

Combinatorial Auctions

This document contains notes from the combinatorial auctions lecture for CS 886.

1 Introduction

Sometimes we want to sell multiple (different) items. How should we do this?

- Sequential auctions
 - But there can be relations between goods
 - Exposure problem
 - Strategic bidding
 - Inefficiencies
- Simultaneous auctions
 - Inefficiencies
 - Not clear what the equilibria look like
- Complicated re-auctioning schemes
- Combinatorial auctions
 - Sell all items at once
 - Agents place bids on *bundles* of items
 - For example
 - * \$1.00 for coffee, and donut.
 - * \$1.00 for coffee and donut, OR, \$1.50 for a tea and cake, but I do not want both.

2 Two perspectives

- Bidders' perspective: "What can I bid on?"
- Auctioneers' perspective (Winner determination problem): Find the allocation that maximizes social welfare.

2.1 Agents' Perspective

An agent will have a valuation function over bundles

$$v_i : 2^X \rightarrow \mathbb{R}$$

where X is the set of all items being auctioned.

There are certain properties of valuation functions:

- No externalities: the valuation function depends only on the set of goods the agent is allocated
- Free disposal: If $S \subseteq T$ then $v_i(S) \leq v_i(T)$
- Nothing-for-nothing: $v_i(\emptyset) = 0$

Other properties describe how a valuation of one good is affected by the presence or absence of another.

- Complementarities: if $v(S \cup T) \geq v(S) + v(T)$ then S, T are complementary.
- Substitutes: $S, T \subseteq X$, $S \cap T = \emptyset$, $v(S \cup T) < v(S) + v(T)$ then S, T are substitutes

The bidder must communicate its valuation information to the auctioneer. The *bidding language* used determines what information can be communicated.

A naive approach is to have the agent specify its valuation for all possible subsets of items. Can we do better?

Desirability

- Conciseness
- Expressiveness
- "Natural"
- Tractable for the auctioneer to process

Possible languages

- Atomic bids: (S, p) where S is the subset and p is the maximum price the bidder is willing to pay for S . We assume that there is an implicit AND between the items in S . That is $S = \{a, b, c\}$ $(S, \$10)$ is interpreted as "I will pay up to \$10 if I get a AND b AND c ."

Not everything can be expressed by atomic bids. For example, additive valuations $v(S) = |S|$ can not be represented by atomic bids.

- OR bids: $(S_1, p_1) \vee (S_2, p_2) \vee \dots$

$v(S) = \max \sum_{i \in W} p_i$ where W is a collection of items $S_j \cap S_i = \emptyset$.

What can OR bids represent? Only bids with no substitutes. For example \$100 for coffee and donut or \$150 for tea and cake, but not both, can not be expressed.

- XOR bids: $(S_1, p_1)\text{XOR}(S_2, p_2)\text{XOR}\dots$

$v(S)$ is defined as $\max_{i|S_i \subseteq S} p_i$

That is, the bidder is willing to accept at most one, but not more than one of the bundles.

Theorem 1 *XOR is fully expressive.*

What about conciseness?

We measure the size of a bid in terms of atomic bids in it. If we consider the additive valuation, $v(S) = |S|$, then if you have m items this can be expressed by

- OR in size m
- XOR in size 2^m

Our bidding languages can also use combinations of operations

- OR-of-XOR: submit an arbitrary number of XOR bids
- XOR-or-OR: submit an arbitrary number of OR bids, and get at most one

Sometimes these languages are concise and sometimes not. XOR-of-OR tends to be less “natural”.

- OR*: simulate XOR by allowing “dummy” bids

$(\{\text{coffee, donut, Dummy}\}, 1.00)\text{OR}(\{\text{tea, cake, Dummy}\}, 1.50)$ forces exclusivity.

Basically, anything we can express in any other language we can express in OR*

- OR-of-XOR in size s then OR* in size s with s dummy bids
- XOR-of-OR in size s then OR* in size e with s^2 dummy bids

2.2 Auctioneer’s Perspective

What sort of auction mechanism should be used? We can use a Generalized Vickrey Auction (GVA), which is a VCG mechanism.

GVA: Find allocation S^* that is feasible and maximizes the sum of total bids

$$S^* = \arg \max_S \sum_i v_i(S)$$

Payments

$$p_i = \sum_{j \neq i} v_j(S') - \sum_{j \neq i} v_j(S^*)$$

where S' is the optimal allocation if agent i did not exist.

Example: 2 items x, y and 3 bidders

- $v_1(\{x, y\}) = 100$
- $v_2(\{x\}) = 75$
- $v_3(\{y\}) = 40$
- all other valuations are 0

Then $S^* =: x \rightarrow 2$ and $y \rightarrow 3$. The payments are $p_1 = 0$, $p_2 = 100 - 40 - 60$ and $p_3 = 100 - 75 = 25$.

Let us look at the auctioneers problem in more detail:

$$\max \sum_{i \in N} \sum_{S \subseteq X} v_i(S) a_{S,i}$$

such that

$$\sum_{S \subseteq X, j \in S} \sum_{i \in N} a_{S,i} \leq 1 \quad \forall j \in X$$

$$\sum_{S \subseteq X} a_{S,i} \leq 1 \quad \forall i \in N$$

$$a_{S,i} \in \{0, 1\} \quad \forall S \subseteq X, i \in N$$

Note: This is an integer program, not a linear program! It is impossible to allocate a fraction of an item to an agent. In particular, this is a set-packing problem, and is NP-hard.

What can be done?

1. Find special cases for which there are poly-time solutions
2. Heuristics and approximations
3. "Brute force"

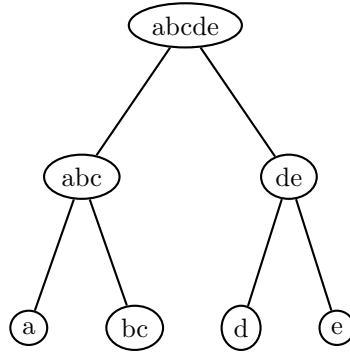
Restricting Classes of Preferences

- Size of a bundle can be at most 2 $\Rightarrow O(n^3)$ where n is the number of items. This can be done by dynamic programming. However, if $|S| \geq 3$ then NP-hard
- Contiguous bids (geographical interpretations):

$A \quad B \quad C \quad D \quad E$

Legal bids are, for example, $\{B\}$, $\{C, D, E\}$, $\{A, B\}$. Illegal bids are, for example, $\{A, C, E\}$, $\{B, D\}$, $\{B, D, E\}$. There is an $O(n^3)$ algorithm.

- Tree-based bids: every 2 sets are either disjoint or one is a subset of another



There is an $O(n^2)$ algorithm.

Using heuristics and approximations

There is a problem here (Nisan and Ronen, 2000, Computationally Feasible VCG Mechanisms).

Result: Any reasonable VCG-based mechanism for combinatorial auctions is not truthful, unless it uses a computationally intractable optimal allocation algorithm.

“Reasonable”: if an item is valued by only a single agent, then that agent gets the item.

Implication: We need to resort to non-VCG mechanisms.

- Lehmann et al (2001) proposed a greedy method that is truthful for single-minded bidders
- Archer and Tardos looked at designing multi-unit combinatorial auctions for single-minded bidders by using a randomized mechanism

1. Let K_j be the number of units available for item j . Define $K'_j = \lfloor (1 - \epsilon) K_j \rfloor$. Solve

$$\max \sum_i v_i x_i$$

such that

$$\sum_{i|j \in S_i} x_i \leq K'_j \quad \forall j$$

$$0 \leq x_i \leq 1$$

This finds the optimal fractional allocation under the constraint that at most K'_j are sold of each item.

2. Round variable x_j to 1 with probability x_j , 0 otherwise

3. Select all agents i with $x_i = 1$ and for whom constraints for all items in their bundle S_i is satisfied
4. Drop each agent with some additional probability θ_i

This is poly-time (linear program) and is truthful in expectation, as well as truthful with high probability. The approximation is $1 + O(\epsilon)$.

2.2.1 “AI Techniques”

This works quite well (IDA*). There are lots of heuristics for node ordering, sophisticated preprocessing, and a careful definition of the search space.

Iterative Combinatorial Auctions

Iterative (or ascending) combinatorial auctions can sometimes simplify both the winner determination problem, as well as the preference elicitation problem for the bidders. There are many varieties of iterative combinatorial auctions, one of which is iBundle (of which there are several variations).

Init: All bundles have price 0

Bid: Agent indicate which bundles they want (they must bid higher than current price to be considered)

Winner determination: Optimal, but the idea is that you are working with a smaller set of bids

Price update: Define $p_i(S)$ to be the price for S by i , and let “unhappy agents” be the agents who have not been allocated a bundle.

$$p^{t+1}(S) = \max \left[p^t(S), \max_{i \in \text{unhappy}} p_i(S) + \epsilon \right]$$

Termination: End the auction when all agents are happy, or all submit the same bids in the sequential rounds.

Agents do not have dominant strategies, but instead the myopic best response is to bid truthfully.