Finite Horizon Bilateral Bargaining (Stahl)

This is an example of a Subgame Perfect Equilibrium (SPE) analysis. [1]

Two players, 1 and 2 bargain on how to split v dollars. The rules are as follows. The game begins in period 1 and player 1 makes an offer of a split to player 2. A split is any real number between $(0, v)$. Player 2 can then accept or reject the split. If she accepts, then the game ends. If she rejects it then the game continues to another round where player 2 gets to make an offer of a split which player 1 can either accept or reject. If no agreement is reached in T periods, then both players get 0. Additionally, there is a discount factor $\delta \in (0, 1)$ so that a dollar received in period t is worth δ^{t-1} dollars in period 1 dollars.

There is a unique SPE, which you can find by backward induction. Assume that T is odd, which means that player 1 makes an offer in period T if no agreement has been reached. Player 2 is willing to accept any offer since she will get zero if she rejects the offer. Therefore, player 1 will offer player 2 zero and player 2 will accept. The payoffs for this equilibrium in the subgame are $(\delta^{T-1}v, 0)$.

Now assume that we are at the subgame starting in period $T-1$ when no previous agreement has been reached. Player 2 gets to make an offer. In a SPE, player 1 will accept any offer which is greater than or equal to the payoff he will get in period T. Thus, player 2 will make an offer to player 1 of y such that $\delta^{T-2}y = \delta^{T-1}v$, and player 1 will accept. This is player 2's best offer among all those that would be accepted. Additionally, player 2 does not want her offer to be rejected since this would lead to period T where her payoff is 0. Therefore, the payoffs in period $T-1$ are $(\delta^{T-2}y, \delta^{T-2}(v - \delta^{T-2}y)$ which is equal to $(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$.

Now assume that we are at the subgame starting in period $T-2$ when no previous agreement has been reached. Player 1 gets to make the offer. In a SPE player 2 will accept any offer which is greater than or equal to what it can get in the next period. Therefore, player 1 will offer x such that $\delta^{T-3} = \delta^{T-2}v - \delta^{T-1}v$ and player 2 will accept. The payoffs are $(\delta^{T-3}(v-x), \delta^{T-3}x)$ which is equal to $(\delta^{T-3}v - \delta^{T-2}v + \delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$.

Continuing on in this manner, we find that the unique SPE when T is odd results in an agreement being reached in period 1, where the payoff to player 1 is

$$\begin{aligned} v^*(T) &= v[1 - \delta + \delta^2 - \ldots + \delta^{T-1}] \\ &= v\left[(1-\delta)\left(\frac{1-\delta^{T-1}}{1-\delta^2}\right) + \delta^{T-1}\right] \end{aligned}$$

and the payoff to player 2 is $v - v^*(T)$.

[1] The presentation comes from *Microeconomic Theory* by Mas-Colell, Whinston, and Green.