Filter Banks and Basis Functions

Filter Banks: Linear filtering of $\vec{s}$ is provided by

$$\vec{a} = F^T \vec{s}, \quad \text{with } F \in \Re^{n^2 \times m}.$$

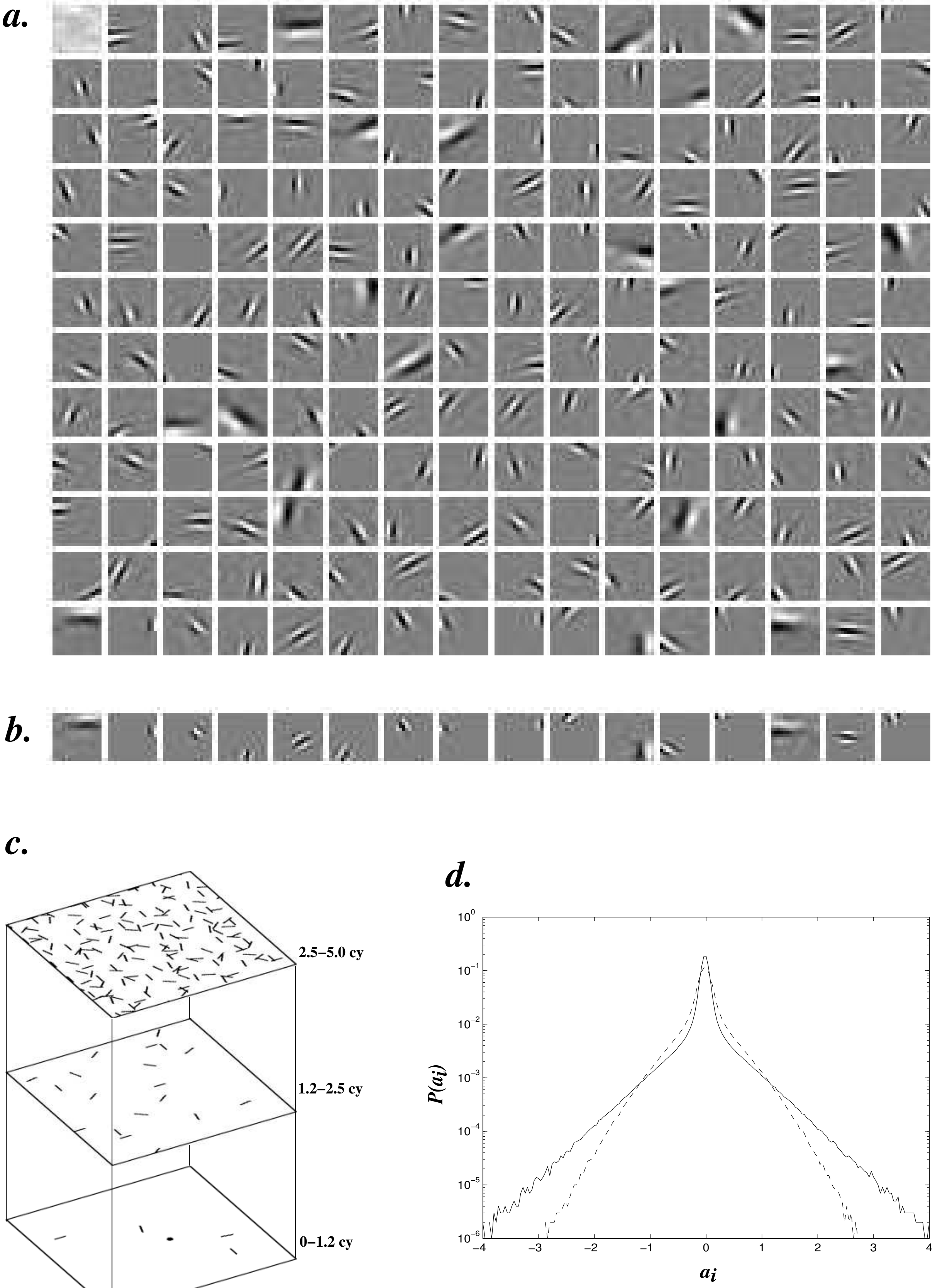
Here:

- The sampled $n \times n$ image $s(\vec{x})$ is taken to be equivalent to the n^2-dimensional column vector, $\vec{s}$, with the elements arranged in the standard raster order.
- The k^{th} element, a_k, is the *filter response*, $\vec{f}_k \cdot \vec{s}$.
- $\vec{f}_k$ defines one location for some convolution kernel.
- m is the number of filter response samples. Typically $m \geq n^2$.

Basis Functions: An expansion of $\vec{s}$ in the basis $\{b_k\}_{k=1}^m$ is given by

$$\vec{s} = B\vec{a}, \quad \text{with } B \in \Re^{n^2 \times m}.$$

- The k^{th} column of B is the *basis function (or vector)* $\vec{b}_k$.
- Typically $m \geq n^2$, although 'view-based' models use $m < n^2$.


a.
b.
c.
2.5–5.0 cy
1.2–2.5 cy
0–1.2 cy
d.
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
P(a_i)
−4 −3 −2 −1 0 1 2 3 4
a_i

Nature's Choice

Filter bank:

- Self-similar copies of a 'mother wavelet'.
- Filters of various scales, positions, and orientations.
- Mother wavelet is roughly a Gabor:

$$Gabox(\vec{x}; \vec{k}, \sigma) = e^{i\vec{k}\cdot\vec{x}} e^{-(x^2+y^2)/(2\sigma^2)}$$

  i.e. a sinsusoidal grating times a Gaussian window.

- The grating has approximately 1.5 cycles across the 2σ window (the filter has roughly a one octave bandwidth).
- An independent component analysis of the ensemble of natural images leads to the above properties [Olshausen and Field, '96].
- Filters appear in quadrature pairs (roughly, even and odd symmetries, but see below).

Self-Inverting Bases

Defn: A filter bank (or, equivalently, a basis set) B is said to be *self-inverting* if $BB^T = rI$. Here $r \in \Re$ is the *redundancy*.

Defn: A filter bank (or, equivalently, a basis set) B is said to be *critically-sampled* if B is a square matrix (i.e. $m = n^2$). B is said to be *over-sampled* for $m > n^2$, and *under-sampled*) for $m < n^2$.

It is possible to have an over-sampled, self-inverting filter bank.

Defn: (From linear algebra.) B is an *orthogonal matrix* if and only if $B^{-1} = B^T$ (in particular, B must be a square matrix).

For self-inverting bases the expansion coefficients can be easily computed. In particular, we require

$$\vec{s} = \frac{1}{r} B\vec{a}.$$

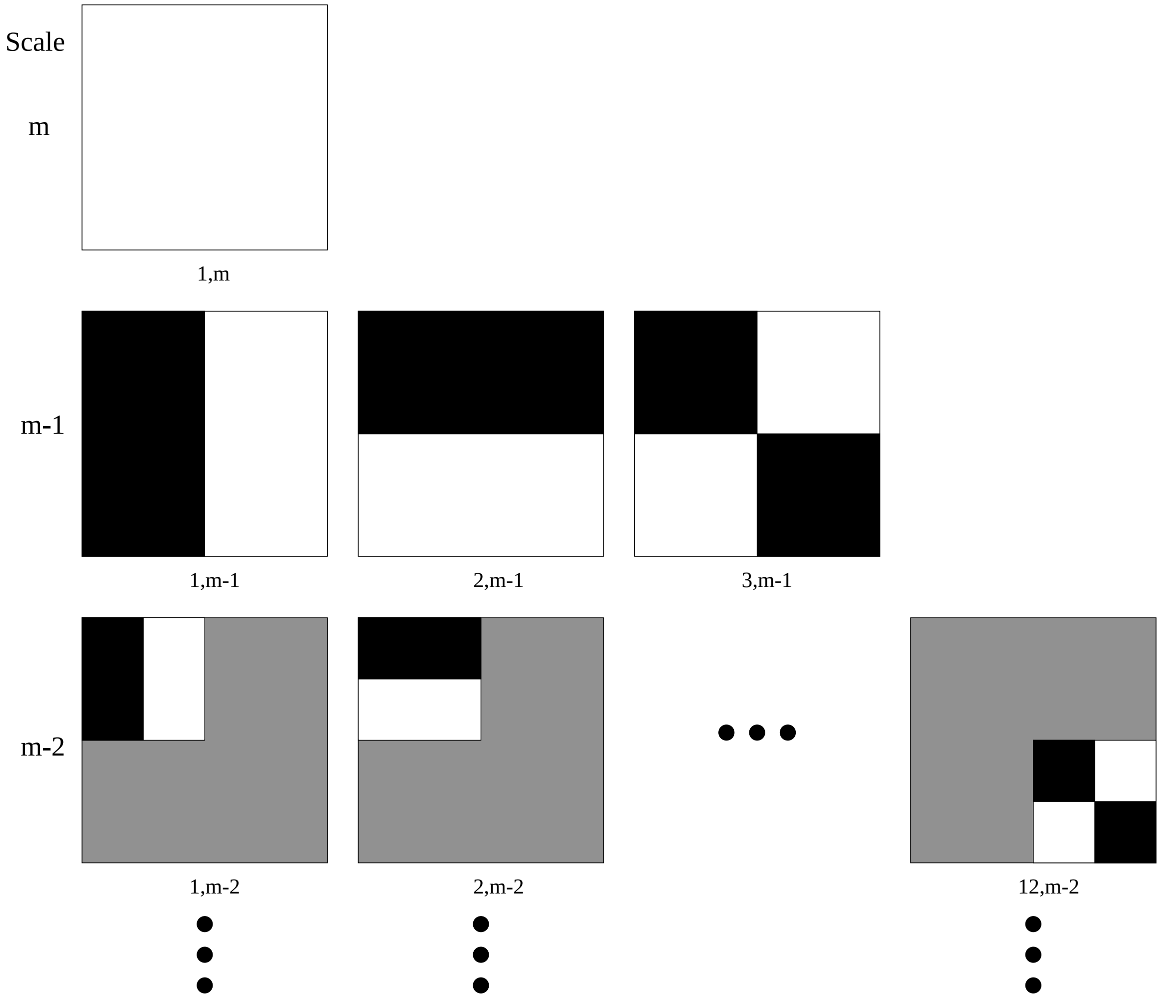
Since $BB^T = rI$ we can choose $\vec{a}$ using the filter bank B^T,

$$\vec{a} = B^T\vec{s}.$$

The coefficients are not unique for the corresponding basis expansions unless the basis is critically (or under) sampled.

Simple Wavelet Example: Haar Basis

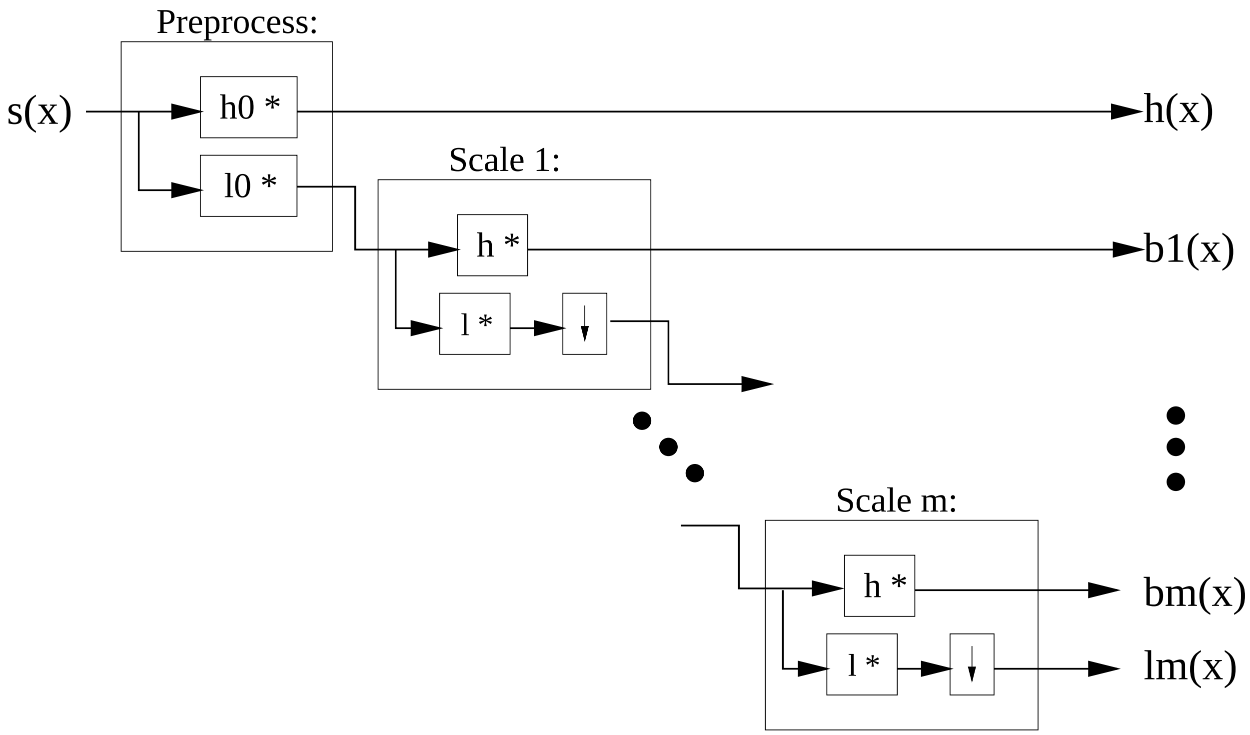
Suppose $n = 2^k$. The basis functions, $\vec{b}_{i,l}$, are defined recursively:



- White denotes +1, black -1, and grey 0.
- Basis functions $\vec{b}_{i,l}$ are mutually orthogonal.
- Subbands are aliased (difference-of-boxes filters).

Self-Inverting Multi-scale Pyramid

Analysis Pyramid $\vec{a} = B^T \vec{s}$



- $h_0(\vec{x})$, $h(\vec{x})$ are high-pass radially-symmetric filter kernels, with Fourier transforms $H_0(\vec{k})$, $H(\vec{k})$,
- $l_0(\vec{x})$, $l(\vec{x})$ are radially-symmetric low-pass kernels, with Fourier transforms $L_0(\vec{k})$, $L(\vec{k})$,
- each pair $(h_0(\vec{x}), l_0(\vec{x})$ and $(h(\vec{x}), l(\vec{x}))$ partitions the Fourier power spectrum:

$$|H_0(\vec{k})|^2 + |L_0(\vec{k})|^2 = 1,$$

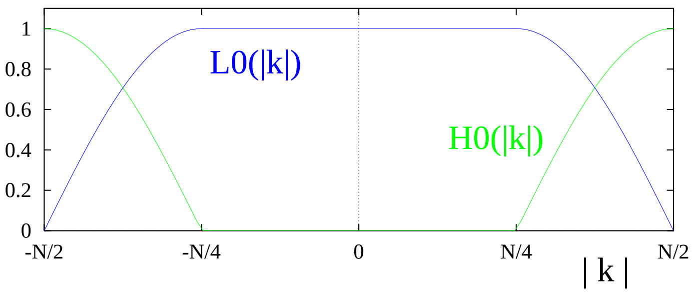
$$|H(\vec{k})|^2 + |L(\vec{k})|^2 = 1.$$

This ensures the pyramid has a simple inverse cascade.

Choice of Cascade Filters

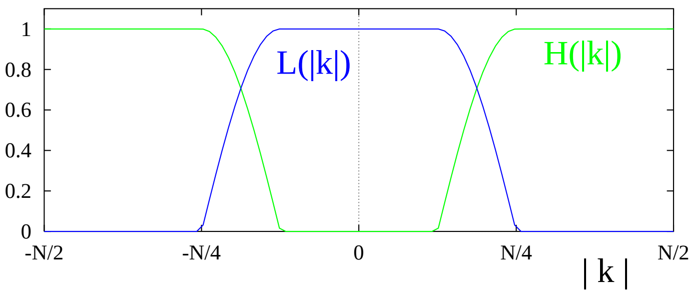
$$H_0(|\vec{k}|) = \begin{cases} cos(\pi - |k|), & \text{for } |k| \in [\pi/2, \pi], \\ 0, & \text{otherwise.} \end{cases}$$

$$L_0(|\vec{k}|) = \begin{cases} sin(\pi - |k|), & \text{for } |k| \in [\pi/2, \pi], \\ 1, & \text{otherwise.} \end{cases}$$

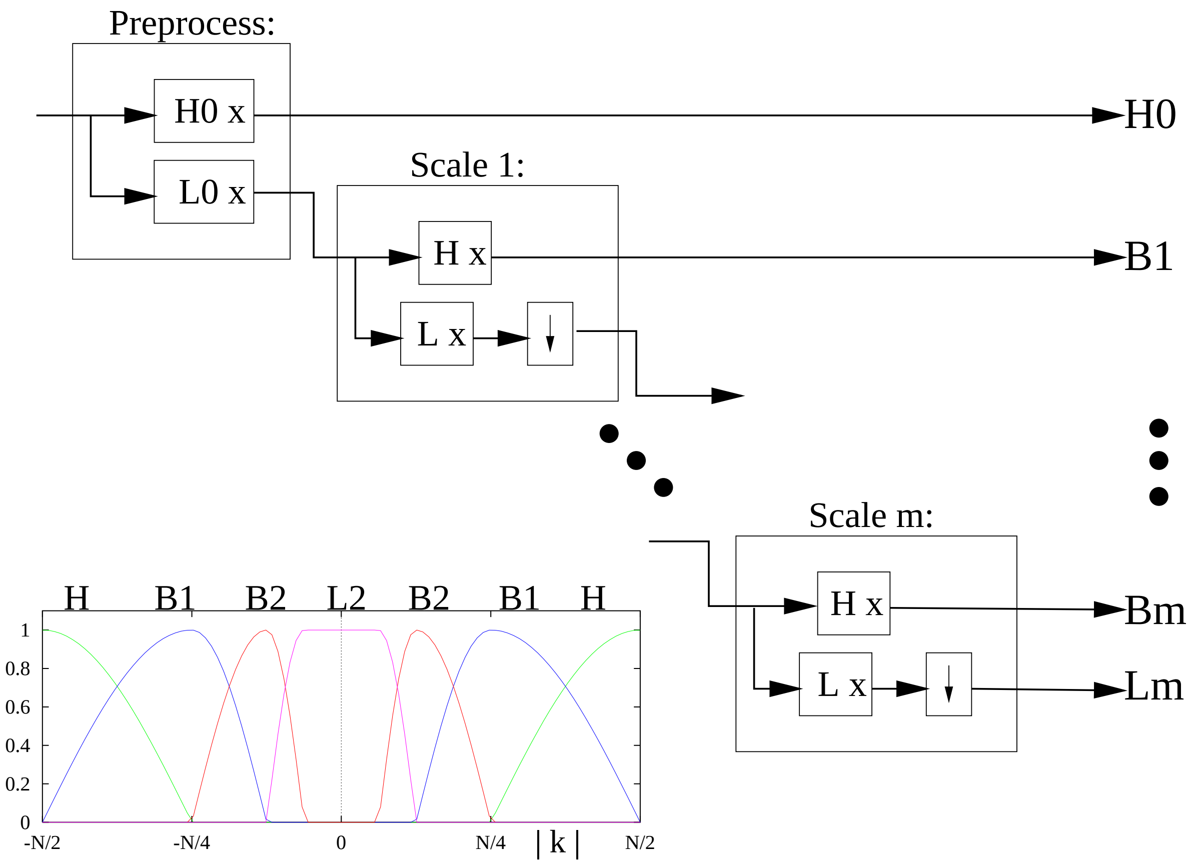


$$H(|\vec{k}|) = \begin{cases} 1, & \text{for } |k| > \pi/2], \\ cos(\pi - 2|k|), & \text{for } |k| \in [\pi/4, \pi/2], \\ 0, & \text{otherwise.} \end{cases}$$

$$L(|\vec{k}|) = \begin{cases} 0, & \text{for } |k| \in [\pi/2, \pi], \\ sin(\pi - 2|k|), & \text{for } |k| \in [\pi/4, \pi/2], \\ 1, & \text{otherwise.} \end{cases}$$



Analysis Pyramid in the Fourier Domain



By the Convolution Theorem,

$$B_1(\vec{k}) = L_0(\vec{k})H(\vec{k})$$

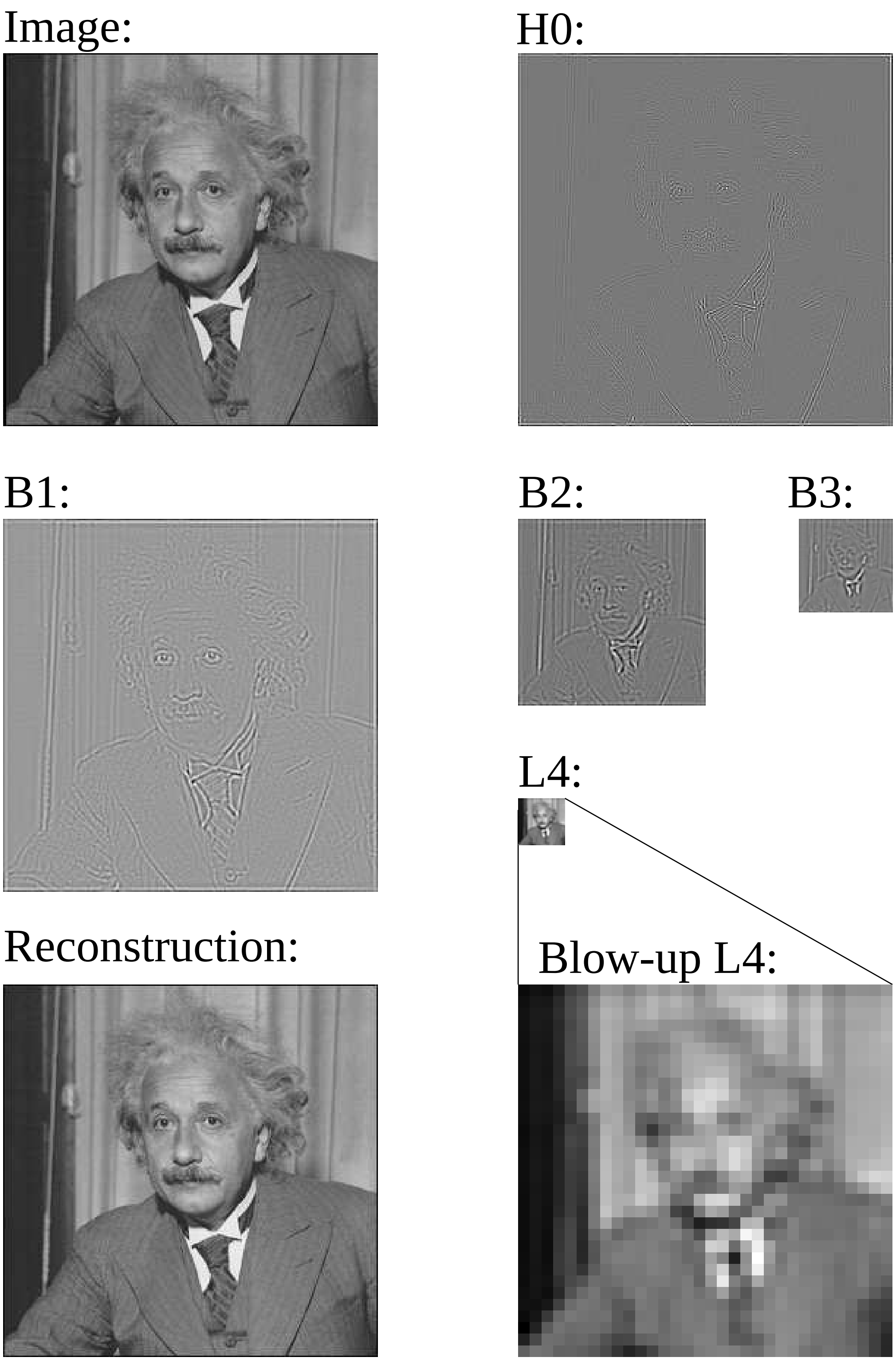
$$B_2(\vec{k}) = L_0(\vec{k})L(\vec{k})H(2\vec{k})$$

$$\vdots \quad \vdots$$

$$B_m(\vec{k}) = L_0(\vec{k})L(\vec{k})\ldots L(2^{m-2}\vec{k})H(2^{m-1}\vec{k})$$

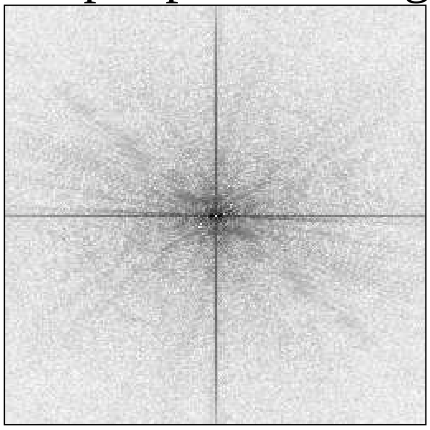
$$L_m(\vec{k}) = L_0(\vec{k})L(\vec{k})\ldots L(2^{m-2}\vec{k})L(2^{m-1}\vec{k})$$

Application of the Analysis Pyramid

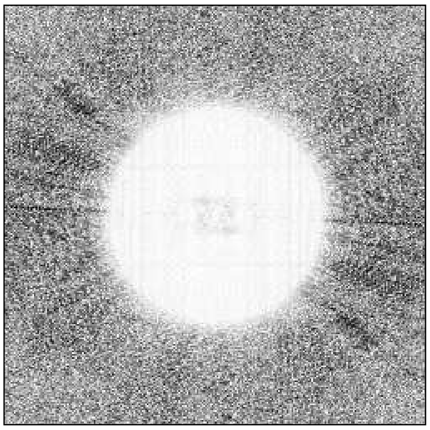


Analysis Pyramid: Fourier Representation

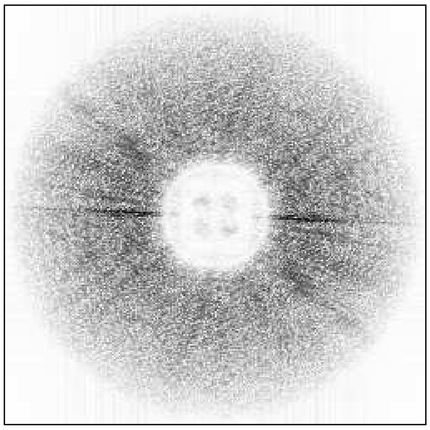
Amp. Spec. of Image



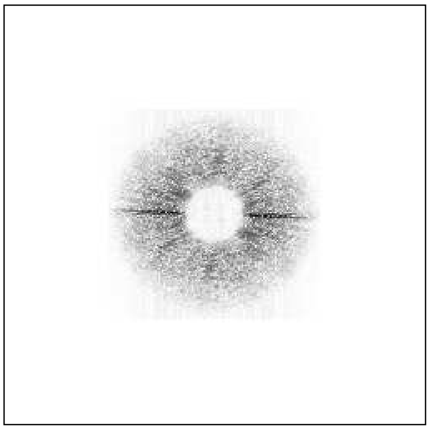
AS H0:



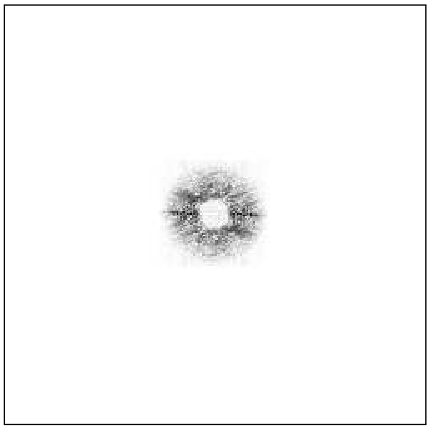
AS B1:



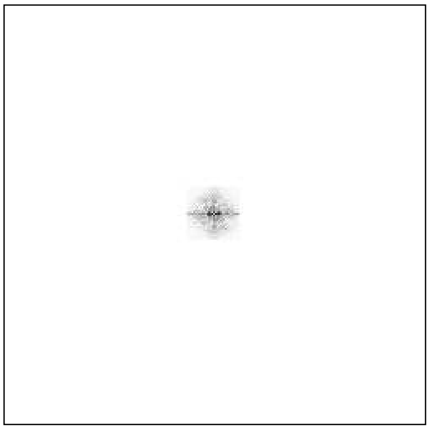
AS B2:



AS B3:



AS L4:



Orientation-Tuned Subbands

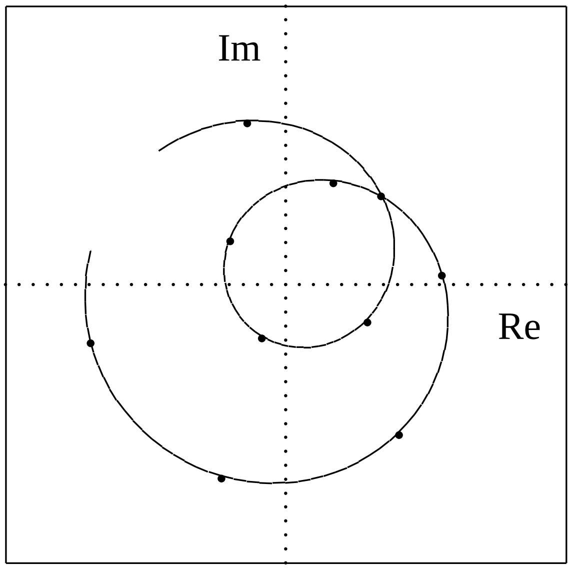
- Additional filter cascades can divide each scale into different spatial orientations.

- Obtain Gabor-like filters using a recursive cascade.

- Can arrange for quadrature pairs.
- Continuously steerable in orientation [Freeman et al, '91].
- Continuously shiftable in space [Simoncelli et al, '92].

Quadrature Pairs

Filtering: Let $g(x) = G * I(x)$ and $h(x) = H * I(x)$, with $G(x)$ and $H(x)$ a quadrature pair (i.e. $\mathcal{G}(k) = \text{sign}(k)\mathcal{H}(k)$).



Complex Polar Representation:

$$\begin{aligned}(G(x) + iH(x)) * I =& g(x) + ih(x)\\ =& e^{i\phi(x)}A(x).\end{aligned}$$

For a bandpass $G(x)$ and $H(x)$:

- **Local Energy:** $A^2(x)$ is a low-pass (slowly varying) signal.
- **Local Phase:** $\phi(x) \approx \phi_0 + k_0(x - x_0)$ provides a roughly linear progression.

Example: G2 and H2

An approximate quadrature pair is given by [Freeman et al, '91]:

$$G2(x, y; \sigma) = [0.9213((x/\sigma)^2 - 1)]e^{-(x^2+y^2)/(2\sigma^2)},$$

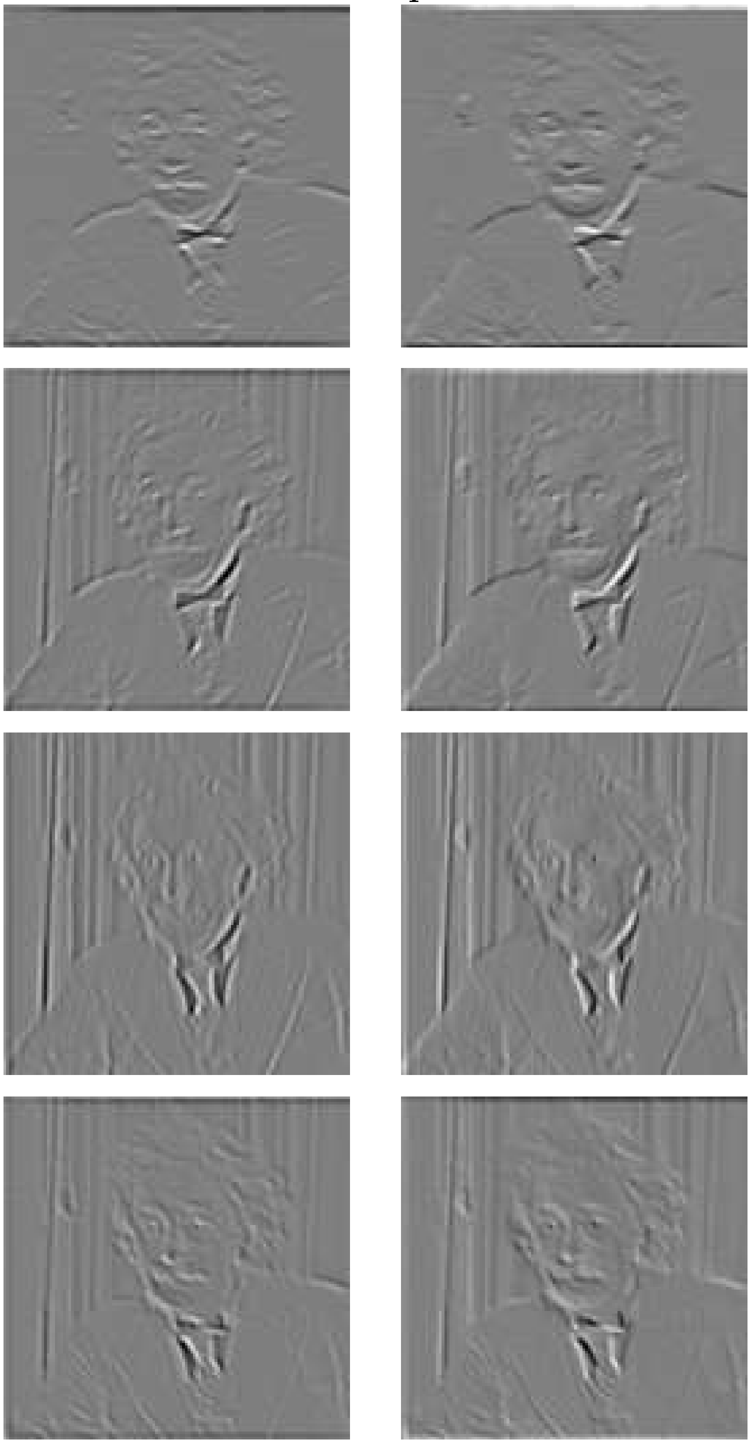
$$H2(x, y; \sigma) = [(x/\sigma)(0.3458(x/\sigma)^2 - 1.559)]e^{-(x^2+y^2)/(2\sigma^2)},$$

Here $G2(x, y; \sigma) \propto \frac{\partial^2}{\partial x^2} e^{-(x^2+y^2)/(2\sigma^2)}$.

An image pyramid using G2 and H2.

- Define the scale using the wavelength, λ, for the most sensitive frequency in the fourier transform. Use $\sigma = \lambda/3.918$.
- Use 4 scales, with wavelengths (say) $\lambda = 4, 8, 16$, and 32 pixels.
- Spatially rotate $G2(x, y)$ and $H2(x, y)$ by 0, 45, 90, 135 degrees.
- Sample each pair at the rate $\lambda/4$ in each direction.
- Efficient implementation uses separable filters.
- The orientation of the filters is continuously steerable from 7 real-valued basis filters (or 4 sampled orientations).

G2 $\lambda = 8.0$ pixels H2



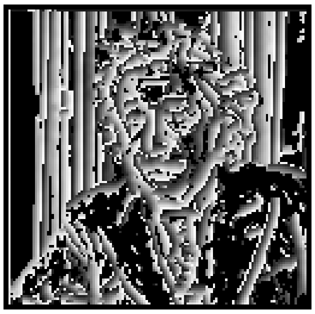
128×128 images

G2 $\lambda = 16.0$ pixels H2

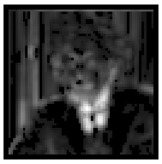
64×64 images

Amp $\lambda = 4.0$ pixels Phase

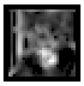

$\lambda = 8.0$ pixels

$\lambda = 16.0$ pixels




$\lambda = 32.0$ pixels




References:

1. C.S. Burrus, R.A. Gopinath, and H. Guo, Introduction to Wavelets and Wavelet Transforms, Prentice Hall.

2. W T Freeman and E H Adelson. The design and use of steerable filters. IEEE Pat. Anal. Mach. Intell., 13(9), 1991, pp.891-906.

3. Olshausen BA, Field DJ (1996). Emergence of simple-cell receptive field properties by learning a sparse code for natural images. Nature, 381: 607-609.
   `URL: /redwood.ucdavis.edu/bruno/papers.html`

4. E P Simoncelli, W T Freeman, E H Adelson, and D J Heeger. Shiftable multi-scale transforms, IEEE Trans. Information Theory, 38(2), 1992, pp.587-607.