Secret sharing on large girth graphs

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Outline

1. Introduction
   - Secret Sharing
   - Graph Secret Sharing
   - Efficiency

2. Method
   - Definitions: complexity
   - Graph Example:
   - Main problem:

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Secret Sharing

Secret Sharing Scheme (Shamir 79)

- Secret: s
- Participants: $P = \{P_1, ..., P_n\}$
- Shares: $\{s_1, ..., s_n\}$
- Access Structure: $\mathcal{A} \subseteq 2^P$

Correctness: Every authorized set $B \in \mathcal{A}$ can recover s.
Privacy: Any unauthorized set $B \notin \mathcal{A}$ cannot learn anything about s.

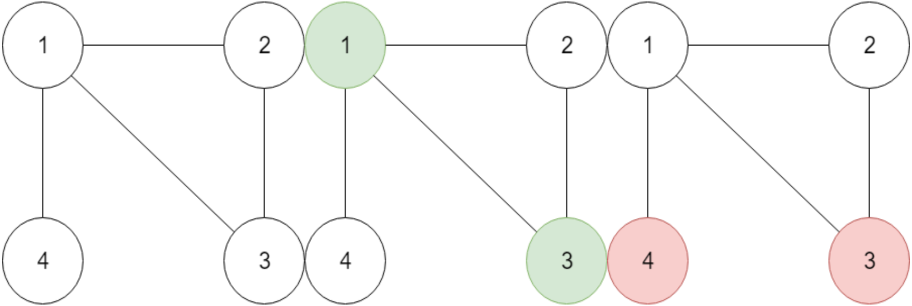
(t,n) Threshold schemes

- Participants: $P = \{P_1, ..., P_n\}$
- Access Structure: $\mathcal{A} = \{A \subseteq P : |A| \geq t\}$

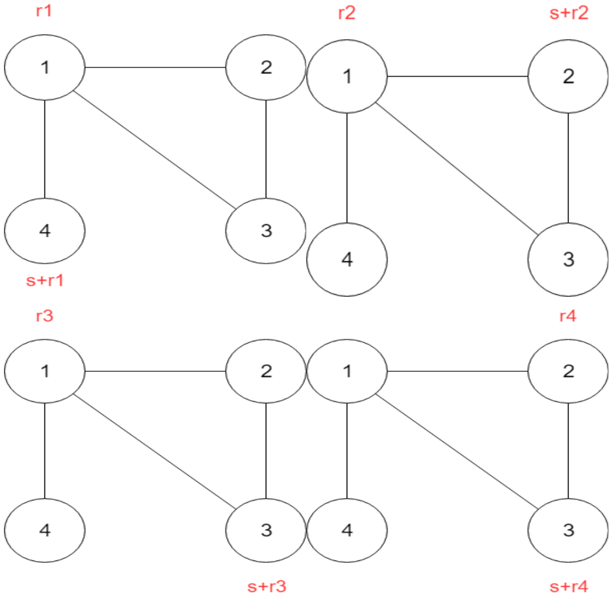
Outline

Graph Secret Sharing Scheme

- Participants: The participants are the vertices of a graph G =(V, E)
- Access Structure: A set of participants is qualified if there is an edge $e \in E$ with endpoints in this set.



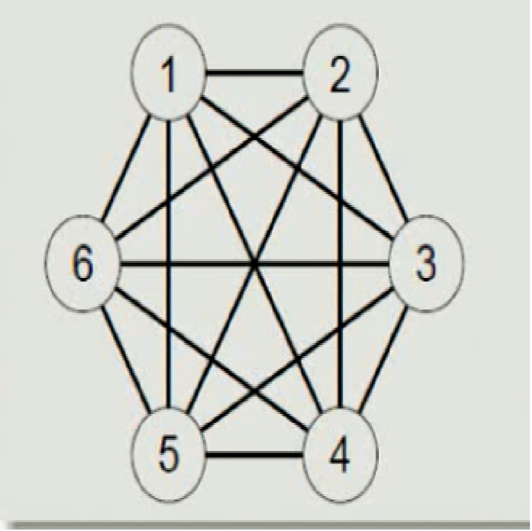
Graph Secret Sharing



All minimal authorized subset are of size 2.
Simple but interesting case
First step of obtaining general results.

The secret s is shared interdependently for every edge.

Graph Threshold Secret Sharing



Clique: It defines threshold access structure of threshold 2.

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Efficiency

Efficiency of a secret sharing scheme

- The efficiency of a secret sharing scheme is measured by:
  The ratio between The maximum size of the shares given to any participant and the size of the secret
- Using Shannons entropy to measure the complexity of a secret sharing scheme

Shannons entropy

Shannons entropy

- Shannons entropy measures the amount of uncertainty of a distribution.
- The requirements of secret sharing can be formulized by using entropy.

Shannons entropy

Let random variable X takes values $x_1, ..., x_n$ with probabilities $p_1, ..., p_n$.
The Shannons entropy of X is defined by
$H(X) = -\sum_{i=1}^{n} p(x_i) log(p(x_i)) = -E(log(Pr[x]))$

Outline

Definitions: complexity

Definitions

- H(.) denotes the Shannon entropy
- Complexity $c(\mathcal{A}) = \inf_{S} \max_{v \in V} \frac{H(\varepsilon_v)}{H(\varepsilon_s)}$
- ideal access structure: when c(A) = 1
- $f : 2^V \to \mathbb{R}^+$ a normalized entropy function
- $f(x) = \frac{H(x)}{H(\varepsilon_s)}$

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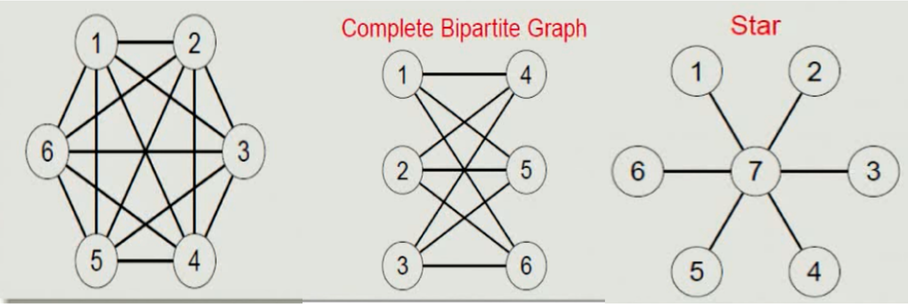
Graph Example:



Graph Example:

Problem

Characterization of ideal schemes?

- matroid theory elements
- This problem isnt meant for this paper

Problem

Estimation/determination of the complexity for a given system

Sporadic Example:

Theorem (Csirmaz, 09)

Let G = (V, E) be a graph of girth at least 6 and with no adjacent vertices of degree at least 3. Then c(G) $= 2 - \frac{1}{d}$, where d is the maximal degree.

Theorem (Csirmaz, 07)

Let$H_d$be the d-dimensional hypercube. Then c$(H_d) = \frac{d}{2}$

Theorem (Csirmaz, 12)

Let T be a tree, with maximal core of size d. Then c(T) $= 2 - \frac{1}{d}$. A subset of the vertices of a tree is a core if it induces a connected subgraph and for each vertex in the subset one finds a neighbor outside the subset.

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main problem:

Problem

Does there exist large girth graphs with large complexity?

- recursive family of d-regular graphs of girth 6 with complexity $(d+1)/2$ (van Dijk and Blundo et al. 95)
- d-dimensional hypercube (girth 4) with complexity $d/2$ (Csirmaz 07)
- graphs of girth at least 6 with no adjacent vertices of degree at least 3 and complexity 2-1/d (Csirmaz, LP 09)
- trees (girth 0) with complexity 2-1/d. (Csirmaz, Tardos 12)

Lower bounds for the complexity:

Entropy method (Blundo, 95)

- $f : 2^V \to \mathbb{R}^+$ a normalized entropy function, such that:
- f is monotone and submodular; moreover f(Ø) = 0;
- f(A) + 1 $\leq$ f(B) if A $\subset$ B, A is independent and B is not (strict monotonicity)
- f(AC) + f(BC) $\geq$ f(C) + f(ABC) + 1 if C is empty or independent, AC and BC are qualified (strict submodularity).
- If for any such function f we have f(v) $\geq \alpha$ for some vertex v of G, then the complexity of G is at least α.

upper bounds for the complexity :

Only solvable for small examples as huge LP problem.

Theorem

- For any normalized entropy function f on Gd :

$$H(X) = -\sum_{v \in G_d} f(v) - f(G_d) \geq \frac{d}{2}|G_d| - 1$$

- For every graph Gd $\in \mathcal{G}_d$:

$$c(Gd) \geq \frac{d+1}{2}$$

Constructions:

Theorem (Stinson,94)

Let G = (V, E) covered by ideal graphs such that every vertex is contained in at most v and every edge is contained in at least e such graphs. Then c(G)$\leq \frac{v}{e}$.

Corollary (Stinson,94)

c(G)$\leq \frac{d+1}{2}$, d is the maximal degree (covering with stars).

Corollary (Pyber, 97)

c(G)$\leq c\frac{n}{log n}$, d is the maximal degree (covering with complete bipartite graphs)).

The graph family $\mathcal{G}_d$:

Recursive construction

- $G_2 = (A_2, B_2)$ is the cycle of even length
- $G_d = (A_d, B_d)$ has been constructed, take several copies of Gd
- G_{d+1} : add an (arbitrary) 1-factor between B_d^i and A_d^{i+1} for all i

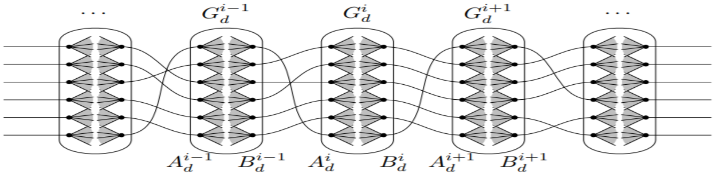

Figure 1: Structure of the graph G_{d+1}

The graph family $\mathcal{G}_d$:

Definition

$\mathcal{G}_d$ consists of all graphs G_d constructed this way

Claim

Every Gd is a d -regular bipartite graph with, and hence $c(G_d) \leq \frac{(d+1)}{2}$ by Stinsons bound

Theorem

For every graph $G_d \in \mathcal{G}_d$:
$c(G_d) = \frac{d+1}{2}$

Summary

- Using the entropy method, it was shown that the general upper bound $(d+1)/2$ on the complexity of graph based secret sharing schemes, known as Stinsons bound, is tight for a large class of inductively defined d-regular bipartite graphs.
- This result refutes the widely believed conjecture that large girth graphs have bounded complexity due to the exponentially diminishing interaction between the shares assigned to the vertices.