CS 886: Game-theoretic methods for computer science

Normal Form Games

Kate Larson

Computer Science
University of Waterloo

Outline

1. Review
   - Normal Form Game
   - Examples
   - Strategies
2. Nash Equilibria
3. Dominant and Dominated Strategies
4. Maxmin and Minmax Strategies

Normal Form

A normal form game is defined by

- Finite set of agents (or players) N, $|N| = n$
- Each agent i has an action space A_i
  - A_i is non-empty and finite
- Outcomes are defined by action profiles ($a = (a_1, \ldots, a_n)$ where a_i is the action taken by agent i
- Each agent has a utility function $u_i : A_1 \times \ldots \times A_n \mapsto \mathbb{R}$

Examples

Prisoners' Dilemma

	C	D
C	a,a	b,c
D	c,b	d,d

$c > a > d > b$

Pure coordination game
$\forall$ action profiles $a \in A_1 \times \ldots \times A_n$ and $\forall i, j$, $u_i(a) = u_j(a)$.

	L	R
L	1,1	0,0
R	0,0	1,1

Agents do not have conflicting interests. There sole challenge is to coordinate on an action which is good for all.

Zero-sum games

$\forall a \in A_1 \times A_2, u_1(a) + u_2(a) = 0$. That is, one player gains at the other player's expense.

Matching Pennies

	H	T
H	1,-1	-1, 1
T	-1,1	1,-1

	H	T
H	1	-1
T	-1	1

Given the utility of one agent, the other's utility is known.

More Examples

Most games have elements of both cooperation and competition.

BoS

	H	S
H	2,1	0,0
S	0,0	1,2

Hawk-Dove

	D	H
D	3,3	1,4
H	4,1	0,0

Strategies

Notation: Given set X, let ΔX be the set of all probability distributions over X.

Definition

Given a normal form game, the set of mixed strategies for agent i is

$$S_i = \Delta A_i$$

The set of mixed strategy profiles is $S = S_1 \times \ldots \times S_n$.

Definition

A strategy s_i is a probability distribution over A_i. $s_i(a_i)$ is the probability action a_i will be played by mixed strategy s_i.

Strategies

Definition

The support of a mixed strategy s_i is

$$\{a_i | s_i(a_i) > 0\}$$

Definition

A pure strategy s_i is a strategy such that the support has size 1, i.e.

$$|\{a_i | s_i(a_i) > 0\}| = 1$$

A pure strategy plays a single action with probability 1.

Expected Utility

The expected utility of agent i given strategy profile s is

$$u_i(s) = \sum_{a \in A} u_i(a) \Pi_{j=1}^{n} s_j(a_j)$$

Example

	C	D
C	-1,-1	-4,0
D	0, -4	-3,-3

Given strategy profile
$s = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{10}, \frac{9}{10}))$

$$u_1 = -1(\frac{1}{2})(\frac{1}{10}) - 4(\frac{1}{2})(\frac{9}{10}) - 3(\frac{1}{2})(\frac{9}{10}) = -3.2$$

$$u_2 = -1(\frac{1}{2})(\frac{1}{10}) - 4(\frac{1}{2})(\frac{1}{10}) - 3(\frac{1}{2})(\frac{9}{10}) = -1.6$$

Best-response

Given a game, what strategy should an agent choose?
We first consider only pure strategies.

Definition

Given a_{-i}, the best-response for agent i is $a_i \in A_i$ such that

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i$$

Note that the best response may not be unique.
A best-response set is

$$B_i(a_{-i}) = \{a_i \in A_i | u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i\}$$

Nash Equilibrium

Definition

A profile a^* is a Nash equilibrium if $\forall i$, a_i^* is a best response to a_{-i}^*. That is

$$\forall i u_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*) \; \forall a_i' \in A_i$$

Equivalently, a^* is a Nash equilibrium if $\forall i$

$$a_i^* \in B(a_{-i}^*)$$

Examples

PD

	C	D
C	-1,-1	-4,0
D	0,-4	-3,-3

BoS

	H	T
H	2,1	0,0
T	0,0	1,2

Matching Pennies

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Nash Equilibria

We need to extend the definition of a Nash equilibrium.
Strategy profile s^* is a Nash equilibrium is for all i

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \; \forall s_i' \in S_i$$

Similarly, a best-response set is

$$B(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \forall s_i' \in S_i\}$$

Examples

Characterization of Mixed Nash Equilibria

s^* is a (mixed) Nash equilibrium if and only if

- the expected payoff, given s^*_{-i}, to every action to which s^*_i assigns positive probability is the same, and
- the expected payoff, given s^*_{-i} to every action to which s^*_i assigns zero probability is at most the expected payoff to any action to which s^*_i assigns positive probability.

Existence

Theorem (Nash, 1950)

Every finite normal form game has a Nash equilibrium.

Proof: Beyond scope of course.
Basic idea: Define set X to be all mixed strategy profiles. Show that it has nice properties (compact and convex).
Define $f : X \mapsto 2^X$ to be the best-response set function, i.e. given s, $f(s)$ is the set all strategy profiles $s' = (s'_1, \ldots, s'_n)$ such that s'_i is i's best response to s'_{-i}.
Show that f satisfies required properties of a fixed point theorem (Kakutani's or Brouwer's).
Then, f has a fixed point, i.e. there exists s such that $f(s) = s$.
This s is mutual best-response – NE!

Interpretations of Nash Equilibria

- Consequence of rational inference
- Focal point
- Self-enforcing agreement
- Stable social convention
- ...

Finding Nash Equilibria

Dominant and Dominated Strategies

For the time being, let us restrict ourselves to pure strategies.

Definition

Strategy s_i is a strictly dominant strategy if for all $s_i' \neq s_i$ and for all s_{-i}

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

Prisoner's Dilemma

	C	D
C	-1,-1	-4,0
D	0, -4	-3,-3

Dominant-strategy equilibria

Dominated Strategies

Definition

A strategy s_i is strictly dominated if there exists another strategy s'_i such that for all s_{-i}

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$$

Definition

A strategy s_i is weakly dominated if there exists another strategy s'_i such that for all s_{-i}

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

with strict inequality for some s_{-i}.

Example

	L	R
U	1,-1	-1,1
M	-1,1	1,-1
D	-2,5	-3,2

	L	R
U	5,1	4,0
M	6,0	3,1
D	6,4	4,4

D is strictly dominated

U and M are weakly dominated

Iterated Deletion of Strictly Dominated Strategies

Algorithm

- Let R_i be the removed set of strategies for agent i
- $R_i = \emptyset$
- Loop
  - Choose i and s_i such that $s_i \in A_i \setminus R_i$ and there exists s_i' such that
    $$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \; \forall s_{-i}$$
  - Add s_i to R_i
  - Continue

Example

	R	C	L
U	3,-3	7,-7	15, -15
D	9,-9	8,-8	10,-10

Some Results

Theorem

If a unique strategy profile s^* survives iterated deletion then it is a Nash equilibrium.

Theorem

If s^* is a Nash equilibrium then it survives iterated elimination.

Weakly dominated strategies cause some problems.

Domination and Mixed Strategies

The definitions of domination (both strict and weak) can be easily extended to mixed strategies in the obvious way.

Theorem

Agent i's pure strategy s_i is strictly dominated if and only if there exists another (mixed) strategy σ_i such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

for all s_{-i}.

Example

	L	R
U	10,1	0,4
M	4,2	4,3
D	0,5	10,2

Strategy $(\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates pure strategy M.

Theorem

If pure strategy s_i is strictly dominated, then so is any (mixed) strategy that plays s_i with positive probability.

Maxmin and Minmax Strategies

- A **maxmin strategy** of player *i* is one that maximizes its worst case payoff in the situation where the other agent is playing to cause it the greatest harm

$$\arg\max_{s_i}\min_{s_{-i}} u_i(s_i, s_{-i})$$

- A **minmax strategy** is the one that minimizes the maximum payoff the other player can get

$$\arg\min_{s_i}\max s_{-i} u_{-i}(s_i, s_{-i})$$

Example

In 2-player games, maxmin value of one player is equal to the minmax value of the other player.

	L	R
U	2,3	5,4
D	0,1	1,2

Calculate maxmin and minmax values for each player (you can restrict to pure strategies).

Zero-Sum Games

- The maxmin value of one player is equal to the minmax value of the other player
- For both players, the set of maxmin strategies coincides with the set of minmax strategies
- Any maxmin outcome is a Nash equilibrium. These are the only Nash equilibrium.

Solving Zero-Sum Games

Let U_i^* be unique expected utility for player i in equilibrium.
Recall that $U_1^* = -U_2^*$.

$$\begin{array}{lll} \text{minimize} & U_1^* & \\ \text{subject to} & \sum_{a_k \in A_2} u_1(a_j, a_k)s_2(a_k) \leq U_1^* & \forall a_j \in A_1 \\ & \sum_{a_k \in A_2} s_2(a_k) = 1 & \\ & s_2(a_k) \geq 0 & \forall a_k \in A_2 \end{array}$$

LP for 2's mixed strategy in equilibrium.

Solving Zero-Sum Games

Let U_i^* be unique expected utility for player *i* in equilibrium. Recall that $U_1^* = -U_2^*$.

$$
\begin{array}{lll}
\text{maximize} & U_1^* & \\
\text{subject to} & \sum_{a_j \in A_1} u_1(a_j, a_k) s_1(a_j) \geq U_1^* & \forall a_k \in A_2 \\
& \sum_{a_j \in A_1} s_1(a_j) = 1 & \\
& s_1(a_j) \geq 0 & \forall a_j \in A_1
\end{array}
$$

LP for 1's mixed strategy in equilibrium.

Two-Player General-Sum Games

LP formulation does not work for general-sum games since agents' interests are no longer diametrically opposed.

Linear Complementarity Problem (LCP)
Find any solution that satisfies

$$
\begin{array}{lr}
\sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) + r_1(a_j) = U_1^* & \forall a_j \in A_1 \\
\sum_{a_j \in A_1} u_2(a_j, a_k) s_1(a_j) + r_2(a_k) = U_2^* & \forall a_k \in A_2 \\
\sum_{a_j \in A_1} s_1(a_j) = 1 \ \sum_{a_k \in A_2} s_2(a_k) = 1 & \\
s_1(a_j) \geq 0, s_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) \geq 0, r_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) s_1(a_j) = 0, r_2(a_k) s_2(a_k) = 0 & \forall a_j \in A_1, a_k \in A_2
\end{array}
$$