Joint Lab Meeting

December 2005

Nearest Polynomials That Factor

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Approximate Factorization Problem [Kaltofen '94]

Given $f = \sum_{i,j} c_{i,j} x^i y^j \in \mathbb{C}[x,y]$ irreducible, find $\tilde{f} \in \mathbb{C}[x,y]$ s.t. $\deg \tilde{f} \leq \deg f$, $\tilde{f}$ factors, and $\|f - \tilde{f}\|$ is minimal.

Problem depends on choice of norm $\|\cdot\|$, and
choice of degree.

Fix

$$\|f\| = \|f\|_2 = \sqrt{\sum_{i,j} |c_{i,j}|^2}$$

Consider different Degree Notions:

Total Degree: $\operatorname{tdeg} f = \max_{\text{monomials } x^i y^j} \{i+j\}$

Rectangular Degree: $\operatorname{mdeg} f = < \deg_x f, \deg_y f >$

Example: $f = x^2 + y^2 - 1$

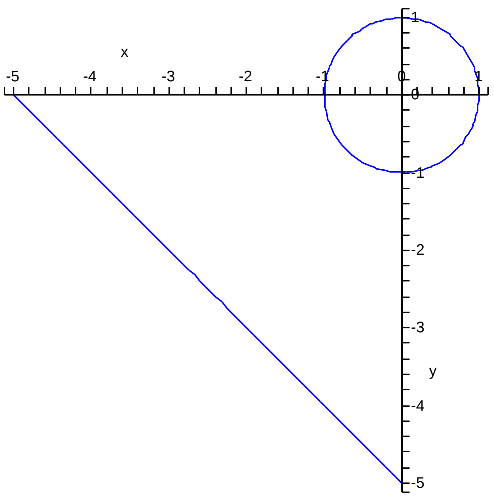
Dangers of Growing the Degree

The condition that $\deg \tilde{f} \leq \deg f$ is essential.
For any polynomial f, $(1+\varepsilon(x+y))\,f$ factors and is arbitrarily nearby.

For our example:

$$f_\varepsilon = (1+\varepsilon(x+y))\,f = \varepsilon x^3 + \varepsilon x^2 y + \varepsilon x y^2 + \varepsilon y^3 + x^2 + y^2 - \varepsilon x - \varepsilon y - 1$$

$$\| f - f_\varepsilon \|_2 = \sqrt{6}\,|\varepsilon|$$



Which Degree?

The polynomial f_ε

has larger total degree **and** rectangular degree than f.

Question:

Is it possible that there is an arbitrarily close $\tilde{f}$ that has larger total degree, but rectangular degree equal $<2,2>$?

Answer: No!

f does not factor iff the *Ruppert matrix* $\mathrm{R}(f)$ is full rank.

The dimensions and structure of $\mathrm{R}(f)$ are determined by the rectangular degree of f.

If $\mathrm{mdeg}\ f = \mathrm{mdeg}\ \tilde{f}$ then $\mathrm{R}(\tilde{f})$ has the same dimensions as $\mathrm{R}(f)$.

Which Degree? (continued)

It is easy to show $\|f - \tilde{f}\|_2 \geq c\,\|\mathrm{R}(f) - \mathrm{R}(\tilde{f})\|_2$. [K&M ISSAC'2003]

So, if $\mathrm{R}(\tilde{f})$ has lower rank than $\mathrm{R}(f)$ then

$$\|\mathrm{R}(f) - \mathrm{R}(\tilde{f})\|_2 \geq \sigma(\mathrm{R}(f)) > 0 \Rightarrow \|f - \tilde{f}\|_2 \geq c_0 > 0.$$

Moral: You should allow the total degree to go up as long as the rectangular degree does not increase, unless you have a compelling reason to preserve the total degree in your approximate factorization.

Question: Does it really make a difference?

Answer: Yes!

Constraining mdeg is less restrictive than constraining tdeg.

For our example, letting the total degree increase gives much better approximate factorizations.

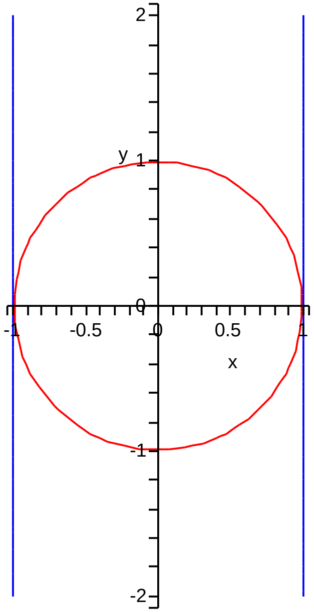
Absolute Nearest: total degree 2

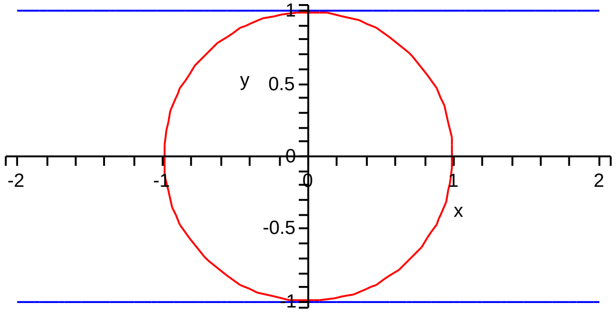
$$f_1 = x + 1,\ f_2 = x - 1$$

$$f_1\, f_2 = x^2 - 1$$

$$\| f_1\, f_2 - (x^2 + y^2 - 1) \|_2 = 1$$

Others at distance 1: $y^2 - 1$ and $x^2 + y^2$





Absolute Nearest: total degree 3

$$f_1 = \left(\left(\sqrt{2}/4 - 1/4\right) a^2 b + \left(1/2 - \sqrt{2}/4\right) ab - 1/2\, b\right) y^2$$

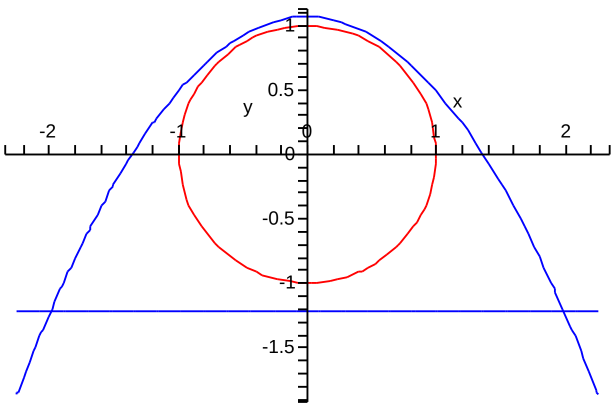
$$+ \left(\left(1/2 - \sqrt{2}/3\right) a^2 - 1/3 + \left(1/3 + \sqrt{2}/6\right) a\right) x + \left(\left(\sqrt{2}/6 - 1/3\right) b + 1/6\, ab + \left(1/3 - \sqrt{2}/3\right) a^2 b\right)$$

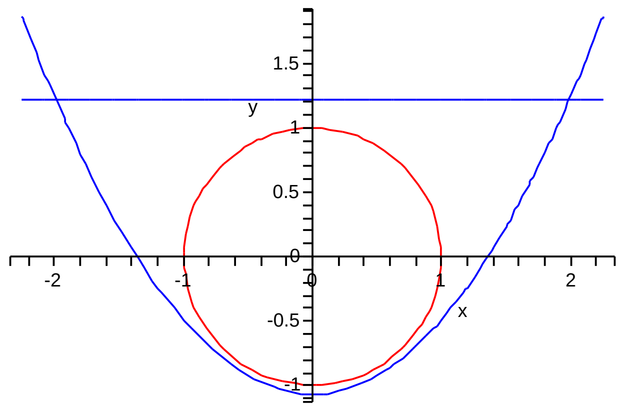
$$f_2 = x + \left(\sqrt{2\,a\,(a^2+2)}\right) / 2\,a$$

$$a = \sqrt[3]{4 + 2\sqrt{2}},\; b = \sqrt{4 + 2\sqrt{2} + 2\,a}$$

$$f_1\, f_2 \approx 1.1022 + 0.1243\,x + .8492\,x^2 + 0.59608\,y^2 - 0.4907\,x y^2$$

$$\|\, f_1\, f_2 - (x^2 + y^2 - 1)\|_2 \approx 0.67272232490474 0$$





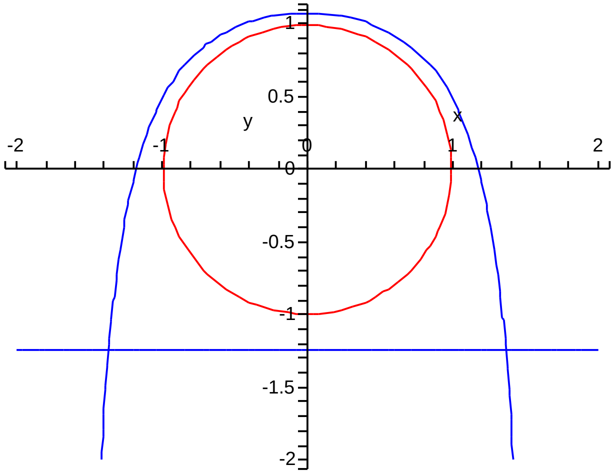
Absolute Nearest: total degree 4

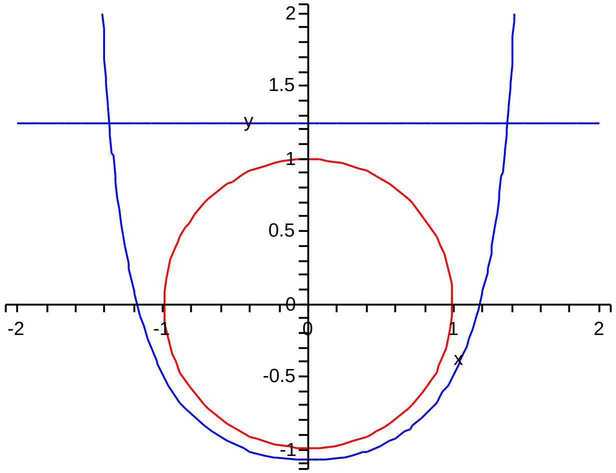
$$f_1 = \left(1/3 - \frac{7}{39}\sqrt{13}\right)x^2y + \left(-1/9\sqrt{3+3\sqrt{13}} - \frac{2}{117}\sqrt{3+3\sqrt{13}}\sqrt{13}\right)x^2 + \left(3/13\sqrt{13}\right)y + \left(1/3\sqrt{3+3\sqrt{13}} - 1/39\sqrt{3+3\sqrt{13}}\sqrt{13}\right)$$

$$f_2 = y - \left(1/3\sqrt{3+3\sqrt{13}}\right)$$

$$f_1\, f_2 \approx -1.1094 - 0.1355\,x + 0.83206\,x^2 + 0.79560\,y^2 - 0.25328\,y^2x - 0.31384\,x^2y^2$$

$$\| f_1\, f_2 - (x^2 + y^2 - 1)\|_2 \approx 0.512801927021870$$





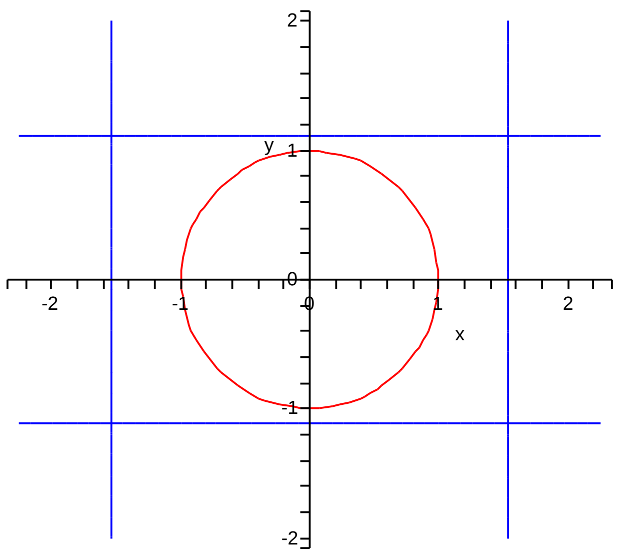
RiSVD finds total degree 4 w/ linear factors

$$f_1 = -0.38903666\,(x+1.54155405)\,(x-1.54155405)$$

$$f_2 = (y-1.11493028)\,(y+1.11493028)$$

$$f_1\,f_2 \approx -1.1507 + .4842\,x^2 + .9257\,y^2 - .3895\,x^2y^2$$

$$\|\,f_1\,f_2 - (x^2+y^2-1)\|_2 \approx 0.66783115070$$



How the Examples were Computed

There is a nice polynomial time algorithm to find the nearest polynomial to f that has a factor of degree 1 [HLK ISSAC'99]
Solve (for example):

$$f - (a_6 xy^2 + a_5 xy + a_4 y^2 + a_3 x + a_2 y + a_1)(x + b_1 y + b_2) = 0$$

a set of linear equations in the a_is. This is a linear system:
$Ma = f$, where M has entries in $\mathbb{R}[b_1, b_2]$.
The residual of the least squares solution of this system,

$$q = \|F - M(M^T M)^{-1} M^T F\|_2,$$

is a rational function in b_1 and b_2 which can be globally minimized by directly solving for and examining its local minima.

Minimize over $\mathbb{C}$ by adding parameters: $b_i \leftarrow c_i + d_i\, i$.