approximation to the infinity boundary. Other assumptions and the boundary  
 237 conditions for  $V$  and  $G$  are the same as those of no bankruptcy case introduced in Section 3.1.2.  
 238 Note that for the bounded control case, the control is finite, thus  $\lim_{w \rightarrow 0}(pw) = 0$  and negative  
 239 wealth is not admissible.

240 We summarize the various cases in Table 1

Case	$\hat{\mathbb{D}}$	$\mathbb{P}$
Bankruptcy	$[w_{\min}, w_{\max}]$	$(-\infty, +\infty)$
No Bankruptcy	$[0, w_{\max}]$	$[0, +\infty)$
Bounded Control	$[0, w_{\max}]$	$[0, p_{\max}]$

TABLE 1: Summary of cases.

## 241 3.2 Special Case: Reduction to the Classic Multi-period Portfolio Selection 242 Problem

243 The classic multi-period portfolio selection problem can be stated as the following: given some  
 244 investment choices (assets) in the market, an investor seeks an optimal asset allocation strategy over  
 245 a period  $T$  with an initial wealth  $\hat{w}_0$ . This problem has been widely studied [24, 32, 21, 23, 6, 22].  
 246 If we use the mean variance approach to solve this problem, then the best strategy  $p^*(w, t)$  can be  
 247 defined as a solution of problem (2.4). We still assume there is one risk free bond and one risky  
 248 asset in the market. In this case,

$$\begin{aligned} dW &= (r + p\xi\sigma)Wdt + p\sigma WdZ_1 , \\ W(t=0) &= \hat{w}_0 > 0 . \end{aligned} \quad (3.19)$$

249 Clearly, the pension plan problem we introduced previously can be reduced to the classic multi-  
 250 period portfolio selection problem by simply setting the contribution rate  $\pi = 0$ . All equations and  
 251 boundary conditions stay the same.

## 252 4 Wealth-to-income Ratio Case

253 In the previous section, we considered the expected value and variance/quadratic variation of the  
 254 terminal wealth in order to construct an efficient frontier. In [10], it is argued that at retirement, a  
 255 pension plan member will be concerned with preservation of her standard of living. This suggests  
 256 measuring wealth in terms of a numeraire computed based on the investor's pre-retirement salary.  
 257 This approach for retirement saving takes into account the stochastic feature of the plan member's  
 258 lifetime salary progression, as well as the stochastic nature of the investment assets.

259 In this section, instead of the terminal wealth, we determine the mean variance efficient strategy  
 260 in terms of the terminal wealth-to-income ratio  $X = \frac{W}{Y}$ , where  $Y$  is the annual salary in the year  
 261 before she retires. As noted in [10], this is consistent with the habit formation model developed in  
 262 [28, 12].

263 In the following, we give a brief overview of the model developed in [10]. We still assume there  
 264 are two underlying assets in the pension plan: one is risk free and the other is risky. Recall from  
 265 equation (2.1) that the risky asset  $S$  follows the Geometric Brownian Motion,

$$dS = (r + \xi\sigma)S dt + \sigma S dZ_1 . \quad (4.1)$$

266 Suppose that the plan member continuously pays into the pension plan at a fraction  $\pi$  of her yearly  
 267 salary  $Y$ , which follows the process

$$dY = (r + \mu_Y)Y dt + \sigma_{Y_0}Y dZ_0 + \sigma_{Y_1}Y dZ_1 , \quad (4.2)$$

268 where  $\mu_Y$ ,  $\sigma_{Y_0}$  and  $\sigma_{Y_1}$  are constants, and  $dZ_0$  is another increment of a Wiener process, which is  
 269 independent of  $dZ_1$ . Let  $p$  denote the proportion of this wealth invested in the risky asset  $S$ , and  
 270 let  $1 - p$  denote the fraction of wealth invested in the risk-free asset. Then

$$\begin{aligned} dW &= (r + p\xi\sigma)W dt + p\sigma W dZ_1 + \pi Y dt , \\ W(t=0) &= \hat{w}_0 \geq 0 . \end{aligned} \quad (4.3)$$

271 Define a new state variable  $X(t) = W(t)/Y(t)$ , then by Ito's Lemma, we obtain

$$\begin{aligned} dX &= [\pi + X(-\mu_Y + p\sigma(\xi - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2)]dt \\ &\quad - \sigma_{Y_0}X dZ_0 + X(p\sigma - \sigma_{Y_1})dZ_1 , \\ X(t=0) &= \hat{x}_0 \geq 0 . \end{aligned} \quad (4.4)$$

272 Let

$$\begin{aligned} \mu_X^p &= \pi + X(-\mu_Y + p\sigma(\xi - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2) \\ (\sigma_X^p)^2 &= (\sigma_X^p(p(t, X(t)), X(t)))^2 \\ &= X^2(\sigma_{Y_0}^2 + (p\sigma - \sigma_{Y_1})^2) , \end{aligned} \quad (4.5)$$

273 then the control problem is to determine the control  $p(t, X(t) = x)$  such that  $p(t, x)$  maximizes

$$\mathcal{V}(x, t) = \sup_{p \in \mathbb{P}} E_{t,x}^p \left\{ X(T) - \lambda \int_t^T e^{2r'(T-s)} (\sigma_X^p)^2 ds \mid X(t) = x \right\}, \quad (4.6)$$

274 subject to stochastic process (4.4), where  $r' = -\mu_Y + \sigma_{Y_0}^2 + \sigma_{Y_1}^2$ . Note that we have posed the  
 275 problem in terms of the future value of the quadratic variation using  $r'$  as the discount factor. For  
 276 the wealth case, with no constraints on the controls, the analytic solution for the time-consistent  
 277 mean variance policy is identical to the mean quadratic strategy (2.12) [7]. However, there does  
 278 not appear to be an analytic solution available for the wealth-to-income ratio case, hence we use  $r'$   
 279 as the effective drift rate (when there is no investment in the risky asset). There are clearly other  
 280 possibilities here.

281 Let  $\tau = T - t$ . Then  $V(x, \tau) = \mathcal{V}(x, t)$  satisfies the HJB equation

$$V_\tau = \sup_{p \in \mathbb{P}} \left\{ \mu_x^p V_x + \frac{1}{2} (\sigma_x^p)^2 V_{xx} - \lambda e^{2r'\tau} (\sigma_x^p)^2 \right\} ; \quad x \in \mathbb{D}, \quad (4.7)$$

282 with terminal condition

$$V(x, 0) = x. \quad (4.8)$$

283 We still use  $\mathbb{D}$  and  $\mathbb{P}$  as the sets of all admissible wealth-to-income ratio and control. As before, we  
 284 let  $\hat{\mathbb{D}}$  be the localized computational domain.

285 We also solve for  $G(x, \tau) = E[X(T)|X(t) = x, p(s \geq t, x) = p^*(s \geq t, x)]$  and  $F(x, \tau) =$   
 286  $E[X(T)^2|X(t) = x, p(s \geq t, x) = p^*(s \geq t, x)]$  using

$$G_\tau = \{ \mu_x^p G_x + \frac{1}{2} (\sigma_x^p)^2 G_{xx} \}_{p(T-\tau, x) = p^*(T-\tau, x)} ; \quad x \in \mathbb{D}, \quad (4.9)$$

$$F_\tau = \{ \mu_x^p F_x + \frac{1}{2} (\sigma_x^p)^2 F_{xx} \}_{p(T-\tau, x) = p^*(T-\tau, x)} ; \quad x \in \mathbb{D}, \quad (4.10)$$

287 with terminal condition

$$G(x, 0) = x. \quad (4.11)$$

$$F(x, 0) = x^2. \quad (4.12)$$

288 We can then use the method described in Section 3 to trace out the efficient frontier solution  
 289 of problem (4.6).

290 We consider the cases: allowing bankruptcy ( $\mathbb{D} = (-\infty, +\infty)$ ,  $\mathbb{P} = (-\infty, +\infty)$ ), no bankruptcy  
 291 ( $\mathbb{D} = [0, +\infty)$ ,  $\mathbb{P} = [0, +\infty)$ ), and bounded control ( $\mathbb{D} = [0, +\infty)$ ,  $\mathbb{P} = [0, p_{\max}]$ ). For computational  
 292 purposes, we localize the problem to  $\hat{\mathbb{D}} = [x_{\min}, x_{\max}]$ , and apply boundary conditions as in  
 293 Section 3.1. More precisely, if  $x = 0$  is a boundary, with  $X < 0$  prohibited, then  $\lim_{w \rightarrow 0} (pw) = 0$ ,  
 294 and hence

$$\begin{aligned} V_\tau(0, \tau) &= \pi V_x, \\ G_\tau(0, \tau) &= \pi G_x, \\ F_\tau(0, \tau) &= \pi F_x. \end{aligned} \quad (4.13)$$

295 The boundary conditions at  $x \rightarrow \pm\infty$  are given in equation (3.14), but using  $x$  instead of  $w$  and  $r'$   
 296 instead of  $r$  with  $k_1 = -\mu_Y + p\sigma(\xi - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2$  and  $k_2 = \sigma_{Y_0}^2 + (p\sigma - \sigma_{Y_1})^2$ .

## 297 5 Discretization of the HJB PDE

298 The numerical scheme to solve the PDEs is similar to the scheme used in [29]. We briefly describe  
 299 the discretization scheme in this section, and refer readers to [29] for details.

300 Define,

$$\mathcal{L}^p V \equiv a(z, p)V_{zz} + b(z, p)V_z, \quad (5.1)$$

301 where

$$z = w; \quad a(z, p) = \frac{1}{2}(\sigma_w^p)^2; \quad b(z, p) = \mu_w^p; \quad d(z, p, \tau) = -\lambda(e^{r\tau}\sigma p w)^2 \quad (5.2)$$

302 (see equation (3.5)) for the wealth case introduced in Section 2; and

$$z = x; \quad a(z, p) = \frac{1}{2}(\sigma_x^p)^2; \quad b(z, p) = \mu_x^p; \quad d(z, p, \tau) = -\lambda e^{2r'\tau}(\sigma_x^p)^2 \quad (5.3)$$

303 (see equation (4.5)) for the wealth-to-income ratio case introduced in Section 4. Then,

$$V_\tau = \sup_{p \in \mathbb{P}} \{ \mathcal{L}^p V + d(z, p, \tau) \}, \quad (5.4)$$

304 and

$$G_\tau = \{ \mathcal{L}^p G \}_{p=p^*}, \quad (5.5)$$

$$F_\tau = \{ \mathcal{L}^p F \}_{p=p^*}. \quad (5.6)$$

305 Define a grid  $\{z_0, z_1, \dots, z_q\}$  with  $z_0 = z_{\min}$ ,  $z_q = z_{\max}$  and let  $V_i^n$  be a discrete approximation  
 306 to  $V(z_i, \tau^n)$ . Set  $P^n = [p_0^n, p_1^n, \dots, p_q^n]'$ , with each  $p_i^n$  a local optimal control at  $(z_i, \tau^n)$ . Let  
 307  $P^* = \{P^0, P^1, \dots, P^N\}$ , where  $\tau^N = T$ . In other words,  $P^*$  contains the discrete optimal controls  
 308 for all  $(i, n)$ . Let  $V^n = [V_0^n, \dots, V_q^n]'$ , and let  $(\mathcal{L}_h^{P^n} V^n)_i$  denote the discrete form of the differential  
 309 operator (5.1) at node  $(z_i, \tau^n)$ . The operator (5.1) can be discretized using forward, backward or  
 310 central differencing in the  $z$  direction to give

$$(\mathcal{L}_h^{P^{n+1}} V^{n+1})_i = \alpha_i^{n+1} V_{i-1}^{n+1} + \beta_i^{n+1} V_{i+1}^{n+1} - (\alpha_i^{n+1} + \beta_i^{n+1}) V_i^{n+1}. \quad (5.7)$$

311 Here  $\alpha_i, \beta_i$  are defined in Appendix A.

312 Equation (5.4) can now be discretized using fully implicit timestepping along with the dis-  
 313 cretization (5.7) to give

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \sup_{P^{n+1} \in \hat{P}} \left\{ (\mathcal{L}_h^{P^{n+1}} V^{n+1})_i + d(z_i, [P^{n+1}]_i, \tau) \right\}, \quad (5.8)$$

314 where  $\hat{P} = \{[p_0, p_1, \dots, p_q]' \mid p_i \in \mathbb{P}, 0 \leq i \leq q\}$ . With  $P^{n+1}$  given from equation (5.8), then  
 315 equations (5.5) and (5.6) can be discretized as

$$\frac{G_i^{n+1} - G_i^n}{\Delta\tau} = \left\{ (\mathcal{L}_h^{P^{n+1}} G^{n+1})_i \right\}, \quad (5.9)$$

$$\frac{F_i^{n+1} - F_i^n}{\Delta\tau} = \left\{ (\mathcal{L}_h^{P^{n+1}} F^{n+1})_i \right\}. \quad (5.10)$$

316 Note that  $\alpha_i^{n+1} = \alpha_i^{n+1}(p_i^{n+1})$  and  $\beta_i^{n+1} = \beta_i^{n+1}(p_i^{n+1})$ , that is, the discrete equation coefficients  
 317 are functions of the local optimal control  $p_i^{n+1}$ . This makes equations (5.8) highly nonlinear in  
 318 general. We use a policy type iteration [29] to solve the non-linear discretized algebraic equation  
 319 (5.8).

320 Given an initial value  $\hat{z}_0$ , we can use the algorithm introduced in [29] to obtain the efficient  
 321 frontier.

## 322 5.1 Convergence to the Viscosity Solution

323 PDEs (5.5) and (5.6) are linear, since the optimal control is pre-computed. We can then obtain  
 324 classical solutions of the linear PDEs (5.5) and (5.6). However, PDE (5.4) is highly nonlinear, so  
 325 the classical solution may not exist in general. In this case, we are seeking the viscosity solution  
 326 [2, 13].

327 In [27], examples were given in which seemingly reasonable discretizations of nonlinear HJB  
 328 PDEs were unstable or converged to the incorrect solution. It is important to ensure that we can  
 329 generate discretizations which are guaranteed to converge to the viscosity solution [2, 13]. In the  
 330 case of bounded controls, on a finite computational domain, the PDE (5.4) satisfies the conditions  
 331 required in [4], so that a strong comparison property holds.

332 In the case of allowing bankruptcy and no bankruptcy (see Table 1), the control  $p$  is unbounded  
 333 near  $w = 0$ , which would violate some of the conditions required in [4]. However, we can avoid these  
 334 difficulties, if we rewrite PDE (5.4) in terms of the control  $q = pw$ . From the analytic solutions,  
 335 we note that  $q$  is bounded on a finite computational domain, hence a strong comparison property  
 336 holds in this case as well. In fact, our numerical implementation for these two cases does in fact  
 337 use  $q = pw$  as the control, as discussed in [30].

338 Following the same proof given in [29], we can show that scheme (5.8) converges to the viscosity  
 339 solution of equation (5.4), assuming that (5.4) satisfies a strong comparison principle. We refer  
 340 readers to [29] for details.

## 341 6 Numerical Results: Mean Quadratic Variation

342 In this section we examine the numerical results for the strategy of minimizing the quadratic  
 343 variation. We consider two risk measures when we construct efficient frontiers. One measure is  
 344 the usual standard deviation, and the other measure is the expected future value of the quadratic  
 345 variation,  $E[\int_0^T (e^{r(T-s)} dW_s)^2]$ . We use the notation  $Q\_std_{t=0,w}^{p^*}[W(T)]$  to denote

$$Q\_std_{t=0,w}^{p^*}[W(T)] = \left( E_{t=0,w}^{p^*} \left\{ \int_0^T (e^{r(T-s)} \sigma p^*(s, W(s)) W(s))^2 ds \mid W(0) = w \right\} \right)^{1/2}. \quad (6.1)$$

### 346 6.1 Wealth Case

347 When bankruptcy is allowed, as pointed out in [7], the mean quadratic variation strategy has  
 348 the same solution as the time-consistent strategy. The analytic solutions for the time-consistent  
 349 strategy are given in Section 7. Given the parameters in Table 2, if  $\lambda = 0.6$ , the exact solution is  
 350  $(Std_{t=0,w}^{p^*}[W(T)], E_{t=0,w}^{p^*}[W(T)]) = (1.24226, 6.41437)$ . Table 3 and 4 show the numerical results.

351 Table 3 reports the value of  $V = E_{t=0,w}^{p*}[W(T) - \lambda \int_0^T (e^{r(T-s)} dW_s)^2]$ , which is the viscosity solution  
352 of the nonlinear HJB PDE (3.3). Table 3 shows that our numerical solution converges to the  
353 viscosity solution at a first order rate. Table 4 reports the value of  $E_{t=0,w}^{p*}[W(T)]$ , which is the  
354 solution of the linear PDE (3.6). We also computed the values of  $E_{t=0,w}^{p*}[W(T)^2]$  (not shown in  
355 Tables), which is the the solution of PDE (3.10). Given  $E_{t=0,w}^{p*}[W(T)^2]$  and  $E_{t=0,w}^{p*}[W(T)]$ , the  
356 standard deviation can now be easily computed, which is also reported in Table 4. The results  
357 show that the numerical solutions of  $\text{Std}_{t=0,w}^{p*}[W(T)]$  and  $E_{t=0,w}^{p*}[W(T)]$  converge to the analytic  
358 values at a first order rate as mesh and timestep size tends to zero.

$r$	0.03	$\xi$	0.33
$\sigma$	0.15	$\pi$	0.1
$T$	20 years	$W(t=0)$	1

TABLE 2: Parameters used in the pension plan examples.

Nodes ( $W$ )	Timesteps	Nonlinear iterations	Normalized CPU Time	$V(w=1, t=0)$	Ratio
1456	320	640	1.	5.49341	
2912	640	1280	4.13	5.49092	
5824	1280	2560	16.31	5.48968	2.008
11648	2560	5120	66.23	5.48906	2.000
23296	5120	10240	268.53	5.48875	2.000
46592	10240	20480	1145.15	5.48860	2.067

TABLE 3: Convergence study, wealth case, allowing bankruptcy. Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 2, with  $\lambda = 0.6$ . Values of  $V = E_{t=0,w}^{p*}[W(T) - \lambda \int_0^T (e^{r(T-t)} dw)^2]$  are reported at  $(W = 1, t = 0)$ . Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter  $h$ . CPU time is normalized. We take the CPU time used for the first test in this table as one unit of CPU time, which uses 1456 nodes for  $W$  grid and 320 timesteps.

Nodes ( $W$ )	Timesteps	$\text{Std}_{t=0,w}^{p*}[W(T)]$	$E_{t=0,w}^{p*}[W(T)]$	Ratio for $\text{Std}_{t=0,w}^{p*}[W(T)]$	Ratio for $E[W(T)]$
1456	320	1.30652	6.41986	1.960	
2912	640	1.27466	6.41711	1.972	
5824	1280	1.25853	6.41574	1.975	2.007
11648	2560	1.25041	6.41505	1.986	1.986
23296	5120	1.24634	6.41471	1.995	2.029
46592	10240	1.244300	6.41454	2.000	1.995098

TABLE 4: Convergence study of the wealth case, allowing bankruptcy. Fully implicit timestepping is applied, using constant timesteps. The parameters are given in Table 2, with  $\lambda = 0.6$ . Values of  $\text{Std}_{t=0,w}^{p*}[W(T)]$  and  $E_{t=0,w}^{p*}[W(T)]$  are reported at  $(W = 1, t = 0)$ . Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter  $h$ . Analytic solution is  $(\text{Std}_{t=0,w}^{p*}[W(T)], E_{t=0,w}^{p*}[W(T)]) = (1.24226, 6.41437)$ .

359 We also solve the problem for the no bankruptcy case and the bounded control case. The  
360 frontiers are shown in Figure 1, with parameters given in Table 2 and ( $W = 1, t = 0$ ). Figure 1  
361 (a) shows the results obtained by using the standard deviation as the risk measure, and Figure 1  
362 (b) shows the results obtained by using the quadratic variation as the risk measure. Note that,  
363 in both figures, the efficient frontiers pass through the same lowest point. At that point, the plan  
364 holder simply invests all her wealth in the risk free bond all the time, so the risk (standard devia-  
365 tion/quadratic variation) is zero. For both risk measures, the frontiers for the allowing bankruptcy  
366 case are straight lines. This result agrees with the results from the pre-commitment strategy [29]  
367 and the time-consistent strategy [30].

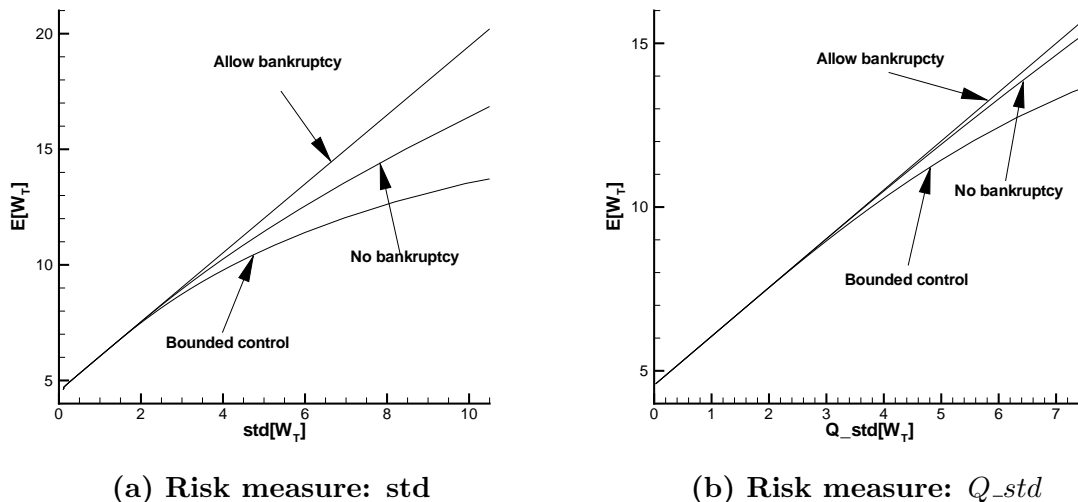
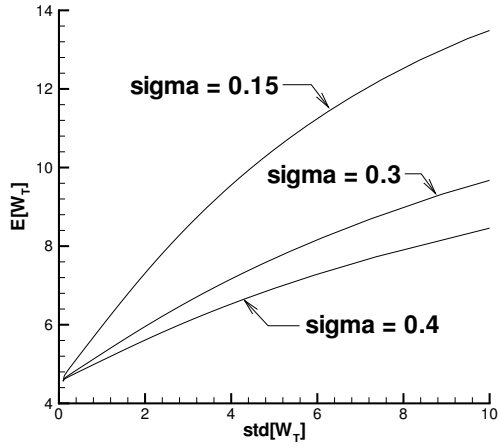


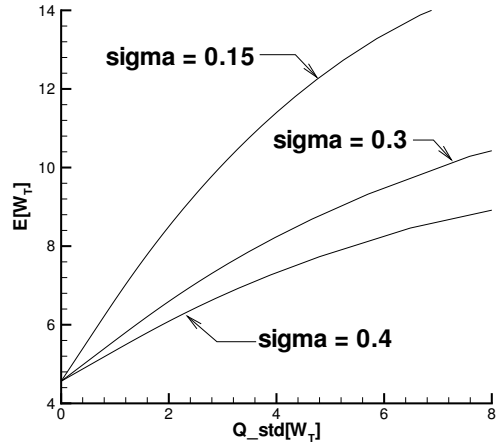
FIGURE 1: *Efficient frontiers (wealth case) for allowing bankruptcy ( $\mathbb{D} = (-\infty, +\infty)$  and  $\mathbb{P} = (-\infty, +\infty)$ ), no bankruptcy ( $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, +\infty)$ ) and bounded control ( $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, 1.5]$ ) cases. Parameters are given in Table 2. Values are reported at ( $W = 1, t = 0$ ). Figure (a) shows the frontiers with risk measure standard deviation. Figure (b) shows the frontiers with risk measure quadratic variation.*

368 Figure 2 shows the effect of varying  $\sigma$  while holding  $\mu_S = r + \xi\sigma$  constant. In this case, the  
369 efficient frontiers for different values of  $\sigma$  are well separated. Figure 3 shows the effect of varying  $\sigma$   
370 while holding  $\xi$  constant. In this case, the curves for different values of  $\sigma$  are much closer together.  
371 Note as well that if the value of  $\sigma$  is increased with  $\mu_S$  fixed, then the efficient frontier moves  
372 downward (Figure 2). On the other hand, as shown in Figure 3, the efficient frontier moves upward  
373 if  $\sigma$  is increased with fixed  $\xi$  (this makes the drift rate  $\mu_S$  increase).

374 Figure 4 shows the values of the optimal control (the investment strategies) at different time  $t$   
375 for a fixed  $T = 20$ . The parameters are given in Table 2, with bounded control ( $p \in [0, 1.5]$ ) and  $\lambda =$   
376  $0.604$ . Under these inputs, if  $W(t = 0) = 1$ ,  $(\text{Std}_{t=0,w}^{p*}[W(T)], E_{t=0,w}^{p*}[W(T)]) = (1.23824, 6.40227)$   
377 and  $Q\_std_{t=0,w}^{p*}[W(T)] = 1.52262$  from the finite difference solution. From this Figure, we can see  
378 that the control  $p$  is an increasing function of time  $t$  for a fixed  $w$ . This agrees with the results  
379 from the pre-commitment [29] and time-consistent strategies [30].

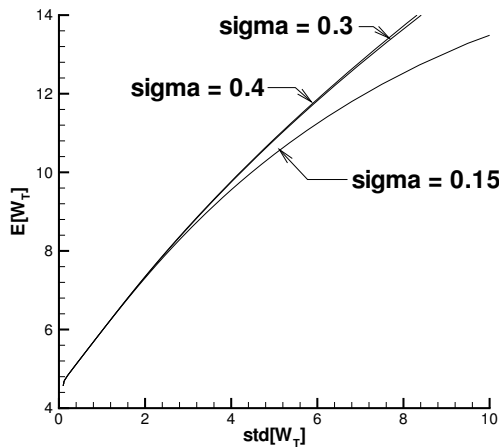


(a) Risk measure: std

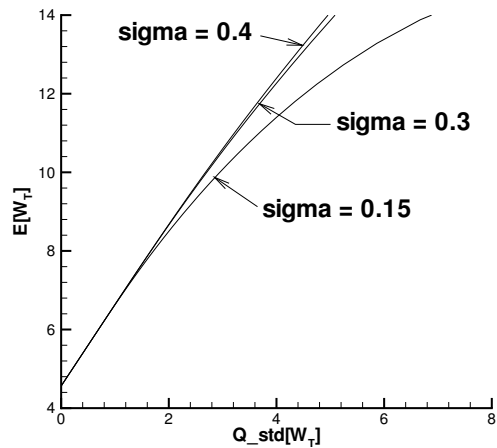


(b) Risk measure:  $Q\_std$

FIGURE 2: *Efficient frontiers (wealth case), bounded control. We fix  $\mu_S = r + \xi\sigma = .08$ , and vary  $\sigma$ . Other parameters are given in Table 2. Values are reported at  $(W = 1, t = 0)$ . Figure (a) shows the frontiers with risk measure standard deviation. Figure (b) shows the frontiers with risk measure quadratic variation.*



(a) Risk measure: std



(b) Risk measure:  $Q\_std$

FIGURE 3: *Efficient frontiers (wealth case), bounded control. We fix  $\xi = 0.33$ , and vary  $\sigma$ . Other parameters are given in Table 2. Values are reported at  $(W = 1, t = 0)$ . Figure (a) shows the frontiers with risk measure standard deviation. Figure (b) shows the frontiers with risk measure quadratic variation.*

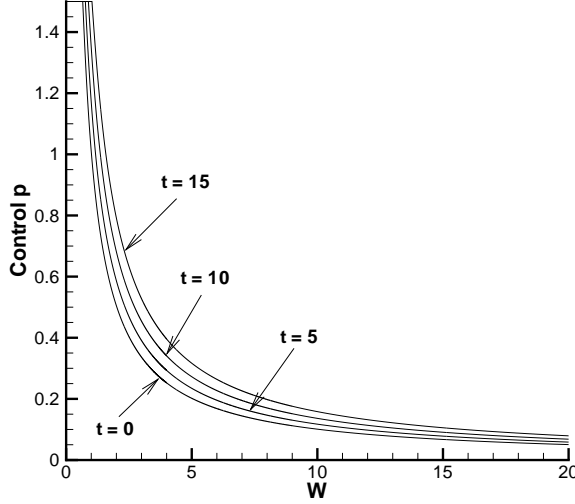


FIGURE 4: Optimal control as a function of  $(W, t)$ , bounded control case. Parameters are given in Table 2, with  $\lambda = 0.604$ . Under these inputs, if  $W(t = 0) = 1$ ,  $(Std_{t=0,w}^{p^*}[W(T)], E_{t=0,w}^{p^*}[W(T)]) = (1.23824, 6.40227)$  and  $Q\_std_{t=0,w}^{p^*}[W(T)] = 1.52262$  from finite difference solution. Mean quadratic variation objective function.

380 **Remark 6.1** As we discussed in Remark 3.3, in the case of bankruptcy prohibition, we have to  
 381 have  $\lim_{w \rightarrow 0} (p^* w) = 0$  so that negative wealth is not admissible. Our numerical tests show that as  
 382  $w$  goes to zero,  $p^* w = O(w^\gamma)$ . For a reasonable range of parameters, we have  $0.9 < \gamma < 1$ . Hence,  
 383 this verifies that the boundary conditions (3.17) ensure that negative wealth is not admissible under  
 384 the optimal strategy. This property also holds for the wealth-to-income ratio case.

## 385 6.2 Multi-period Portfolio Selection

386 As discussed in Section 3.2, the wealth case can be reduced to the classic multi-period portfolio  
 387 selection problem. Efficient frontier solutions of a particular multi-period portfolio selection problem  
 388 are shown in Figure 5, with parameters in Table 2 but with  $\pi = 0$ . Again, we consider three cases:  
 389 allowing bankruptcy, no bankruptcy, and bounded control cases. Figure 5 (a) shows the results  
 390 obtained by using the standard deviation as the risk measure, and Figure 5 (b) shows the results  
 391 obtained by using the quadratic variation as the risk measure. As for the wealth case, in both  
 392 figures, the frontiers for the allowing bankruptcy case are straight lines.

## 393 6.3 Wealth-to-income Ratio Case

394 In this section, we examine the wealth-to-income ratio case. Tables 6 and 7 show the numerical  
 395 results for the bounded control case, using parameters in Table 5. Table 6 reports the value of  
 396  $V = E_{t=0,x}^{p^*}[X(T) - \lambda \int_0^T (e^{r(T-s)} dX_s)^2]$ , which is the viscosity solution of nonlinear HJB PDE  
 397 (4.7). Table 7 reports the value of  $E_{t=0,x}^{p^*}[X(T)]$ , which is the solution of the linear PDE (4.9). We

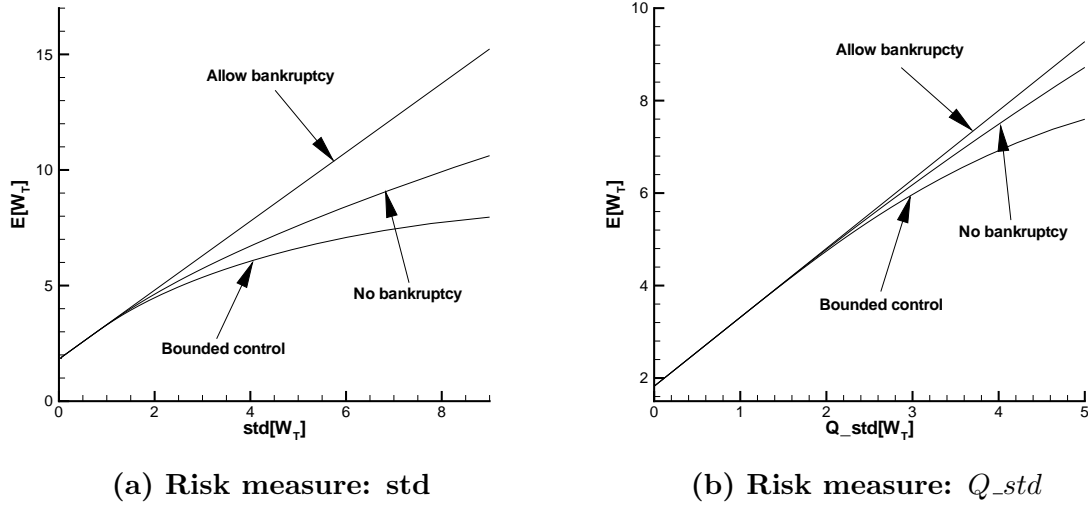


FIGURE 5: *Efficient frontiers (multi-period portfolio selection) for allowing bankruptcy ( $\mathbb{D} = (-\infty, +\infty)$  and  $\mathbb{P} = (-\infty, +\infty)$ ), no bankruptcy ( $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, +\infty)$ ) and bounded control ( $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, 1.5]$ ) cases. Parameters are given in Table 2 but with  $\pi = 0$ . Values are reported at  $(W = 1, t = 0)$ . Figure (a) shows the frontiers with risk measure standard deviation. Figure (b) shows the frontiers with risk measure quadratic variation.*

$\mu_y$	0.	$\xi$	0.2
$\sigma$	0.2	$\sigma_{Y1}$	0.05
$\sigma_{Y0}$	0.05	$\pi$	0.1
$T$	20 years	$\lambda$	0.25
$\mathbb{Q}$	$[0, 1.5]$	$\mathbb{D}$	$[0, +\infty)$

TABLE 5: *Parameters used in the pension plan examples.*

398 also computed the values of  $E_{t=0,x}^{p^*}[X(T)^2]$  (not shown in tables), which is the the solution of PDE  
 399 (4.10).

400 Given  $E_{t=0,x}^{p^*}[X(T)^2]$  and  $E_{t=0,x}^{p^*}[X(T)]$ , the standard deviation is easily computed. This is also  
 401 reported in Table 7. The results show that the numerical solutions of  $V$  and  $E_{t=0,x}^{p^*}[X(T)]$  converges  
 402 at a first order rate as mesh and timestep size tends to zero.

403 Efficient frontiers are shown in Figure 6, using parameters in Table 5 with  $(X(0) = 0.5; t = 0)$ .  
 404 Figure 6 (a) shows the results obtained by using the standard deviation as the risk measure, and  
 405 Figure 6 (b) shows the results obtained by using the quadratic variation as the risk measure.  
 406 Note that, although the frontiers in both figures pass through the same lowest point, unlike the  
 407 wealth case, the minimum standard deviation/quadratic variation for all strategies are no longer  
 408 zero. Since the plan holder's salary is stochastic (equation (4.2)) and the salary risk cannot be  
 409 completely hedged away, there is no risk free strategy.

410 Figure 7 shows the values of the optimal control (the investment strategies) at different time  
 411  $t$  for a fixed  $T = 20$ . The parameters are given in Table 5, with  $\lambda = 0.2873$ . Under these inputs,

Nodes (W)	Timesteps	Nonlinear iterations	Normalized CPU Time	$V(w = 1, t = 0)$	Ratio
177	80	160	0.21	3.26653	
353	160	320	1.	3.26534	
705	320	640	3.86	3.26476	2.052
1409	640	1280	15.00	3.26447	2.000
2817	1280	2560	56.79	3.26433	2.071
5633	2560	5120	239.79	3.26426	2.000
11265	5120	10240	966.29	3.26422	1.750

TABLE 6: *Convergence study. quadratic variation, Bounded Control. Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 5, with  $\lambda = 0.2873$ . Values of  $V = E_{t=0,x}^{p*}[X(T) - \lambda \int_0^T (e^{r(T-s)} dX^2)]$  Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter  $h$ . CPU time is normalized. We take the CPU time used for the second test in this table as one unit of CPU time, which uses 353 nodes for  $X$  grid and 160 timesteps.*

Nodes (W)	Timesteps	$\text{Std}_{t=0,x}^{p*}[W(T)]$	$E_{t=0,x}^{p*}[W(T)]$	Ratio for $\text{Std}_{t=0,x}^{p*}[W(T)]$	Ratio for $E[W(T)]$
177	80	1.39064	3.69771		
353	160	1.35723	3.69524		
705	320	1.34035	3.69403	1.979	2.041
1409	640	1.33187	3.69343	1.991	2.017
2817	1280	1.32762	3.69313	1.995	2.000
5633	2560	1.32549	3.69298	1.995	2.000
11265	5120	1.32443	3.69291	2.009	2.143

TABLE 7: *Convergence study, wealth-to-income ratio case, bounded control. Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 5, with  $\lambda = 0.2873$ . Values of  $\text{Std}_{t=0,x}^{p*}[X(T)]$  and  $E_{t=0,x}^{p*}[X(T)]$  are reported at  $(X = 0.5, t = 0)$ . Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter  $h$ .*

412 if  $X(t = 0) = 0.5$ ,  $(\text{Std}_{t=0,x}^{p*}[X(T)], E_{t=0,x}^{p*}[X(T)]) = (1.32443, 3.69291)$  and  $Q\text{-std}_{t=0,w}^{p*}[X(T)] =$   
413  $1.49213$  from the finite difference solution. Similar to the wealth case, we can see that the control  
414  $p$  is a increasing function of time  $t$  for a fixed  $x$ .

415 **Remark 6.2 (Behaviour of the control as a function of time)** *The optimal strategy for the*  
416 *wealth-to-income ratio case, based on a power law utility function [10] has the property that, for*  
417 *fixed  $x$ , the control  $p$  is a decreasing function of time. In other words, if the wealth-to-income ratio*  
418 *is static, the investor reduces the weight in the risky asset as time goes on [10]. Using the mean*  
419 *quadratic variation criterion, the optimal strategy is to increase the weight in the risky asset if the*  
420 *wealth-to-income ratio is static.*

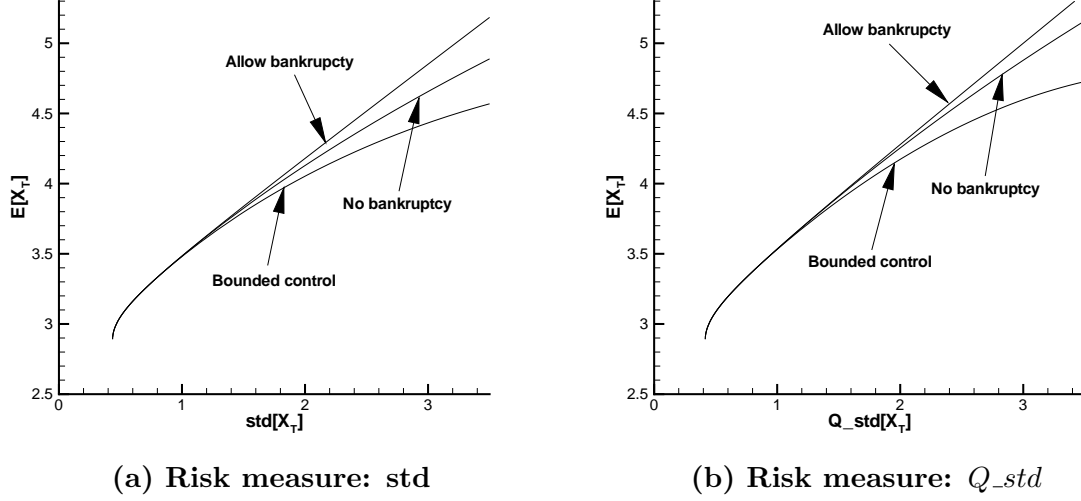


FIGURE 6: *Efficient frontiers (wealth-to-income ratio) for allowing bankruptcy ( $\mathbb{D} = (-\infty, +\infty)$  and  $\mathbb{P} = (-\infty, +\infty)$ ), no bankruptcy ( $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, +\infty)$ ) and bounded control ( $\mathbb{D} = [0, +\infty)$  and  $\mathbb{P} = [0, 1.5]$ ) cases. Parameters are given in Table 5. Values are reported at ( $W = 1, t = 0$ ). Figure (a) shows the frontiers with risk measure standard deviation. Figure (b) shows the frontiers with risk measure quadratic variation.*

## 421 7 Comparison of Various Strategies

422 In this section, we compare the three strategies: pre-commitment, time-consistent and quadratic  
 423 variation strategies. We remind the reader that the pre-commitment solutions are computed using  
 424 the methods in [29], and the time-consistent strategies are computed using the methods in [30].  
 425 The mean quadratic variation results are computed using the techniques developed in this article.

### 426 7.1 Wealth Case

427 We first study the wealth case for the three strategies. Figure 8 shows the frontiers for the case  
 428 of allowing bankruptcy for the three strategies. The analytic solution for the pre-commitment  
 429 strategy is given in [19],

$$\begin{cases} Var^{t=0}[W(T)] = \frac{e^{\xi^2 T} - 1}{4\lambda^2} \\ E^{t=0}[W(T)] = \hat{w}_0 e^{rT} + \pi \frac{e^{rT} - 1}{r} + \sqrt{e^{\xi^2 T} - 1} \text{Std}(W(T)) \end{cases}, \quad (7.1)$$

430 and the optimal control  $p$  at any time  $t \in [0, T]$  is

$$p^*(t, w) = -\frac{\xi}{\sigma w} \left[ w - (\hat{w}_0 e^{rt} + \frac{\pi}{r}(e^{rt} - 1)) - \frac{e^{-r(T-t) + \xi^2 T}}{2\lambda} \right]. \quad (7.2)$$

431 Extending the results from [5], we can obtain the analytic solution for the time-consistent

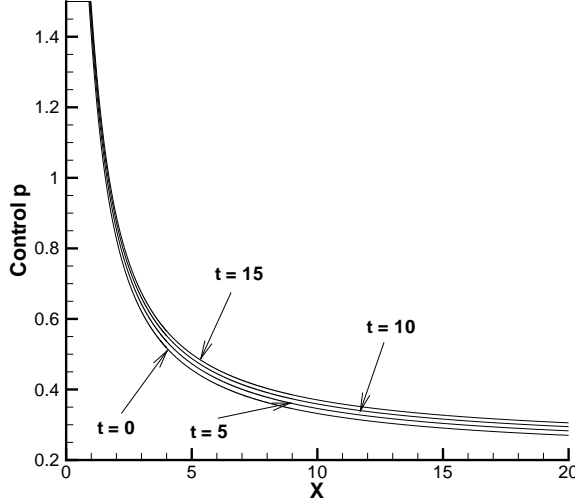


FIGURE 7: *Optimal control as a function of  $(X, t)$ , mean quadratic variation, wealth-to-income ratio with bounded control. Parameters are given in Table 5, with  $\lambda = 0.2873$ . Under these inputs, if  $X(t = 0) = 0.5$ ,  $(Std_{t=0,x}^{p^*}[X(T)], E_{t=0,x}^{p^*}[X(T)]) = (1.32443, 3.69291)$  and  $Q\text{-}std_{t=0,x}^{p^*}[X(T)] = 1.49213$  from finite difference solution.*

432 strategy,

$$\begin{cases} Var_{t=0,\hat{w}_0}[W(T)] = \frac{\xi^2}{4\lambda^2} T \\ E_{t=0,\hat{w}_0}[W(T)] = \hat{w}_0 e^{rT} + \pi \frac{e^{rT}-1}{r} + \xi \sqrt{T} \text{Std}(W(T)) \end{cases}, \quad (7.3)$$

433 and the optimal control  $p$  at any time  $t \in [0, T]$  is

$$p^*(t, w) = \frac{\xi}{2\lambda\sigma w} e^{-r(T-t)}. \quad (7.4)$$

434 Figure 8 shows that the frontiers for the time-consistent strategy and the mean quadratic  
 435 variation strategy are the same. This result agrees with the result in [7]. It is also interesting to  
 436 observe that this control is also identical to the control obtained using a utility function of the form  
 437 [14]

$$U(w) = -\frac{e^{-2\lambda w}}{2\lambda}. \quad (7.5)$$

438 Figure 8 also shows that the pre-commitment strategy dominates the other strategies, according  
 439 to the mean-variance criterion. The three frontiers are all straight lines, and pass the same point  
 440 at  $(\text{Std}(W(T)), E(W(T))) = (0, \hat{w}_0 e^{rT} + \pi \frac{e^{rT}-1}{r})$ . At that point, the plan holder simply invests all  
 441 her wealth in the risk free bond, so the standard deviation is zero.

442 **Remark 7.1** *It appears that in general, the the investment policies for time consistent mean vari-*  
 443 *ance and mean quadratic variation strategies are not the same. These two strategies do give rise*

444 to the same policy in the unconstrained (allow bankruptcy) case. When we apply constraints to  
 445 the investment strategy, the optimal polices are different, but quite close (see the numerical results  
 446 later in this Section). However, as noted in [7], there exists some standard time consistent control  
 447 problem which does give rise to the same control. But, as pointed out in [7], it is not obvious how  
 448 to find this equivalent problem.

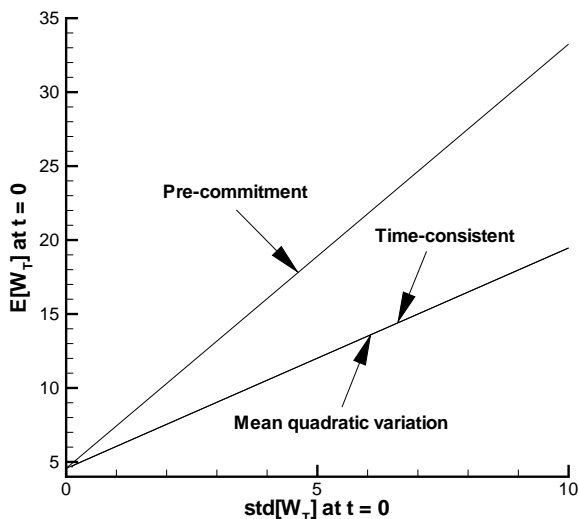


FIGURE 8: Comparison of three strategies: wealth case, allowing bankruptcy. Parameters are given in Table 2.

449 Figure 9 (a) shows a comparison for the three strategies for the no bankruptcy case, and Figure  
 450 9 (b) is for the bounded control case. We can see that the pre-commitment strategy dominates the  
 451 other strategies. The mean quadratic variation strategy dominates the time-consistent strategy. For  
 452 the bounded control case, the three frontiers have the same end points. The lower end corresponds  
 453 to the most conservative strategy, i.e. the whole wealth is invested in the risk free bond at any  
 454 time. The higher end corresponds to the most aggressive strategy, i.e. choose the control  $p$  to be  
 455 the upper bound  $p_{max}(= 1.5)$  at any time. Figure 8 and 9 show that the difference between the  
 456 frontiers for the three strategies becomes smaller after adding constraints.

457 Since the frontiers for the time-consistent strategy and the mean quadratic variation strategy  
 458 are very close for the bounded control case, it is desirable to confirm that the small difference is  
 459 not due to computational error. In Table 8, we show a convergence study for both time-consistent  
 460 strategy and mean quadratic variation strategy. The parameters are given in Table 2. We fix  
 461  $\text{Std}_{t=0,w}^{p^*}[W(T)] = 5$ . Table 8 shows that the two strategies converge to different expected terminal  
 462 wealth.

463 It is not surprising that the pre-commitment strategy dominates the other strategies, since  
 464 the pre-commitment strategy is the strategy which optimizes the objective function at the initial  
 465 time ( $t = 0$ ). However, as discussed in Section 1, in practice, there are many reasons to choose a  
 466 time-consistent strategy or a mean quadratic variation strategy.

Refine	Time-consistent $E_{t=0,w}^{p*}[W(T)]$	Mean Quadratic Variation $E_{t=0,w}^{p*}[W(T)]$
0	10.3570	10.4337
1	10.4508	10.5537
2	10.5055	10.6035
3	10.5319	10.6273
4	10.5448	10.6390
5	10.55139	10.6447

TABLE 8: *Convergence study, wealth case, bounded control. Fully implicit timestepping is applied, using constant timesteps. The parameters are given in Table 2. We fix  $\text{Std}_{t=0,w}^{p*}[W(T)] = 5$  for both time-consistent and mean quadratic variation strategies. Values of  $\text{Std}_{t=0,w}^{p*}[W(T)]$  and  $E_{t=0,w}^{p*}[W(T)]$  are reported at  $(W = 1, t = 0)$ . On each refinement, new nodes are inserted between each coarse grid node, and the timestep is divided by two. Initially (zero refinement), for time-consistent strategy, there are 41 nodes for the control grid, 182 nodes for the wealth grid, and 80 timesteps; for mean quadratic variation strategy, there are 177 nodes for the wealth grid, and 80 timesteps.*

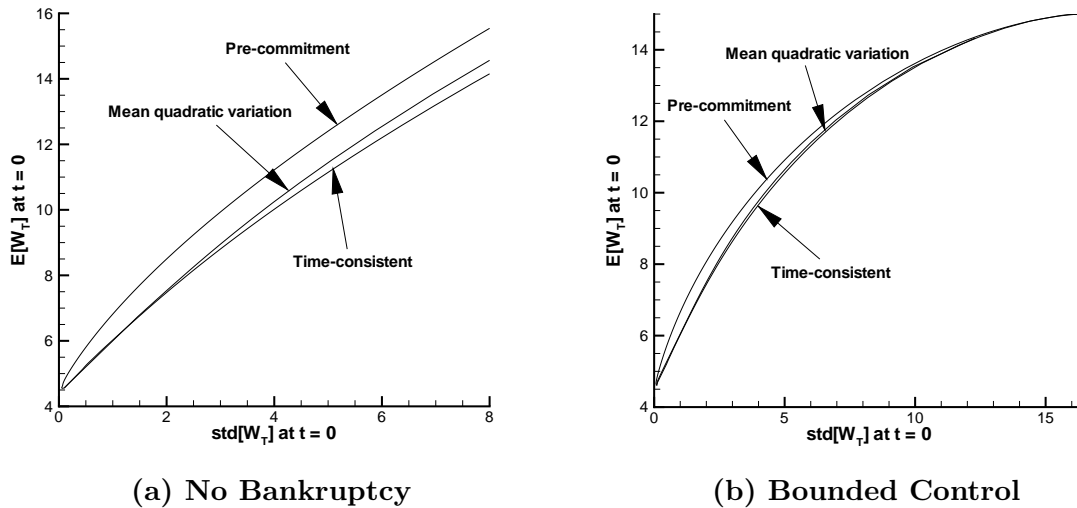


FIGURE 9: *Comparison of three strategies: wealth case. (a): no bankruptcy case; (b): bounded control case. The parameters are given in Table 2.*

467 In Figure 10, we compare the control policies for the three strategies. The parameters are given  
468 in Table 2, and we use the wealth case with bounded control ( $p \in [0, 1.5]$ ). We fix  $\text{Std}_{t=0,x}^{p*}[W(T)] \simeq$   
469 8.17 for this test. Figure 10 shows that the control policies given by the three strategies are  
470 significantly different. This is true even for the bounded control case, where the expected values  
471 for the three strategies are similar for fixed standard deviation (see Figure 9 (b)). Figure 10 (a)  
472 shows the control policies at  $t = 0^+$ .

473 We can interpret Figure 10 as follows. Suppose initially  $W(t = 0) = 1$ . If at the instant right  
474 after  $t = 0$ , the value for  $W$  jumps to  $W(t = 0^+)$ , Figure 10 (a) shows the control policies for

475 all  $W(t = 0^+)$ . We can see that once the wealth  $W$  is large enough, the control policy for the  
476 pre-commitment strategy is to invest all wealth in the risk free bond. The reason for this is that for  
477 the pre-commitment strategy, there is an effective investment target given at  $t = 0$ , which depends  
478 on the value of  $\lambda$ . Once the target is reached, the investor will not take any more risk and switch  
479 all wealth into bonds. However, there is no similar effective target for the time-consistent or the  
480 mean quadratic variation cases, so the control never reaches zero. Figure 10 (b) shows the mean  
481 of the control policies versus time  $t \in [0, T]$ . The mean of both policies are decreasing functions of  
482 time, i.e. all strategies are less risky (on average) as maturity is approached. We use Monte-Carlo  
483 simulations to obtain Figure 10 (b). Using the parameters in Table 2, we solve the stochastic  
484 optimal control problem (2.12) with the finite difference scheme introduced in Section 5, and store  
485 the optimal strategies for each  $(W = w, t)$ . We then carry out Monte-Carlo simulations based on  
486 the stored strategies with  $W(t = 0) = 1$  initially. At each time step, we can get the control  $p$  for  
487 each simulation. We then can obtain the mean of  $p$  for each time step.

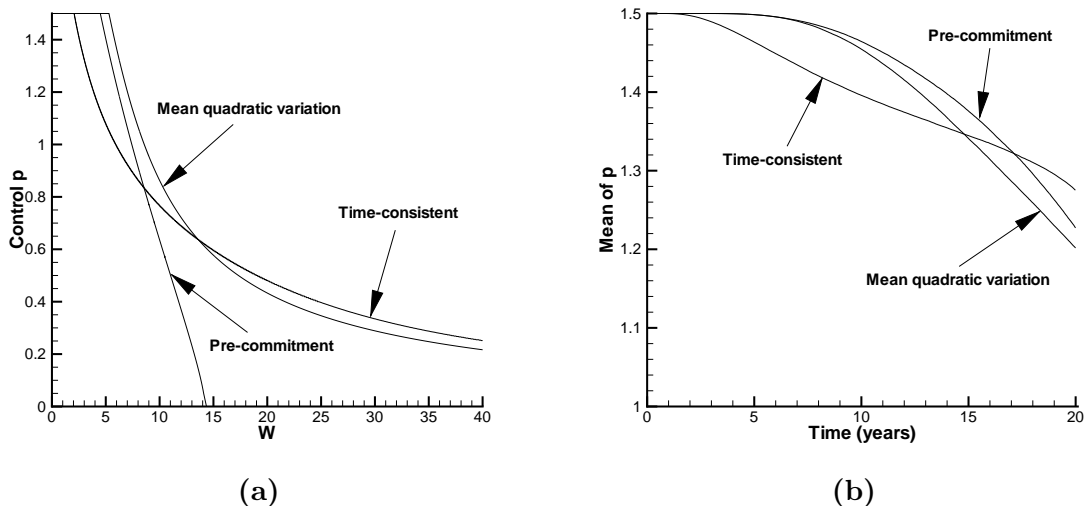


FIGURE 10: Comparison of the control policies: wealth case with bounded control ( $p \in [0, 1.5]$ ). Parameters are given in Table 2. We fix  $std_{t=0,w}^{p^*}[W(T)] \simeq 8.17$  for this test. More precisely, from our finite difference solutions,  $(Std_{t=0,w}^{p^*}[W(T)], E_{t=0,w}^{p^*}[W(T)]) = (8.17479, 12.7177)$  for the mean quadratic variation strategy;  $(Std_{t=0,w}^{p^*}[W(T)], E_{t=0,w}^{p^*}[W(T)]) = (8.17494, 12.6612)$  for the time-consistent strategy; and  $(Std_{t=0,w}^{p^*}[W(T)], E_{t=0,w}^{p^*}[W(T)]) = (8.17453, 12.8326)$  for the pre-commitment strategy. Figure (a) shows the control policies at  $t = 0^+$ ; Figure (b) shows the mean of the control policies versus time  $t \in [0, T]$ .

## 488 7.2 Wealth-to-income Ratio Case

489 Figure 11 and 12 shows a comparison for the three strategies for the wealth-to-income ratio case.  
490 Figure 11 is for bankruptcy case, Figure 12 (a) is for no bankruptcy case, and Figure 12 (b)  
491 is for the bounded control case. Similar to the allowing bankruptcy case, the pre-commitment  
492 strategy dominates the other strategies. Note that unlike the wealth case, the frontiers for the

493 three strategies do not have the common lower end point. As discussed in Section 6.3, no risk free  
 494 strategy exists in this case because of the salary risk. Furthermore, since the salary is correlated  
 495 with the stock index ( $\sigma_{Y_1} \neq 0$ ), in order to (partially) hedge the salary risk, the most conservative  
 496 policy is not to invest all money in the bond ( $p = 0$ ) all the time. The three strategies have different  
 497 views of risk, hence their most conservative investment policies would be different. Therefore, their  
 498 minimum risks (in terms of standard deviation) are different. Also note that, the frontiers given  
 499 by the time-consistent strategy and the mean quadratic variation strategy are very close, almost  
 500 on top of each other.

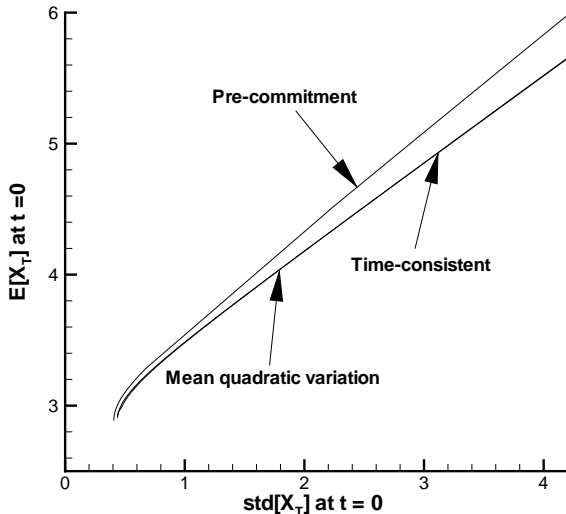


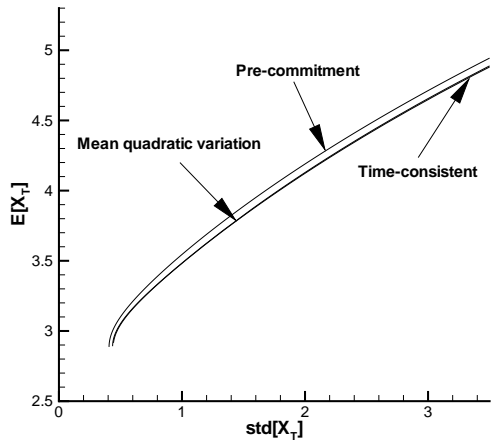
FIGURE 11: Comparison of three strategies: wealth-to-income ratio case, allowing Bankruptcy. Parameters are given in Table 5.

501 Similar to the wealth case, Figure 13 shows a comparison of the control policies for the three  
 502 strategies. Parameters are given in Table 5, and we use wealth case with bounded control ( $p \in$   
 503  $[0, 1.5]$ ). We fix  $\text{Std}_{t=0,x}^{p^*}[X(T)] \simeq 3.24$  for this test. The comparison shows that although the three  
 504 strategies have a similar pair of expected value and standard deviation, the control policies are  
 505 significantly different.

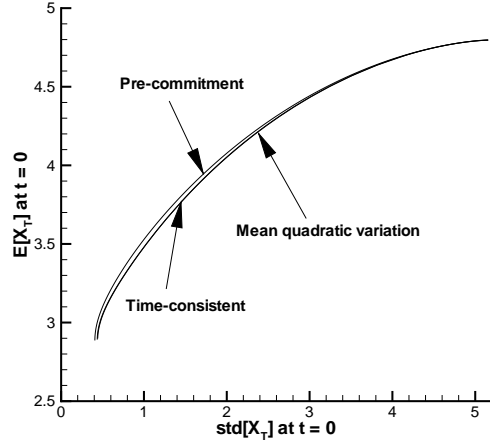
506 **Remark 7.2 (Average strategy)** From Remark 6.2, we note that if the wealth-to-income ratio is  
 507 static, the optimal strategy (under the mean-quadratic-variation criteria) is to increase the weight in  
 508 the risky asset. This is also observed for the pre-commitment and time consistent policies [29, 30].  
 509 Nevertheless, for all three optimal strategies, the mean optimal policy is to decrease the weight in  
 510 the risky asset as  $t \rightarrow T$ .

## 511 8 Conclusion

512 In this article, we consider three mean variance like strategies: a pre-commitment strategy, a  
 513 time-consistent strategy (as defined in [5]) and a mean quadratic variation strategy. Although the

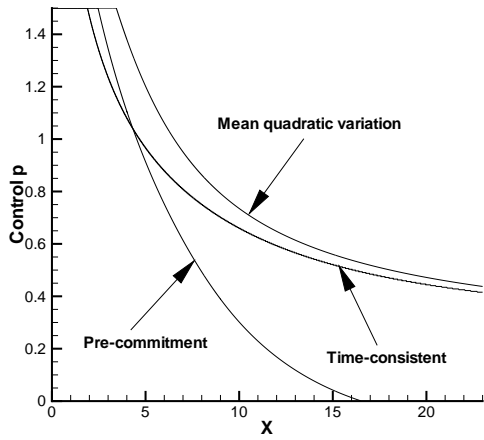


(a) No Bankruptcy

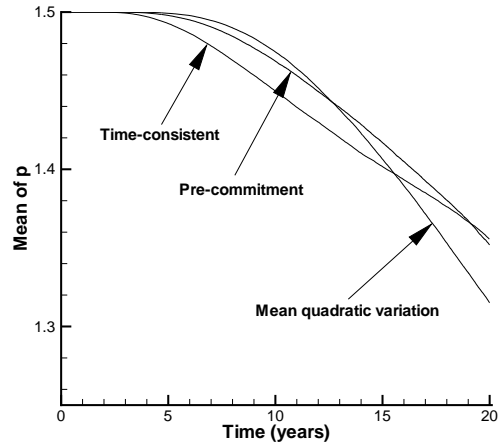


(b) Bounded Control

FIGURE 12: Comparison of the three strategies: wealth-to-income ratio case. (a): no bankruptcy case; (b): bounded control case. Parameters are given in Table 5.



(a)



(b)

FIGURE 13: Comparison of the control policies: wealth-to-income ratio case with bounded control ( $p \in [0, 1.5]$ ). Parameters are given in Table 5. We fix  $std_{t=0,x}^{p^*}[X(T)] \simeq 3.24$  for this test. More precisely, from our finite difference solutions,  $(Std_{t=0,x}^{p^*}[X(T)], E_{t=0,x}^{p^*}[X(T)]) = (3.24214, 4.50255)$  for the mean quadratic variation strategy;  $(Std_{t=0,x}^{p^*}[X(T)], E_{t=0,x}^{p^*}[X(T)]) = (3.24348, 4.50168)$  for the time-consistent strategy; and  $(Std_{t=0,x}^{p^*}[X(T)], E_{t=0,x}^{p^*}[X(T)]) = (3.24165, 4.50984)$  for the pre-commitment strategy. Figure (a) shows the control policies at  $t = 0^+$ ; Figure (b) shows the mean of the control policies versus time  $t \in [0, T]$ .

514 pre-commitment strategy dominates the other strategies, in terms of an efficient frontier solution,  
 515 it is not time-consistent.

516 In practice, many investors may choose a time-consistent strategy. However, for both pre-  
 517 commitment and time-consistent strategies, the risk is only measured in terms of the standard  
 518 deviation at the end of trading. Practitioners might prefer to control the risk during the whole  
 519 investment period [16]. The mean quadratic variation strategy controls this risk.

520 In this paper, we consider two cases for a pension plan investment strategy: the wealth case and  
 521 the wealth-to-income ratio case. We study three types of constraints on the strategy: the allowing  
 522 bankruptcy case, a no bankruptcy case, and a bounded control case.

523 We have implemented numerical schemes for the pre-commitment strategy and the time-consistent  
 524 strategy in [29, 30]. In this paper, we extend the method in [29] to solve for the optimal strategy  
 525 for the mean quadratic variation problem. The equation for the value function is in the form of a  
 526 nonlinear HJB PDE. We use a fully implicit method to solve the nonlinear HJB PDE. It can be  
 527 shown that our numerical scheme converges to the viscosity solution. Numerical examples confirm  
 528 that our method converges to the analytic solution where available.

529 We carry out a comparison of the three mean variance like strategies. For the allowing  
 530 bankruptcy case, analytic solutions exist for all strategies. Furthermore, the time-consistent strat-  
 531 egy and the mean quadratic variation strategy have the same solution. However, when additional  
 532 constraints are applied to the control policy, analytic solutions do not exist in general.

533 After realistic constraints are applied, the frontiers for all three strategies are very similar. In  
 534 particular, the mean quadratic variation strategy and the time consistent mean variance strategy  
 535 (with constraints) produce very similar frontiers. However, the investment policies are quite differ-  
 536 ent, for all three strategies. This suggests that the choice among various strategies cannot be made  
 537 by only examining the efficient frontier, but rather should be based on the qualitative behavior of  
 538 the optimal policies.

## 539 A Discrete Equation Coefficients

540 Let  $p_i^n$  denote the optimal control  $p^*$  at node  $i$ , time level  $n$  and set

$$a_i^{n+1} = a(z_i, p_i^n), \quad b_i^{n+1} = b(z_i, p_i^n), \quad c_i^{n+1} = c(z_i, p_i^n). \quad (\text{A.1})$$

541 Then, we can use central, forward or backward differencing at any node.

542 Central Differencing:

$$\begin{aligned} \alpha_{i,central}^n &= \left[ \frac{2a_i^n}{(z_i - z_{i-1})(z_{i+1} - z_{i-1})} - \frac{b_i^n}{z_{i+1} - z_{i-1}} \right] \\ \beta_{i,central}^n &= \left[ \frac{2a_i^n}{(z_{i+1} - z_i)(z_{i+1} - z_{i-1})} + \frac{b_i^n}{z_{i+1} - z_{i-1}} \right]. \end{aligned} \quad (\text{A.2})$$

543 Forward/backward Differencing: ( $b_i^n > 0$  /  $b_i^n < 0$ )

$$\begin{aligned} \alpha_{i,forward/backward}^n &= \left[ \frac{2a_i^n}{(z_i - z_{i-1})(z_{i+1} - z_{i-1})} + \max\left(0, \frac{-b_i^n}{z_i - z_{i-1}}\right) \right] \\ \beta_{i,forward/backward}^n &= \left[ \frac{2a_i^n}{(z_{i+1} - z_i)(z_{i+1} - z_{i-1})} + \max\left(0, \frac{b_i^n}{z_{i+1} - z_i}\right) \right]. \end{aligned} \quad (\text{A.3})$$

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