CS340 Notes, Lecture 9A

Solving Linear Homogeneous Recurrence Relations

Assume we have 1 pair of rabbits to begin with. Each month, the female of a pair gives birth to a new pair of rabbits of opposite sex. Beginning with the second month, each new pair also gives birth to a pair of rabbits eah month. Fibonacci number $f(n)$ is the number of pairs of rabbits at the beginning of month n. It can be expressed as

$$f(n) = f(n-1) + f(n-2)$$

This is *linear*: if $s(n)$ is a solution, i.e.

$$s(n) = s(n-1) + s(n-2)$$

then $cs(n)$ is also a solution since

$$cs(n) = cs(n-1) + cs(n-2).$$

And if $s_1(n)$ and $s_2(n)$ are solutions, then so is $s_1(n) + s_2(n)$, since from

$$\begin{aligned} s_1(n) &= s_1(n-1) + s_1(n-2) \\ s_2(n) &= s_2(n-1) + s_2(n-2) \end{aligned}$$

We conclude

$$\begin{aligned} s_1(n) + s_2(n) = & \quad [s_1(n-1) + s_2(n-1)] \\ & + \quad [s_1(n-2) + s_2(n-2)] \end{aligned}$$

In fact, for any c_1, c_2, $c_1 s_1(n) + c_2 s_2(n)$ is a solution.

In order to solve this kind of recurrence relations, we need to find two special solutions. One way is to assume $f(n) = x^n$, $x \neq 0$. Then we have:

$$x^n = x^{n-1} + x^{n-2}$$

Since $x \neq 0$, we just need to solve:

$$x^2 - x - 1 = 0$$

to get $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$.

Thus both

$$f(n) = (\frac{1+\sqrt{5}}{2})^n$$

and

$$f(n) = (\frac{1-\sqrt{5}}{2})^n$$

are solutions of $f(n) = f(n-1) + f(n-2)$.

Thus for any c_1, c_2,

$$f(n) = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$$

is a solution to $f(n) = f(n-1) + f(n-2)$ in general. But since we have boundary conditions $f(0) = 1$ and $f(1) = 1$, we can solve for c_1 and c_2, using the following two equations:

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1\frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2} &= 1 \end{aligned}$$

This gives $c_1 = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2}$ and $c_2 = \frac{-1}{\sqrt{5}}\frac{1-\sqrt{5}}{2}$. Thus the final solution is:

$$f(n) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} + \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}.$$