THE MAPLE COMPUTER ALGEBRA SYSTEM

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1 Introduction

Maple is a computer algebra system that provides a unified environment for symbolic, numeric, and graphical computation. It has its own programming language, an interactive environment, a library of mathematical software, and both published and on-line documentation. It comes with a large library of specialized mathematical packages. In this paper we will give a brief introduction to some of the capabilities of the Maple computer algebra system.

Maple developed out of a research project started in 1980 by Gaston Gonnet and Keith Geddes at the University of Waterloo. The central focus of the project has been, and continues to be, the design and implementation of the Maple computer algebra system [1]. The original motivation of the project was to create a computer algebra system that was able to run on the smaller computers that were starting to gain prominence at that time. Maple has a small compiled kernel written in C with the remaining mathematical knowledge coded using its own user-level programming language. The small kernel facilitates portability over a wide range of computer platforms (Unix, Windows, MacIntosh) and allows Maple to run on computers with only modest memory capacity. Its large mathematical library, on the other hand, allows Maple to be powerful enough to meet the mathematical needs of a wide range of users. Maple has a large world-wide base of users in both research laboratories and classrooms.

2 Basic Facilities

2.1 Numerical Computation

The arithmetic supported by systems such as Maple is exact arithmetic rather than the floating-point arithmetic used by typical calculators. In addition, the arithmetic is not limited to the few digits that one encounters with calculators.

```
> 42!;
```

$$1405006117752879898543142606244511569936384000000000$$

```
> ifactor( " );
```

$$(2)^{39}(3)^{19}(5)^{9}(7)^{6}(11)^{3}(13)^{3}(17)^{2}(19)^{2}(23)(29)(31)(37)(41)$$

```
> ( 1117 / 23125 + 441/7)^3 - 1/5;
```

$$\frac{3099310420652919863}{12366455078125}$$

Real numbers are not required to be floating-point numbers in Maple; rather they can be symbolic quantities. Of course, such symbols have properties known to the Maple system. For example,

```
> sqrt( 2 );
```

$$\sqrt{2}$$

```
>  " ^ 2 ;
```

$$2$$

and

```
> sin( 3*Pi );
```

$$0$$

In the above examples " denotes the previous expression. One can also obtain floating-point approximations to real quantities using the command `evalf` (for evaluation over the floating-point numbers):

```
> Pi^2;
```

$$\pi^2$$

```
> evalf( " );
```

$$9.869604404$$

Maple also supports arbitrary-precision arithmetic, that is, the number of digits to be carried can be set by the user. (The default is 10 digits.) For example, a more accurate decimal expansion of π^2 is given by

```
> evalf( "" , 50 );
```

$$9.8696044010893586188344909998761511353136994072408$$

Additional number systems handled by Maple include complex numbers

```
> ( 2 + 3 * I ) / ( 4 - 5 * I );
```

$$-\frac{7}{41}+\frac{22}{41}I$$

where I denotes the square root of -1, and integers modulo m for a specified integer m.

When computing in exact symbolic mode, one may wish to compute exact roots of polynomials and to manipulate such quantities. It is not sufficient to represent such values by numerical approximations. In mathematics there is a formal structure, namely algebraic extension fields, for the representation and manipulation of roots of polynomials. In Maple, algebraic number arithmetic is invoked via the `evala` evaluator. For example, we can let α denote a root of the irreducible polynomial $x^5 + 2x + 2$ by

```
> alias( alpha = RootOf( x^5 + 2*x + 2, x ) ):
```

and then compute the following expression in $\mathbf{Q}(\alpha)$:

```
> (alpha^4 - alpha + 3)^2;
```

$$\left(\alpha^4 - \alpha + 3\right)^2$$

```
>  evala( " );
```

$$4\,\alpha^4 - 2\,\alpha^3 + \alpha^2 - 2\,\alpha + 13$$

Computer algebra algorithms that require the manipulation of roots of polynomials (e.g. integration of rational functions, laplace transforms, etc.) often make use of the `RootOf` construct.

2.2 Symbolic Computation

One of the main differences between computer algebra systems and traditional calculators is the ability to compute with symbols. In Maple, one can do algebraic manipulations such as expanding and factoring polynomials.

```
> (x^2*y + y^2 - 1)^3;
```

$$(\,x^2\,y + y^2 - 1\,)^3$$

```
> expand( " );
```

$$x^6\,y^3 + 3\,x^4\,y^4 - 3\,x^4\,y^2 + 3\,x^2\,y^5 - 6\,x^2\,y^3 + 3\,x^2\,y + y^6 - 3\,y^4 + 3\,y^2 - 1$$

```
> factor( " );
```

$$(\,x^2\,y + y^2 - 1\,)^3$$

and normalizing rational functions

```
> ( x^5 - y^5)/ (x^3 - y^3);
```

$$\frac{x^5 - y^5}{x^3 - y^3}$$

```
> normal( " );
```

$$\frac{x^4 + y\,x^3 + x^2\,y^2 + x\,y^3 + y^4}{x^2 + x\,y + y^2}$$

Similar operations are also possible for symbolic quantities that do not represent polynomial data. For example,

```
> sin( x + 2*y );
```

$$\sin(\,x + 2\,y\,)$$

```
> expand( " );
```

$$2\sin(\,x\,)\cos(\,y\,)^2 - \sin(\,x\,) + 2\cos(\,x\,)\sin(\,y\,)\cos(\,y\,)$$

```
> combine( " , trig );
```

$$\sin(\,x + 2\,y\,)$$

Operations such as factorization and taking the greatest common divisor can be performed in domains such as $\mathbf{Q}[\mathrm{x}]$. In addition, one can compute in domains such as $\mathbf{Z}_p[x]$

```
> x^24 - 2;
```

$$x^{24} - 2$$

```
> Factor( " ) mod 7;
```

$$(\,x^3 + 4\,)\,(\,x^6 + 2\,x^3 + 2\,)\,(\,x^6 + 2\,)\,(\,x^3 + 3\,)\,(\,x^6 + 5\,x^3 + 2\,)$$

```
> Gcd( "" , x^7 - x^6 + 5*x - 5) mod 7;
```

$$x^6 + 5$$

Maple also has a wide variety of solvers. For example, exact solutions for equations are computed as in

```
> solve( x^3+x^2*a+4*x*a+2*x^2+4*a = 0, x);
```

$$-2, -\frac{1}{2}\,a + \frac{1}{2}\,\sqrt{a^2 - 8\,a}, -\frac{1}{2}\,a - \frac{1}{2}\,\sqrt{a^2 - 8\,a}$$

while floating-point solutions are computed via the `fsolve` procedure.

```
> fsolve( x^5 + 2*x + 2, x, complex );
```

$$-.8174710190, -.6111632249 - .9892348119\,I, -.6111632249 + .9892348119\,I,$$
$$1.019898734 - .8770734498\,I, 1.019898734 + .8770734498\,I$$

Other solvers include `rsolve` for solving recurrence equations

```
>  rsolve({f(n) = -3*f(n-1) - 2*f(n-2), f(1) = 1, f(2)=1}, {f});
```

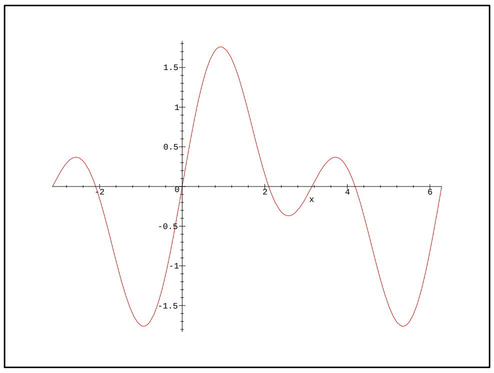
$$\{\mathrm{f}(\,n\,) = -3\,(\,-1\,)^n + (\,-2\,)^n\}$$

`dsolve` for symbolic or numeric solution of differential equations, `pdesolve` for partial differential equations, `linsolve` for linear equations, `isolve` for equations over the integers, and `msolve` for equations over a finite field to name just a few.

2.3 Graphics

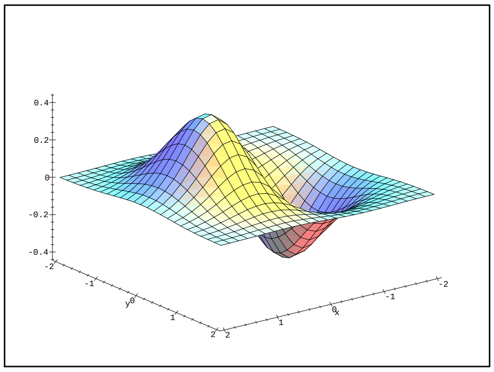
Maple supports a wide variety of graphic functions for visualizing mathematical expressions. One can plot in two-dimensions

```
> plot( sin(x) + sin(2*x), x=-Pi..2*Pi );
```



or three-dimensions

```
> plot3d( x*exp(-x^2 - y^2), x=-2..2, y=-2..2 );
```

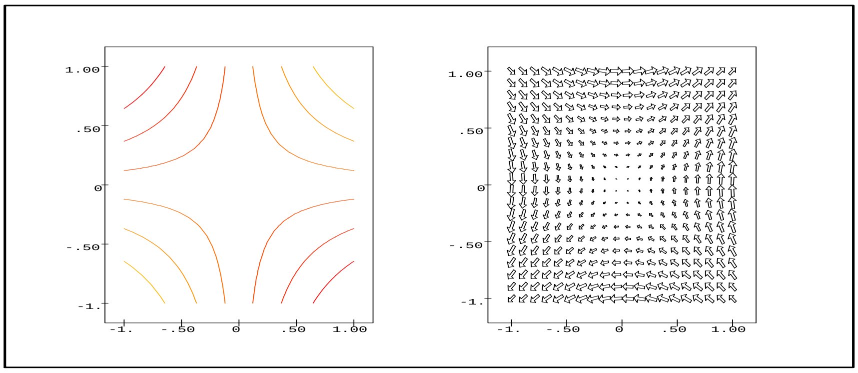


Alternate coordinate systems, drawing styles, and other optional arguments such as axis style, coloring, etc., are available for these commands. There are also a number of specialized plotting routines available in the `plots` and `plottools` packages. The `plots` package includes commands to generate contours, points, gradient, and other vector fields, implicit plots in both two and three dimensions, along with many other types of plots. For example, one can generate the contour plot and the gradient plot of the surface of $z = sin(xy)$

```
> with(plots):
> p:=contourplot(sin(x*y), x= -1..1, y=-1..1, axes=boxed,contours=7):
> q:=gradplot(sin(x*y), x= -1..1,y=-1..1,axes=boxed,arrows=THICK):
```

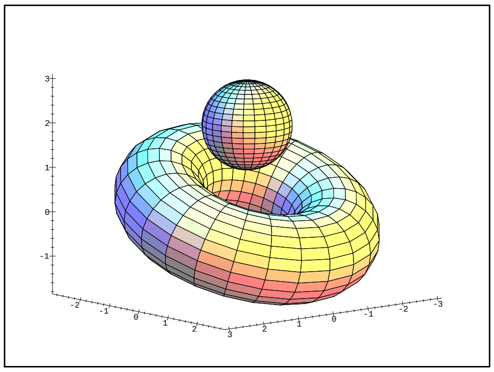
and display the two plots side by side in an array of plots via

```
> display( array( [p, q] ) );
```



The `plottools` package allows one to manipulate with basic graphical objects, for example

```
> with(plottools):
> display( sphere([ 0, 0, 2]) ,  rotate( torus([0,0,0]), Pi/8,0,0) );
```



Additional objects include geometric shapes such as the tetrahedron, octahedron, and ellipses, to name just a few. One can rotate, scale, and translate such objects or transform them using any function.

2.4 Programming

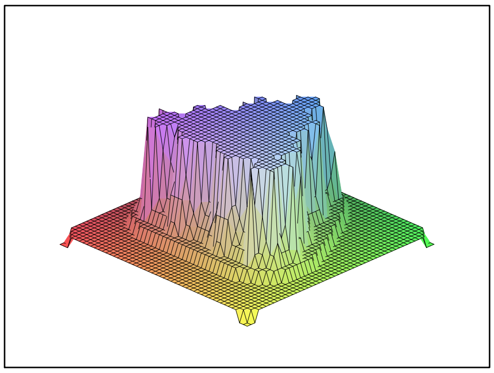
Maple comes with its own programming language. This language is Pascal-like with all the usual constructions such as `if-then-else` and `for` and `while` loops. For example, the procedure

```
> mandelbrotSet := proc(x, y)
>     local z, m;
>         z := evalf( x + y*I );  m := 0;
>         to 10 while abs(z) < 2 do
>             z := z^4 + ( x + y*I );
```

```
>                 m := m + 1
>             od;
>             m
>     end:
```

counts the number of iterations (to a maximum of ten) needed to exit the disk of radius two from various starting points. In this case one can visualize such a function using `plot3d`:

```
> plot3d(  mandelbrotSet , -1.5..1.5, -1.5..1.5, grid = [50, 50] );
```



In addition to the basic facilities found in most computer languages, Maple supports a large set of programming tools and constructions that work with mathematical expressions.

2.5 Language Translation

One very useful feature of Maple is its ability to translate mathematical expressions into alternate computer languages. For example, the mathematical expression

```
> s := ln(x)^2 + 2*ln(x)^3;
```

$$s := \ln(\,x\,)^2 + 2\ln(\,x\,)^3$$

can be differentiated

```
> diff(s,x);
```

$$2\,\frac{\ln(\,x\,)}{x} + 6\,\frac{\ln(\,x\,)^2}{x}$$

and then converted to a Fortran expression

```
> fortran(");

      t0 = 2*alog(x)/x+6*alog(x)**2/x
```

One can ask that the Fortran code be optimized in the sense of pulling out common subexpressions to achieve more efficient evaluation.

```
> fortran(",optimized);

      t1 = alog(x)
      t2 = 1/x
      t4 = t1**2
      t6 = 2*t1*t2+6*t4*t2
```

There are similar commands for translating Maple expressions into the C programming language. In addition, one can convert Maple expressions into text processing languages such as `latex` or `eqn`:

```
>  linalg[hilbert](4);
```

$$\left[\begin{array}{cccc} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{array}\right]$$

```
> latex( " );

\left [\begin {array}{cccc} 1&1/2&1/3&1/4
\\\noalign{\medskip}1/2&1/3&1/4&1/5\\\noalign{\medskip}1
/3&1/4&1/5&1/6\\\noalign{\medskip}1/4&1/5&1/6&1/7
\end {array}\right ]
```

This allows users to incorporate the results of Maple computations into their documents in an error-free manner.

3 Calculus

Since Maple has the ability to do algebraic manipulations, it does not come as a surprise that one can preform many computations from calculus.

```
> Limit( sin(Pi*x) / (x-1), x=1 ) = limit( sin(Pi*x) / (x-1), x=1 );
```

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x-1} = -\pi$$

```
> f := x^(x^x);
```

$$f := x^{(x^x)}$$

```
> diff(f, x);
```

$$x^{(x^x)}\left(x^x(\ln(x)+1)\ln(x)+\frac{x^x}{x}\right)$$

```
> series(cos(x)*exp( a*x^2), x);
```

$$1+\left(a-\frac{1}{2}\right)x^2+\left(\frac{1}{2}a^2-\frac{1}{2}a+\frac{1}{24}\right)x^4+\mathrm{O}(x^6)$$

Even indefinite integration

```
> g := sin(a*x)*exp(-b*x);
```

$$g:=\sin(x\,a)\,\mathrm{e}^{(-b\,x)}$$

```
> int(g, x);
```

$$-\frac{a\,\mathrm{e}^{(-b\,x)}\cos(x\,a)}{b^2+a^2}-\frac{b\,\mathrm{e}^{(-b\,x)}\sin(x\,a)}{b^2+a^2}$$

is possible with Maple. One can verify the answer by differentiating

```
> diff(" , x);
```

$$\frac{a^2\,\mathrm{e}^{(-b\,x)}\sin(x\,a)}{b^2+a^2}+\frac{b^2\,\mathrm{e}^{(-b\,x)}\sin(x\,a)}{b^2+a^2}$$

and then simplifying to get the original function

```
> simplify( " );
```

$$\sin(x\,a)\,\mathrm{e}^{(-b\,x)}$$

The topic of indefinite integration is an interesting one because Maple does not rely only on heuristics (i.e. intelligent guess work) to solve integrals. Rather, it uses the Risch integration algorithm [2], a decision procedure for determining if closed form solutions exist. Thus, for the integral

```
> int( exp(-x^3)*x^4*log(x), x);
```

$$\int \mathrm{e}^{(-x^3)}\,x^4\ln(x)\,dx$$

it is not simply the case that Maple cannot find an answer. Rather, Maple has used the Risch algorithm to prove that no closed form solution exists, at least no closed form solution in terms of elementary functions such as exponentials, logarithms, and algebraic functions.

One can perform manipulations with integrals even when no closed form solution can be determined. For example, one can differentiate or take series expansions

```
> series( " , x, 14);
```

$$\left(\frac{1}{5}\ln(x)-\frac{1}{25}\right)x^5+\left(-\frac{1}{8}\ln(x)+\frac{1}{64}\right)x^8+\left(\frac{1}{22}\ln(x)-\frac{1}{242}\right)x^{11}+\mathrm{O}(x^{14})$$

In addition, not finding a closed form expression for an indefinite integral does not necessarily imply that there is no closed form expression for a corresponding definite integral. Thus we have

```
> int( exp(-x^3)*x^4*log(x), x=0..infinity );
```

$$\frac{1}{9}\Gamma\left(\frac{2}{3}\right) - \frac{2}{27}\Gamma\left(\frac{2}{3}\right)\gamma + \frac{1}{81}\Gamma\left(\frac{2}{3}\right)\pi\sqrt{3} - \frac{1}{9}\Gamma\left(\frac{2}{3}\right)\ln(3)$$

One can also calculate formulas for special integrals, for example elliptic integrals. Such integrals, which appear in numerous applications (e.g. computing the arc length of a trigonometric function) reduce to a normal form in terms of Legendre elliptic functions. We can show one of these reduction formulas by taking advantage of Maple's `assume` facility. For example,

```
> assume( k>0, k<1 );
> Int(x^2/sqrt( (1-x^2)*(1-k^2*x^2)), x= 0..1/k);
```

$$\int_0^{\frac{1}{k\tilde{}}} \frac{x^2}{\sqrt{(1-x^2)(1-k\tilde{}^{\,2}x^2)}}\,dx$$

```
> value( " );
```

$$\frac{\mathrm{EllipticK}(k\tilde{})}{k\tilde{}^{\,2}} - \frac{\mathrm{EllipticE}(k\tilde{})}{k\tilde{}^{\,2}} - I\,\mathrm{EllipticPi}\left(1-k\tilde{}^{\,2}, \sqrt{1-k\tilde{}^{\,2}}\right)$$

gives a formula in the case where the integrand crosses one of the branch cuts of the square root function.

4 Conclusions

We have presented a brief introduction to the functionality which is included in the Maple computer algebra system. We have not described such topics as special functions, their manipulation and evaluation to arbitrary precision over the complex plane, advanced graphics, linear algebra, algebraic functions, solvers such as those for exact solutions of partial differential equations, and many other topics.

References

[1] B.W. Char, K.O. Geddes, W.M. Gentleman, and G.H. Gonnet. The design of Maple: A compact, portable, and powerful computer algebra system. Appears in *Computer Algebra (Proceedings of EUROCAL '83)*, J.A. van Hulzen (ed.), Lecture Notes in Computer Science, No. 162, Springer-Verlag, Berlin, 1983, pp. 101-115.

[2] K.O. Geddes, S.R. Czapor and G. Labahn. Algorithms for Computer Algebra. Kluwer Academic Publishers, Boston, 1992, 585 pages.