# Efficient Enumeration of Regular Languages

Margareta Ackerman     Jeffrey Shallit

School of Computer Science, University of Waterloo

July 5, 2007

# Cross-Section Enumeration Problem

- We represent regular languages via nondeterministic finite automata (NFA)
- The $n^{th}$ cross-section of a language is $L$ is $L \cap \Sigma^n$.

# Cross-Section Enumeration Problem

- We represent regular languages via nondeterministic finite automata (NFA)
- The $n^{th}$ cross-section of a language is $L$ is $L \cap \Sigma^n$.

## Definition

Cross-section Enumeration Problem: List all words in a regular language of length $n$ in lexicographical order.

# Cross-Section Enumeration Problem

- We represent regular languages via nondeterministic finite automata (NFA)
- The $n^{th}$ cross-section of a language is $L$ is $L \cap \Sigma^n$.

## Definition

Cross-section Enumeration Problem: List all words in a regular language of length $n$ in lexicographical order.

## Example

$L(N) = (0+1)^*$
The $2^{nd}$-cross-section is: $00, 01, 10, 11$.

# Cross-Section Enumeration Problem

- We represent regular languages via nondeterministic finite automata (NFA)
- The $n^{th}$ cross-section of a language is $L$ is $L \cap \Sigma^n$.

## Definition

Cross-section Enumeration Problem: List all words in a regular language of length $n$ in lexicographical order.

## Example

$L(N) = (0+1)^*$
The $2^{nd}$-cross-section is: $00, 01, 10, 11$.

## Example

$L(N) = \{w \in \{0,1\}^* \mid |w|_1 \equiv 0 \mod 3\}$
The $4^{th}$-cross-section is: $0000, 0111, 1011, 1101, 1110$.

# Enumeration Problem

## Definition

Given words $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m$, $u < v$ according to *radix order* if $n < m$ or if $n = m$, $u \neq v$, and $u_i < v_i$ for the minimal $i$ where $u_i \neq v_i$.

# Enumeration Problem

## Definition

Given words $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m$, $u < v$ according to *radix order* if $n < m$ or if $n = m$, $u \neq v$, and $u_i < v_i$ for the minimal $i$ where $u_i \neq v_i$.

## Definition

Enumeration Problem: List the first $m$ words in a regular language in radix order.

# Enumeration Problem

## Definition

Given words $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m$, $u < v$ according to *radix order* if $n < m$ or if $n = m$, $u \neq v$, and $u_i < v_i$ for the minimal $i$ where $u_i \neq v_i$.

## Definition

Enumeration Problem: List the first $m$ words in a regular language in radix order.

## Example

$L(N) = (0 + 1)^*$
Enumerate the first 5 words in $L(N)$: $\epsilon, 0, 1, 00, 01$.

# Enumeration Problem

## Definition

Given words $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m$, $u < v$ according to *radix order* if $n < m$ or if $n = m$, $u \neq v$, and $u_i < v_i$ for the minimal $i$ where $u_i \neq v_i$.

## Definition

Enumeration Problem: List the first $m$ words in a regular language in radix order.

## Example

$L(N) = (0+1)^*$
Enumerate the first 5 words in $L(N)$: $\epsilon, 0, 1, 00, 01$.

## Example

$L(N) = \{w \in \{0,1\}^* \mid |w|_1 \equiv 0 \mod 3\}$
Enumerate the first 10 words in $L(N)$:
$\epsilon, 0, 00, 000, 111, 0000, 0111, 1011, 1101, 1110$.

# Applications

- Correctness testing, provide evidence that an NFA generates the expected language.

# Applications

- Correctness testing, provide evidence that an NFA generates the expected language.
- Ex. Verify that an NFA has the property that the least word not accepted is exponential in length (construction by Shallit).

# Applications

- Correctness testing, provide evidence that an NFA generates the expected language.
- Ex. Verify that an NFA has the property that the least word not accepted is exponential in length (construction by Shallit).
- Test if two NFAs accept the same language.

# Applications

- Correctness testing, provide evidence that an NFA generates the expected language.
- Ex. Verify that an NFA has the property that the least word not accepted is exponential in length (construction by Shallit).
- Test if two NFAs accept the same language.
- If sufficiently many words are generated, then we can verify if two NFAs accept the same language (Conway, 1971).

# Applications

Given an NFA on $s$ states, decide if every word it accepts is a power (a string of the form $x^n, |x| \geq 1, n \geq 2$).

- If every word is a power, then the NFA accepts no more than $7s$ words of each length, and further, if it accepts a non-power, it must accept a non-power of length $< 3s$ (Anderson, Rampersad, Santean, and Shallit.)

# Applications

Given an NFA on $s$ states, decide if every word it accepts is a power (a string of the form $x^n, |x| \geq 1, n \geq 2$).

- If every word is a power, then the NFA accepts no more than $7s$ words of each length, and further, if it accepts a non-power, it must accept a non-power of length $< 3s$ (Anderson, Rampersad, Santean, and Shallit.)
- We get an efficient algorithm by enumerating all the words of length $1, 2, \ldots, 3s - 1$ and testing if each is a power, stopping if the length of any cross-section exceeds $7s$.

# Previous Work - Grail

- Grail+ 3.0, a symbolic computation environment, implements an enumeration algorithm under the function `fmenum`.

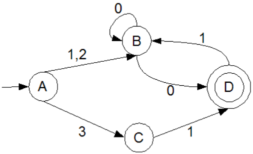
# Previous Work - Grail

- Grail+ 3.0, a symbolic computation environment, implements an enumeration algorithm under the function `fmenum`.
- The function `fmenum` performs Breadth-First-Search on the tree of paths that can be traversed on an NFA.

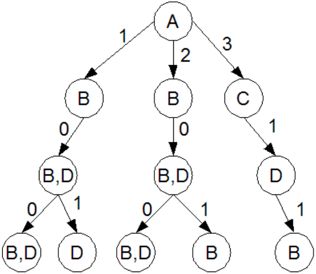
# Previous Work - Grail

- Grail+ 3.0, a symbolic computation environment, implements an enumeration algorithm under the function `fmenum`.
- The function `fmenum` performs Breadth-First-Search on the tree of paths that can be traversed on an NFA.

Enumerate the $3^{rd}$ cross-section of the following NFA.



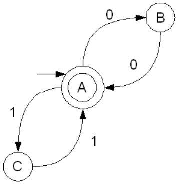

(e) NFA




(f) Illustration of Grail algorithm

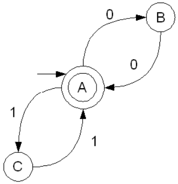
# Previous Work - Grail

- Algorithm may do exponential work for empty output. For the NFA below, it will take $\Theta(n2^{n/2})$ operations to enumerate the $n^{th}$-cross-section, where $n$ is odd.



# Previous Work - Grail

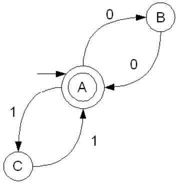
- Algorithm may do exponential work for empty output. For the NFA below, it will take $\Theta(n2^{n/2})$ operations to enumerate the $n^{th}$-cross-section, where $n$ is odd.




- Takes $O(s^2\sigma^{n+1})$ operations to enumerate the $n^{th}$ cross-section.
  $s$: number of states. $\sigma$: alphabet size.

# Previous Work - Grail

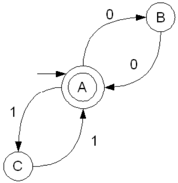
- Algorithm may do exponential work for empty output. For the NFA below, it will take $\Theta(n2^{n/2})$ operations to enumerate the $n^{th}$-cross-section, where $n$ is odd.




- Takes $O(s^2\sigma^{n+1})$ operations to enumerate the $n^{th}$ cross-section.
  $s$: number of states. $\sigma$: alphabet size.
- Takes $O(s^2\sigma^{k+1})$ operations to enumerate the first $m$ words accepted by the NFA.
  $k$: length of words in last cross-section examined.

# Previous Work - Grail

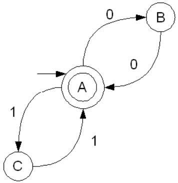
- Algorithm may do exponential work for empty output. For the NFA below, it will take $\Theta(n2^{n/2})$ operations to enumerate the $n^{th}$-cross-section, where $n$ is odd.




- Takes $O(s^2\sigma^{n+1})$ operations to enumerate the $n^{th}$ cross-section.
  $s$: number of states. $\sigma$: alphabet size.
- Takes $O(s^2\sigma^{k+1})$ operations to enumerate the first $m$ words accepted by the NFA.
  $k$: length of words in last cross-section examined.
- Works well in practice for small input sizes.

# Previous Work - Grail

- Algorithm may do exponential work for empty output. For the NFA below, it will take $\Theta(n2^{n/2})$ operations to enumerate the $n^{th}$-cross-section, where $n$ is odd.




- Takes $O(s^2\sigma^{n+1})$ operations to enumerate the $n^{th}$ cross-section.
  $s$: number of states. $\sigma$: alphabet size.
- Takes $O(s^2\sigma^{k+1})$ operations to enumerate the first $m$ words accepted by the NFA.
  $k$: length of words in last cross-section examined.
- Works well in practice for small input sizes.
- We found a few bugs in Grail. Words are not always output in lexicographical order and for some NFAs words are missing from the enumeration.

# Previous Work - Mäkinen's algorithm

- Mäkinen finds an efficient algorithm in the unit-cost model ("On lexicographic enumeration of regular and context-free languages." *Acta Cybernetica*, **13** (1997), 55–61.)

# Previous Work - Mäkinen's algorithm

- Mäkinen finds an efficient algorithm in the unit-cost model ("On lexicographic enumeration of regular and context-free languages." *Acta Cybernetica*, **13** (1997), 55–61.)
- Mäkinen's algorithm uses dynamic programming. It finds the minimal words of length 1 through $n$ starting at each state.

# Previous Work - Mäkinen's algorithm

- Mäkinen finds an efficient algorithm in the unit-cost model ("On lexicographic enumeration of regular and context-free languages." *Acta Cybernetica*, **13** (1997), 55–61.)
- Mäkinen's algorithm uses dynamic programming. It finds the minimal words of length 1 through $n$ starting at each state.
- Originally analyzed in the unit-cost model.

# Previous Work - Mäkinen's algorithm

- Mäkinen finds an efficient algorithm in the unit-cost model ("On lexicographic enumeration of regular and context-free languages." *Acta Cybernetica*, **13** (1997), 55–61.)
- Mäkinen's algorithm uses dynamic programming. It finds the minimal words of length 1 through $n$ starting at each state.
- Originally analyzed in the unit-cost model.
- We analyze its running time in the bit-complexity model and modify the algorithm.

# Previous Work - Mäkinen's algorithm

- Mäkinen finds an efficient algorithm in the unit-cost model ("On lexicographic enumeration of regular and context-free languages." *Acta Cybernetica*, **13** (1997), 55–61.)
- Mäkinen's algorithm uses dynamic programming. It finds the minimal words of length 1 through $n$ starting at each state.
- Originally analyzed in the unit-cost model.
- We analyze its running time in the bit-complexity model and modify the algorithm.
- Mäkinen's cross-section enumeration algorithm uses $O(s^2 n^2 + \sigma s^2 t)$ operations. *Algorithm is quadratic in n.*

# Previous Work - Mäkinen's algorithm

- Mäkinen finds an efficient algorithm in the unit-cost model ("On lexicographic enumeration of regular and context-free languages." *Acta Cybernetica*, **13** (1997), 55–61.)
- Mäkinen's algorithm uses dynamic programming. It finds the minimal words of length 1 through $n$ starting at each state.
- Originally analyzed in the unit-cost model.
- We analyze its running time in the bit-complexity model and modify the algorithm.
- Mäkinen's cross-section enumeration algorithm uses $O(s^2n^2 + \sigma s^2 t)$ operations. *Algorithm is quadratic in n.*
- Mäkinen's enumeration algorithm uses $O(s^2 e + \sigma s^2 t)$ operations.
  $e$: number of empty cross-sections. $n$: length of words in cross-section.
  $t$: output size. $s$: number of states in the NFA. $\sigma$: size of the alphabet of the NFA.

# Previous Work

Pál Dömösi gives a cross-section enumeration algorithm, where finding each consecutive word is superexponential in the size of the cross-section (Dömösi, 1998.)

# Contributions

1. New algorithm for cross-section enumeration that is linear in $n$.

# Contributions

1. New algorithm for cross-section enumeration that is linear in $n$.
2. Variants of Mäkinen's algorithm with best practical performance.

# Contributions

1. New algorithm for cross-section enumeration that is linear in $n$.
2. Variants of Mäkinen's algorithm with best practical performance.
3. Extensive performance analysis.

# Algorithm Framework

- We describe an algorithm for finding the minimal word in a cross-section.

# Algorithm Framework

- We describe an algorithm for finding the minimal word in a cross-section.
- Then, we use that algorithm for finding the rest of the words in a cross-section.

# Algorithm Framework

- We describe an algorithm for finding the minimal word in a cross-section.
- Then, we use that algorithm for finding the rest of the words in a cross-section.
- We construct an enumeration algorithm through repeated application of the cross-section enumeration algorithm.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

- Starting at the start state, we traverse the NFA only down paths that will lead to a final state in the required number of steps.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

- Starting at the start state, we traverse the NFA only down paths that will lead to a final state in the required number of steps.
- We need a method of determining whether a word of length $i$ can be completed to a word of length $n$ in $n - i$ steps from a given state.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

- Starting at the start state, we traverse the NFA only down paths that will lead to a final state in the required number of steps.
- We need a method of determining whether a word of length $i$ can be completed to a word of length $n$ in $n - i$ steps from a given state.
- Precompute $M$, the adjacency matrix of the NFA; $M_{p,q} = 1$ if there is a transition from state $p$ to state $q$, and $M_{p,q} = 0$ otherwise.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

- Starting at the start state, we traverse the NFA only down paths that will lead to a final state in the required number of steps.
- We need a method of determining whether a word of length $i$ can be completed to a word of length $n$ in $n-i$ steps from a given state.
- Precompute $M$, the adjacency matrix of the NFA; $M_{p,q} = 1$ if there is a transition from state $p$ to state $q$, and $M_{p,q} = 0$ otherwise.
- $M^i_{p,q} = 1$ (using bit-arithmetic) if and only if there is a path from state $p$ to state $q$ of length exactly $i$.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

## Definition

A state $q$ in an NFA $N$ is *$i$-complete* if there is a path in $N$ of length $i$ starting at $q$ and ending at a final state.

- Notice that $M^i$ enables us to determine if a given state is $i$-complete.

# Lookahead-Matrix Algorithm For Finding the Minimal Word

## Definition

A state $q$ in an NFA $N$ is *$i$-complete* if there is a path in $N$ of length $i$ starting at $q$ and ending at a final state.

- Notice that $M^i$ enables us to determine if a given state is $i$-complete.
- To find the minimal word of length $n$:
  - Compute $M, M^2, \ldots, M^n$ using boolean matrix multiplication.
  - Find the set of $(n-1)$-complete states, $S_1$, reachable from the start state on the minimal possible symbol $a_1$.
  - Then find the set of $(n-2)$-complete states, $S_2$, reachable from any state in $S_1$ on the minimal possible symbol $a_2$.
  - Repeat this process for a total of $n$ iterations.
  - Then $a_1 a_2 \cdots a_n$ is the minimal word of length $n$.

# Finding the Minimal Word

## Algorithm

*minWordLM(n,N)*

*INPUT: A nonnegative integer $n$ and an NFA $N$.*
*OUTPUT: The minimal word of length $n$ accepted by $N$. Updates state stack $S$ for a potential subsequent call to minWord or nextWord.*

```
Compute M, M^2, ..., M^n
S_0 = {s_0}
IF M^n_{q,f} = 0 for all f ∈ F, q ∈ S_0
   return NULL
w = empty word
FOR i ← 0 ··· n − 1
     a_{i+1} = min(a ∈ Σ | ∃u ∈ S_i, f ∈ F where M^{n−1−i}_{v,f} = 1 for some v ∈ δ(u, a))
     w = w a_{i+1}
     S_{i+1} = {v ∈ ∪_{u∈S_i} δ(u, a_{i+1}) | M^{n−1−i}_{v,f} = 1 for some f ∈ F}
     IF i ≠ n − 1
        push(S, S_{i+1})
return w
```

Compute $M, M^2, \ldots, M^n$

$S_0 = \{s_0\}$

IF $M^n_{q,f} = 0$ for all $f \in F, q \in S_0$
  return NULL

$w$ = empty word

FOR $i \leftarrow 0 \cdots n-1$

  $a_{i+1} = \min(a \in \Sigma \mid \exists u \in S_i, f \in F \text{ where } M^{n-1-i}_{v,f} = 1 \text{ for some } v \in \delta(u,a))$

  $w = wa_{i+1}$

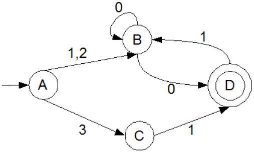
  $S_{i+1} = \{v \in \cup_{u \in S_i} \delta(u, a_{i+1}) \mid M^{n-1-i}_{v,f} = 1 \text{ for some } f \in F\}$

  IF $i \neq n-1$
    push($S$,$S_{i+1}$)

return $w$

# Finding the Minimal Word

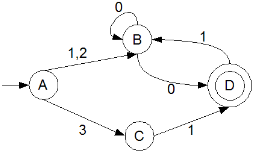
Find the minimal word $w$ of length 4 accepted by the following NFA.




$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, M^3 = M^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

# Finding the Minimal Word

Find the minimal word $w$ of length 4 accepted by the following NFA.

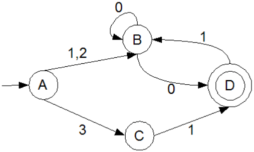



$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, M^3 = M^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

1. State $A$ is 4-complete. Therefore, $w$ exists.
   Set $w = \epsilon$, $S = \{\{A\}\}$.

# Finding the Minimal Word

Find the minimal word $w$ of length 4 accepted by the following NFA.

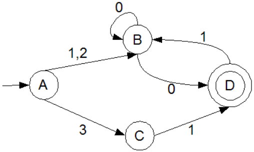



$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, M^3 = M^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

1. State $A$ is 4-complete. Therefore, $w$ exists.
   Set $w = \epsilon$, $S = \{\{A\}\}$.
2. State $B$ is 3-complete and reachable from $A$ on the minimal* char.
   Set $w = 1$, $S = \{\{A\}, \{B\}\}$.

# Finding the Minimal Word

Find the minimal word $w$ of length 4 accepted by the following NFA.

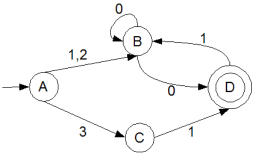



$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, M^3 = M^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

1. State $A$ is 4-complete. Therefore, $w$ exists.
   Set $w = \epsilon$, $S = \{\{A\}\}$.
2. State $B$ is 3-complete and reachable from $A$ on the minimal* char.
   Set $w = 1$, $S = \{\{A\}, \{B\}\}$.
3. States $B$ and $D$ are 2-complete and reachable from $B$ on the minimal* char.
   Set $w = 10$, $S = \{\{A\}, \{B\}, \{B, D\}\}$.

# Finding the Minimal Word

Find the minimal word $w$ of length 4 accepted by the following NFA.

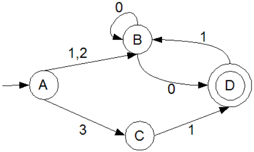



$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, M^3 = M^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

1. State $A$ is 4-complete. Therefore, $w$ exists.
   Set $w = \epsilon$, $S = \{\{A\}\}$.
2. State $B$ is 3-complete and reachable from $A$ on the minimal* char.
   Set $w = 1$, $S = \{\{A\}, \{B\}\}$.
3. States $B$ and $D$ are 2-complete and reachable from $B$ on the minimal* char.
   Set $w = 10$, $S = \{\{A\}, \{B\}, \{B, D\}\}$.
4. State $B$ is 1-complete and reachable from {B, D}.
   Set $w = 100$, $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$.

# Finding the Minimal Word

Find the minimal word $w$ of length 4 accepted by the following NFA.




$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, M^3 = M^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

1. State $A$ is 4-complete. Therefore, $w$ exists.
   Set $w = \epsilon$, $S = \{\{A\}\}$.
2. State $B$ is 3-complete and reachable from $A$ on the minimal* char.
   Set $w = 1$, $S = \{\{A\}, \{B\}\}$.
3. States $B$ and $D$ are 2-complete and reachable from $B$ on the minimal* char.
   Set $w = 10$, $S = \{\{A\}, \{B\}, \{B, D\}\}$.
4. State $B$ is 1-complete and reachable from {B, D}.
   Set $w = 100$, $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$.
5. State $D$ is final and reachable from $B$. Set $w = 1000$.

# Complexity of Finding the Minimal Word

- Since the matrices require $O(s^2n)$ space, `minWordLM` uses $O(s^2n)$ space.

# Complexity of Finding the Minimal Word

- Since the matrices require $O(s^2 n)$ space, `minWordLM` uses $O(s^2 n)$ space.
- The best bound for matrix multiplication is $O(s^{2.376})$ (Coppersmith, 90).

# Complexity of Finding the Minimal Word

- Since the matrices require $O(s^2 n)$ space, `minWordLM` uses $O(s^2 n)$ space.
- The best bound for matrix multiplication is $O(s^{2.376})$ (Coppersmith, 90).
- All operations other than the matrix multiplication in the algorithm cost $O(\sigma s^2 n)$.

# Complexity of Finding the Minimal Word

- Since the matrices require $O(s^2 n)$ space, `minWordLM` uses $O(s^2 n)$ space.
- The best bound for matrix multiplication is $O(s^{2.376})$ (Coppersmith, 90).
- All operations other than the matrix multiplication in the algorithm cost $O(\sigma s^2 n)$.

**Theorem**

*The Lookahead-Matrix algorithm finds the minimal word of length n in $O(s^{2.376} n + \sigma s^2 n)$ time and $O(s^2 n)$ space.*

# Finding the Next Word in a Cross-Section

- When looking for the minimal word, store the sets of states $\{s_0\}, S_1, S_2, \ldots, S_{n-1}$ on a state stack.

# Finding the Next Word in a Cross-Section

- When looking for the minimal word, store the sets of states $\{s_0\}, S_1, S_2, \ldots, S_{n-1}$ on a state stack.
- Pop $S_{n-1}$ from the state stack and check if it can reach any final state reachable from a character other than $a_n$.

# Finding the Next Word in a Cross-Section

- When looking for the minimal word, store the sets of states $\{s_0\}, S_1, S_2, \ldots, S_{n-1}$ on a state stack.
- Pop $S_{n-1}$ from the state stack and check if it can reach any final state reachable from a character other than $a_n$.
- If so, complete the word $a_1 a_2 \ldots a_{n-1}$ with the minimal 1-character word $> a_n$ reachable from $S_{n-1}$.

# Finding the Next Word in a Cross-Section

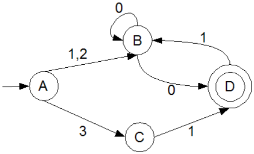
- When looking for the minimal word, store the sets of states $\{s_0\}, S_1, S_2, \ldots, S_{n-1}$ on a state stack.
- Pop $S_{n-1}$ from the state stack and check if it can reach any final state reachable from a character other than $a_n$.
- If so, complete the word $a_1 a_2 \ldots a_{n-1}$ with the minimal 1-character word $> a_n$ reachable from $S_{n-1}$.
- If not, pop $S_{n-2}$ and look for 1-complete states reachable from $S_{n-2}$ on a character $> a_{n-1}$.

# Finding the Next Word in a Cross-Section

- When looking for the minimal word, store the sets of states $\{s_0\}, S_1, S_2, \ldots, S_{n-1}$ on a state stack.
- Pop $S_{n-1}$ from the state stack and check if it can reach any final state reachable from a character other than $a_n$.
- If so, complete the word $a_1 a_2 \ldots a_{n-1}$ with the minimal 1-character word $> a_n$ reachable from $S_{n-1}$.
- If not, pop $S_{n-2}$ and look for 1-complete states reachable from $S_{n-2}$ on a character $> a_{n-1}$.
- Continue until the next word is found or the state stack is empty.

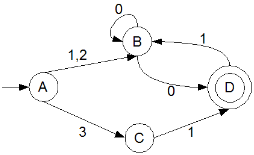
# Finding the Next Word

Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.



# Finding the Next Word

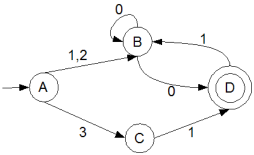
Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.




1. Pop $\{B\}$ from $S$. Set $u = 100$, $a = 0$. We cannot reach a final state from $\{B\}$ on a character $> a$.

# Finding the Next Word

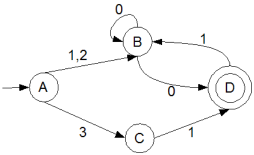
Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.




1. Pop $\{B\}$ from $S$. Set $u = 100$, $a = 0$. We cannot reach a final state from $\{B\}$ on a character $> a$.
2. Pop $\{B, D\}$ from $S$. Remove last character from $u$ and assign it to $a$. We get $u = 10$, $a = 0$. State $B$ is 1-complete and reachable from state $D$ on character $1 > a$.

# Finding the Next Word

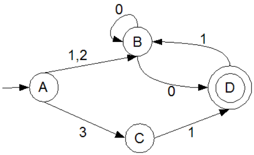
Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.




1. Pop $\{B\}$ from $S$. Set $u = 100$, $a = 0$. We cannot reach a final state from $\{B\}$ on a character $> a$.
2. Pop $\{B, D\}$ from $S$. Remove last character from $u$ and assign it to $a$. We get $u = 10$, $a = 0$. State $B$ is 1-complete and reachable from state $D$ on character $1 > a$.
3. Push $\{B, D\}$ and $\{B\}$ onto $S$. Append 1 to $u$, giving $u = 101$.

# Finding the Next Word

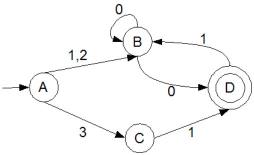
Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.




1. Pop $\{B\}$ from $S$. Set $u = 100$, $a = 0$. We cannot reach a final state from $\{B\}$ on a character $> a$.
2. Pop $\{B, D\}$ from $S$. Remove last character from $u$ and assign it to $a$. We get $u = 10$, $a = 0$. State $B$ is 1-complete and reachable from state $D$ on character $1 > a$.
3. Push $\{B, D\}$ and $\{B\}$ onto $S$. Append 1 to $u$, giving $u = 101$.
4. Find the minimal word of length 1 from $\{B\}$ to a final state and append it to $u$.

# Finding the Next Word

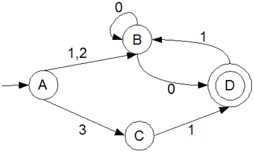
Given the minimal word $w = 1000$, and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the second word $u$ of length 4 accepted by the following NFA.




1. Pop $\{B\}$ from $S$. Set $u = 100$, $a = 0$. We cannot reach a final state from $\{B\}$ on a character $> a$.
2. Pop $\{B, D\}$ from $S$. Remove last character from $u$ and assign it to $a$. We get $u = 10$, $a = 0$. State $B$ is 1-complete and reachable from state $D$ on character $1 > a$.
3. Push $\{B, D\}$ and $\{B\}$ onto $S$. Append 1 to $u$, giving $u = 101$.
4. Find the minimal word of length 1 from $\{B\}$ to a final state and append it to $u$.
5. We get $u = 1010$ and $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$.

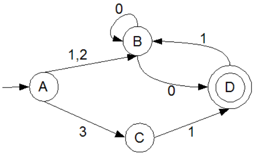
# Finding the Next Word

Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.



# Finding the Next Word

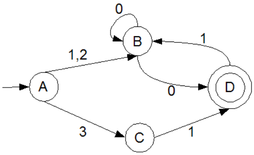
Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.

# Finding the Next Word

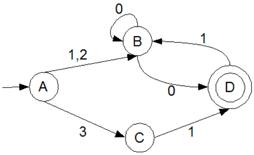
Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.
2. Pop $\{B, D\}$ from $S$. Now $v = 10$, $a = 1$.

# Finding the Next Word

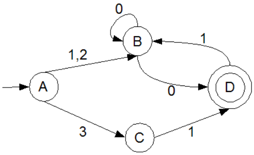
Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.
2. Pop $\{B, D\}$ from $S$. Now $v = 10$, $a = 1$.
3. Pop $\{B\}$ from $S$. Now $v = 1$, $a = 0$.

# Finding the Next Word

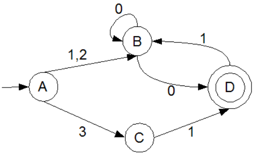
Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.
2. Pop $\{B, D\}$ from $S$. Now $v = 10$, $a = 1$.
3. Pop $\{B\}$ from $S$. Now $v = 1$, $a = 0$.
4. Pop $\{A\}$ from $S$. Now $v = \epsilon$, $a = 1$. State $B$ is a 3-complete state reachable from $\{A\}$ on character $2 > a$.

# Finding the Next Word

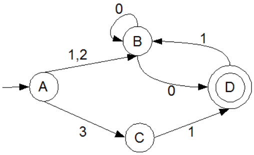
Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.
2. Pop $\{B, D\}$ from $S$. Now $v = 10$, $a = 1$.
3. Pop $\{B\}$ from $S$. Now $v = 1$, $a = 0$.
4. Pop $\{A\}$ from $S$. Now $v = \epsilon$, $a = 1$. State $B$ is a 3-complete state reachable from $\{A\}$ on character $2 > a$.
5. Push $\{A\}$ and $\{B\}$ onto $S$. Append 2 to $v$, giving $v = 2$.

# Finding the Next Word

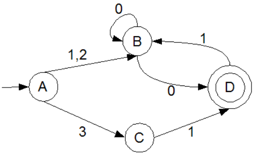
Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.
2. Pop $\{B, D\}$ from $S$. Now $v = 10$, $a = 1$.
3. Pop $\{B\}$ from $S$. Now $v = 1$, $a = 0$.
4. Pop $\{A\}$ from $S$. Now $v = \epsilon$, $a = 1$. State $B$ is a 3-complete state reachable from $\{A\}$ on character $2 > a$.
5. Push $\{A\}$ and $\{B\}$ onto $S$. Append 2 to $v$, giving $v = 2$.
6. Find the minimal word of length 3 from $\{B\}$ to a final state and append it to $v$.

# Finding the Next Word

Given word $u = 1010$ and state stack $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$, find the next word $v$ of length 4 accepted by the NFA.




1. Pop $\{B\}$ from $S$. Set $v = 101$, $a = 0$.
2. Pop $\{B, D\}$ from $S$. Now $v = 10$, $a = 1$.
3. Pop $\{B\}$ from $S$. Now $v = 1$, $a = 0$.
4. Pop $\{A\}$ from $S$. Now $v = \epsilon$, $a = 1$. State $B$ is a 3-complete state reachable from $\{A\}$ on character $2 > a$.
5. Push $\{A\}$ and $\{B\}$ onto $S$. Append 2 to $v$, giving $v = 2$.
6. Find the minimal word of length 3 from $\{B\}$ to a final state and append it to $v$.
7. Thus, $v = 2000$ and $S = \{\{A\}, \{B\}, \{B, D\}, \{B\}\}$.

# Lookahead-Matrix Cross-Section Enumeration Algorithm

- The algorithm `CrossSectionLM` finds the minimal word in the cross-section and repeatedly applies the algorithm for finding the next word in the cross-section, until the state stack is empty.

# Lookahead-Matrix Cross-Section Enumeration Algorithm

- The algorithm `CrossSectionLM` finds the minimal word in the cross-section and repeatedly applies the algorithm for finding the next word in the cross-section, until the state stack is empty.
- Finding the minimal word costs $O(s^{2.376}n + \sigma s^2 n)$.

# Lookahead-Matrix Cross-Section Enumeration Algorithm

- The algorithm `CrossSectionLM` finds the minimal word in the cross-section and repeatedly applies the algorithm for finding the next word in the cross-section, until the state stack is empty.
- Finding the minimal word costs $O(s^{2.376}n + \sigma s^2 n)$.
- Finding the remaining words costs $O(\sigma s^2 t)$.

# Lookahead-Matrix Cross-Section Enumeration Algorithm

- The algorithm `CrossSectionLM` finds the minimal word in the cross-section and repeatedly applies the algorithm for finding the next word in the cross-section, until the state stack is empty.
- Finding the minimal word costs $O(s^{2.376}n + \sigma s^2 n)$.
- Finding the remaining words costs $O(\sigma s^2 t)$.

**Theorem**

*The algorithm* `crossSectionLM` *uses* $O(s^{2.376}n + \sigma s^2 t)$ *operations.*

$t$: output size. $s$: number of states. $\sigma$: alphabet size. $n$: length of words in cross-section.

# Lookahead-Matrix Enumeration Algorithm

- The algorithm `enumLM` consists of repeated applications of `CrossSectionLM`.

# Lookahead-Matrix Enumeration Algorithm

- The algorithm enumLM consists of repeated applications of CrossSectionLM.
- If an empty cross-section is encountered in enumLM, the algorithm performs $O(s^{2.376})$ operations to determine that.

# Lookahead-Matrix Enumeration Algorithm

- The algorithm `enumLM` consists of repeated applications of `CrossSectionLM`.
- If an empty cross-section is encountered in `enumLM`, the algorithm performs $O(s^{2.376})$ operations to determine that.
- The algorithm `enumLM` does $O(s^{2.376} + \sigma s^2 t)$ operations for each non-empty cross-section.

# Lookahead-Matrix Enumeration Algorithm

- The algorithm `enumLM` consists of repeated applications of `CrossSectionLM`.
- If an empty cross-section is encountered in `enumLM`, the algorithm performs $O(s^{2.376})$ operations to determine that.
- The algorithm `enumLM` does $O(s^{2.376} + \sigma s^2 t)$ operations for each non-empty cross-section.

**Theorem**

*The algorithm* `enumLM` *uses* $O(s^{2.376}(c+e) + \sigma s^2 t)$ *operations.*

$e$: number of empty cross-sections encountered throughout the enumeration.
$c$: number of non-empty cross-sections encountered throughout the enumeration.
$t$: output size. $s$: number of states. $\sigma$: alphabet size.

- The number of consecutive empty cross-sections is at most $s-1$. Thus, $e < cs < ts$.

# Mäkinen's Algorithms

- Uses dynamic programming to find the minimal word in each cross-section.

# Mäkinen's Algorithms

- Uses dynamic programming to find the minimal word in each cross-section.
- Originally, algorithms were analyzed in the unit-cost model.

# Mäkinen's Algorithms

- Uses dynamic programming to find the minimal word in each cross-section.
- Originally, algorithms were analyzed in the unit-cost model.
- We give variants of Mäkinen's algorithms which we analyze in the bit-complexity model.

# Mäkinen's Minimal Word Algorithm

## Algorithm

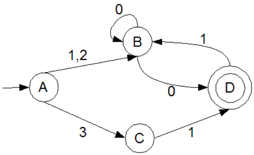
*minWordMäkinen(n, N)*

*INPUT: A positive integer* $n$ *and an NFA* $N$*.*
*OUTPUT: Table* $A^{min}[1 \cdots n]$ *for each state* $A \in Q$ *where* $A^{min}[i]$ *is the minimal word of length* $i$ *starting at state* $A$*.*

```
FOR each A ∈ Q
    IF for all a ∈ Σ, δ(A, a) ∩ F = ∅
        A^min[1] = NULL
    ELSE
        A^min[1] = min{a ∈ Σ | δ(A, a) ∩ F ≠ ∅}
FOR i ← 2 ··· n
    FOR each A ∈ Q
        min = NULL
        FOR each B ∈ Q and minimal a ∈ Σ such that B ∈ δ(A, a)
            IF B^min[i − 1] ≠ NULL
                IF aB^min[i − 1] < min OR min = NULL
                    min ← aB^min[i − 1]
        A^min[i] = min
RETURN {A^min | A ∈ Q}
```

# Mäkinen's Minimal Word Algorithm

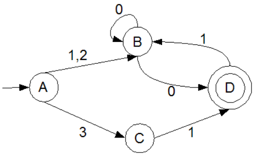
Find the minimal word of length 4 in the NFA.




| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | | | |
| B | 0 | | | |
| C | 1 | | | |
| D | x | | | |

# Mäkinen's Minimal Word Algorithm

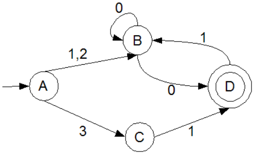
Find the minimal word of length 4 in the NFA.




| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | 10 | | |
| B | 0 | 00 | | |
| C | 1 | x | | |
| D | x | 10 | | |

# Mäkinen's Minimal Word Algorithm

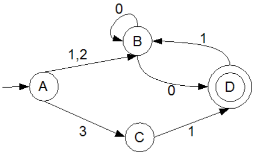
Find the minimal word of length 4 in the NFA.




| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | 10 | 100 | |
| B | 0 | 00 | 000 | |
| C | 1 | x | 110 | |
| D | x | 10 | 100 | |

# Mäkinen's Minimal Word Algorithm

Find the minimal word of length 4 in the NFA.




| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | 10 | 100 | 1000 |
| B | 0 | 00 | 000 | 0000 |
| C | 1 | x | 110 | 1100 |
| D | x | 10 | 100 | 1000 |

# Mäkinen's Minimal Word Algorithm

- Finding the minimal word using Mäkinen's algorithm costs $O(n)$ and uses $O(n)$ space in the *unit-cost model* and when the size of the NFA is constant.

# Mäkinen's Minimal Word Algorithm

- Finding the minimal word using Mäkinen's algorithm costs $O(n)$ and uses $O(n)$ space in the *unit-cost model* and when the size of the NFA is constant.
- We analyze the algorithm in the *bit-complexity model* and integrate other parameters into our analysis.

# Mäkinen's Minimal Word Algorithm

- Finding the minimal word using Mäkinen's algorithm costs $O(n)$ and uses $O(n)$ space in the *unit-cost model* and when the size of the NFA is constant.
- We analyze the algorithm in the *bit-complexity model* and integrate other parameters into our analysis.
- Concatenation of words can be performed in constant time by changing the mode of storage: Instead of storing a word $w$ of length $i$ in $A^{min}[i]$, store the pair $(a, B)$ such that $w = aB^{min}[i-1]$.

| states/length | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | x | (1, B) | (1, B) | (1, B) |
| B | 0 | (0, B) | (0, B) | (0, B) |
| C | 1 | x | x | x |
| D | x | (1, B) | (1, B) | (1, B) |

- With this modification, Mäkinen's algorithm uses $\Theta(sn)$ space to find the minimal word.

# Mäkinen's Minimal Word Algorithm

- Since the states of an NFA can form a complete graph, the worst-case running time is $O(s^2 n^2)$.

# Mäkinen's Minimal Word Algorithm

- Since the states of an NFA can form a complete graph, the worst-case running time is $O(s^2 n^2)$.
- This worst-case is reached in the figure below.

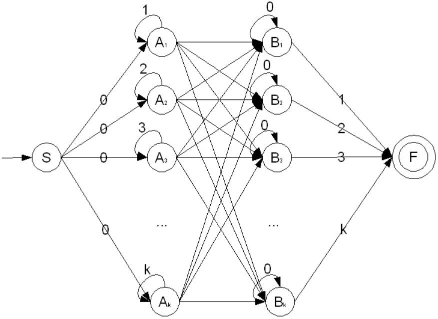



Figure: $\delta(A_i, a_i) = \{B_1, B_2, \ldots, B_k\}$ for all distinct $a_i$.

# Mäkinen's Cross-Section Enumeration Algorithms

- Mäkinen's original cross-section enumeration algorithm requires subset-construction to work on NFAs.

# Mäkinen's Cross-Section Enumeration Algorithms

- Mäkinen's original cross-section enumeration algorithm requires subset-construction to work on NFAs.
- To overcome this constraint, find consecutive words in a cross-section using the same method as in the Lookahead-Matrix algorithm.

# Mäkinen's Cross-Section Enumeration Algorithms

- Mäkinen's original cross-section enumeration algorithm requires subset-construction to work on NFAs.
- To overcome this constraint, find consecutive words in a cross-section using the same method as in the Lookahead-Matrix algorithm.
- Mäkinen's original cross-section algorithm determines when a cross-section has been fully enumerated by precomputing the maximal word in the cross-section.

# Mäkinen's Cross-Section Enumeration Algorithms

- Mäkinen's original cross-section enumeration algorithm requires subset-construction to work on NFAs.
- To overcome this constraint, find consecutive words in a cross-section using the same method as in the Lookahead-Matrix algorithm.
- Mäkinen's original cross-section algorithm determines when a cross-section has been fully enumerated by precomputing the maximal word in the cross-section.
- Recall that Lookahead-Matrix determines that a cross-section has been enumerated when the state stack is empty.

# Mäkinen's Cross-Section Enumeration Algorithms

- When Mäkinen's original cross-section termination method is used, we call the algorithm `crossSectionMäkinenI`.

# Mäkinen's Cross-Section Enumeration Algorithms

- When Mäkinen's original cross-section termination method is used, we call the algorithm `crossSectionMäkinenI`.
- When the Lookahead-Matrix cross-section termination is used on Mäkinen's algorithm, we call the algorithm `crossSectionMäkinenII`.

# Mäkinen's Cross-Section Enumeration Algorithms

- When Mäkinen's original cross-section termination method is used, we call the algorithm `crossSectionMäkinenI`.
- When the Lookahead-Matrix cross-section termination is used on Mäkinen's algorithm, we call the algorithm `crossSectionMäkinenII`.
- Both `crossSectionMäkinenI` and `crossSectionMäkinenII` use $O(s^2n^2 + \sigma s^2 t)$ operations.
  $t$: output size. $s$: number of states. $\sigma$: alphabet size.

# Mäkinen's Cross-Section Enumeration Algorithms

- When Mäkinen's original cross-section termination method is used, we call the algorithm `crossSectionMäkinenI`.
- When the Lookahead-Matrix cross-section termination is used on Mäkinen's algorithm, we call the algorithm `crossSectionMäkinenII`.
- Both `crossSectionMäkinenI` and `crossSectionMäkinenII` use $O(s^2 n^2 + \sigma s^2 t)$ operations.
  $t$: output size. $s$: number of states. $\sigma$: alphabet size.
- In practice, `crossSectionMäkinenI` and `crossSectionMäkinenII` perform differently.

# Mäkinen's Enumeration Algorithm

- The algorithm `enumMäkinenI` consists of repeated calls to `crossSectionMäkinenI`.

# Mäkinen's Enumeration Algorithm

- The algorithm `enumMäkinenI` consists of repeated calls to `crossSectionMäkinenI`.
- Similarly, `enumMäkinenII` consists of repeated calls to `crossSectionMäkinenII`.

# Mäkinen's Enumeration Algorithm

- The algorithm `enumMäkinenI` consists of repeated calls to `crossSectionMäkinenI`.
- Similarly, `enumMäkinenII` consists of repeated calls to `crossSectionMäkinenII`.
- Both algorithms use $O(\sigma s^2 t + s^2 e)$ operations.
  $e$: number of empty cross-sections. $t$: output size.
  $s$: number of states. $\sigma$: alphabet size.

# Complexity Summary

| | Cross-Section | Enum |
|---|---|---|
| Lookahead-Matrix | $O(s^{2.376}n + \sigma s^2 t)$ | $O(s^{2.376}(c + e) + \sigma s^2 t)$ |
| Mäkinen | $O(s^2 n^2 + \sigma s^2 t)$ | $O(s^2 e + \sigma s^2 t)$ |
| Grail | $O(s^2 \sigma^{n+1})$ | $O(s^2 \sigma^{k+1})$ |

$e$: number of empty cross-sections. $c$: number of non-empty cross-sections.
$t$: output size. $s$: number of states. $\sigma$: alphabet size.
$n$: length of words in cross-section. $k$: length of words in last cross-section examined.

# Experimental Framework

- We ran tests to determine the algorithms' average performance in practice.

# Experimental Framework

- We ran tests to determine the algorithms' average performance in practice.
- For each test, 100 NFAs were randomly generated.

# Experimental Framework

- We ran tests to determine the algorithms' average performance in practice.
- For each test, 100 NFAs were randomly generated.
- For each NFA, we ran each algorithm up to 10 times and recorded the average running time.

# Experimental Framework

- We ran tests to determine the algorithms' average performance in practice.
- For each test, 100 NFAs were randomly generated.
- For each NFA, we ran each algorithm up to 10 times and recorded the average running time.
- To test worst-case practical performance, we also tested the algorithms on the NFA that accepts $1^*$, the NFA that accepts $(0+1)^*$, and NFAs that cause Mäkinen's cross-section algorithms to run in quadratic time.

# Experimental Framework

- We ran tests to determine the algorithms' average performance in practice.
- For each test, 100 NFAs were randomly generated.
- For each NFA, we ran each algorithm up to 10 times and recorded the average running time.
- To test worst-case practical performance, we also tested the algorithms on the NFA that accepts $1^*$, the NFA that accepts $(0+1)^*$, and NFAs that cause Mäkinen's cross-section algorithms to run in quadratic time.
- For comparison and correctness testing, we implemented a naive algorithm that generates all words over $\Sigma^*$ and checks which are accepted by the input NFA.

# Experimental Results

- The naive algorithms perform reasonably well on small NFAs when the alphabet is of size less than 3, but usually slower than the other algorithms.
- With an alphabet size greater than 3, the naive algorithms are unreasonably slow.

# Experimental Results

- The naive algorithms perform reasonably well on small NFAs when the alphabet is of size less than 3, but usually slower than the other algorithms.
- With an alphabet size greater than 3, the naive algorithms are unreasonably slow.
- The Grail algorithms tend to perform well on small input size.
- The Grail algorithms outperform the other enumeration algorithms on $1^*$.

# Experimental Results

- The naive algorithms perform reasonably well on small NFAs when the alphabet is of size less than 3, but usually slower than the other algorithms.
- With an alphabet size greater than 3, the naive algorithms are unreasonably slow.
- The Grail algorithms tend to perform well on small input size.
- The Grail algorithms outperform the other enumeration algorithms on $1^*$.
- Naive and Grail algorithms are significantly slower than the Lookahead-Matrix algorithm and Mäkinen's algorithms on most NFAs.

# Experimental Results

- Mäkinen's algorithms and lookahead-matrix are poorly suited for sparse languages, like $1^*$.

# Experimental Results

- Mäkinen's algorithms and lookahead-matrix are poorly suited for sparse languages, like $1^*$.
- MäkinenII is significantly more efficient than MäkinenI on NFAs with unary alphabets.

# Experimental Results

- Mäkinen's algorithms and lookahead-matrix are poorly suited for sparse languages, like $1^*$.
- MäkinenII is significantly more efficient than MäkinenI on NFAs with unary alphabets.
- On NFAs where Mäkinen's cross-section enumeration algorithms are quadratic in $n$, `crossSectionLM` performs significantly better than Mäkinen's cross-section algorithms (at times over 50 times faster).

# Experimental Results

- Mäkinen's algorithms and lookahead-matrix are poorly suited for sparse languages, like $1^*$.
- MäkinenII is significantly more efficient than MäkinenI on NFAs with unary alphabets.
- On NFAs where Mäkinen's cross-section enumeration algorithms are quadratic in $n$, `crossSectionLM` performs significantly better than Mäkinen's cross-section algorithms (at times over 50 times faster).
- On average, the Matrix-Lookahead algorithms perform almost as well as the MäkinenII algorithms and better than the MäkinenI algorithms.
- On average, MäkinenII performs best.

# Conclusions

- The algorithm `crossSectionLM` has the best worst-case complexity and the best worst-case running time in practice.

# Conclusions

- The algorithm `crossSectionLM` has the best worst-case complexity and the best worst-case running time in practice.
- The MäkinenII algorithms for both enumeration problems have the best average-case running times in practice.

# Future Work

- Improve on the running time of enumMäkinen for the enumeration problem.

# Future Work

- Improve on the running time of enumMäkinen for the enumeration problem.
- Find heuristics to further improve the running time of the algorithms in practice. For instance, check the density of the language and select algorithm based on the density.

# Future Work

- Improve on the running time of enumMäkinen for the enumeration problem.
- Find heuristics to further improve the running time of the algorithms in practice. For instance, check the density of the language and select algorithm based on the density.
- Prove lower bounds for the enumeration and cross-section enumeration problems.