Introduction to Quantum Information Processing

CS 667 / PH 767 / CO 681 / AM 871

Lecture 18 (2009)

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Grover’s quantum search algorithm

Quantum search problem

Given: a black box computing $f : \{0,1\}^n \rightarrow \{0,1\}$

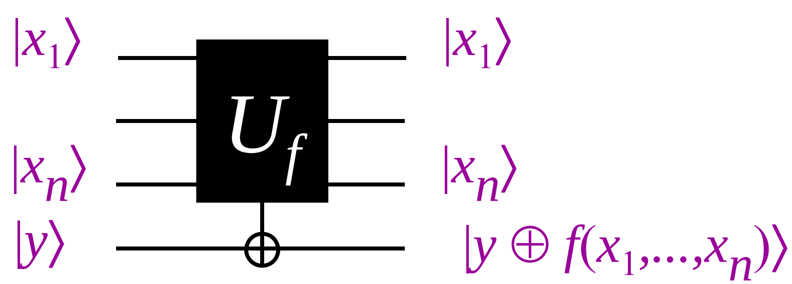
Goal: determine if f is ***satisfiable*** (if $\exists x \in \{0,1\}^n$ s.t. $f(x) = 1$)

In positive instances, it makes sense to also ***find*** such a satisfying assignment x

Classically, using probabilistic procedures, order 2^n queries are necessary to succeed—even with probability ¾ (say)

Grover's ***quantum*** algorithm that makes only $O(\sqrt{2^n})$ queries

Query:



[Grover '96]

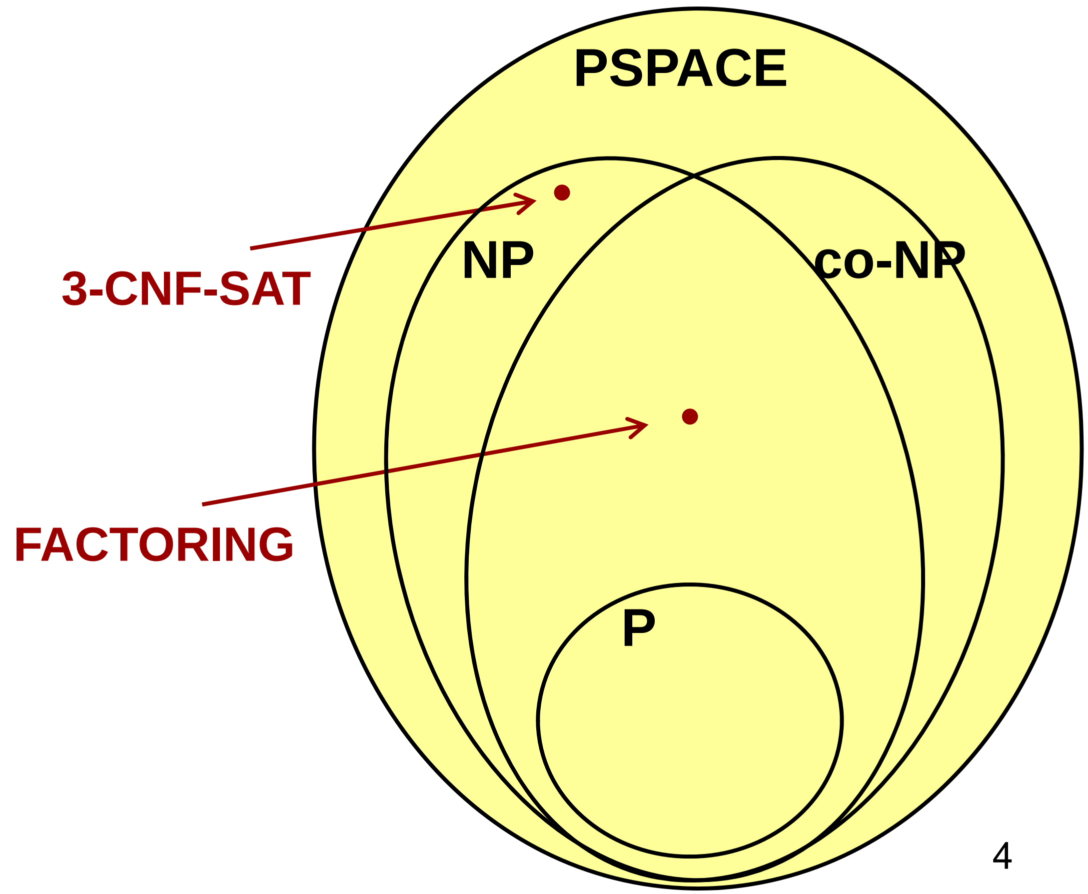
Applications of quantum search

The function f could be realized as a **3-CNF formula**:

$$f\left(x_1,\ldots,x_n\right)=\left(x_1\vee\overline{x}_3\vee x_4\right)\wedge\left(\overline{x}_2\vee x_3\vee\overline{x}_5\right)\wedge\cdots\wedge\left(\overline{x}_1\vee x_5\vee\overline{x}_n\right)$$

Alternatively, the search could be for a certificate for any problem in **NP**

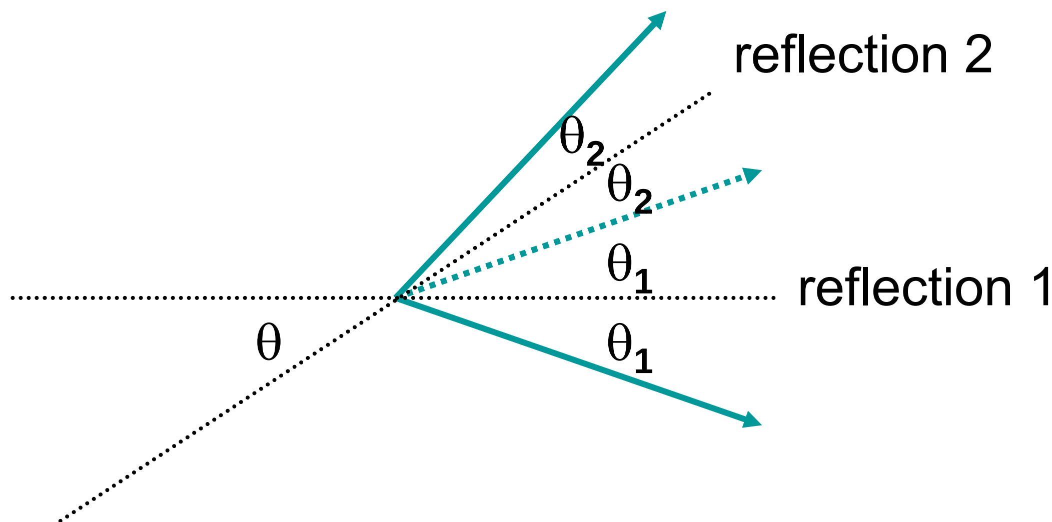
The resulting quantum algorithms appear to be ***quadratically*** more efficient than the best classical algorithms known



Prelude to Grover's algorithm:

two reflections = a rotation

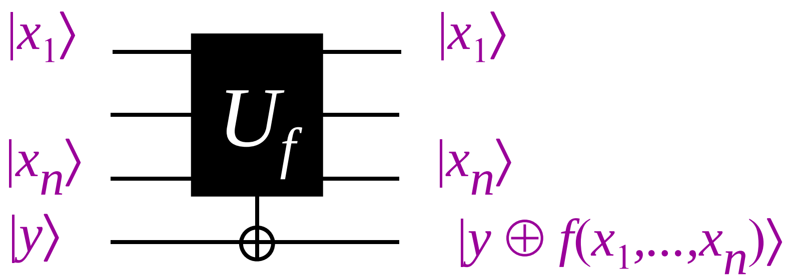
Consider two lines with intersection angle θ:



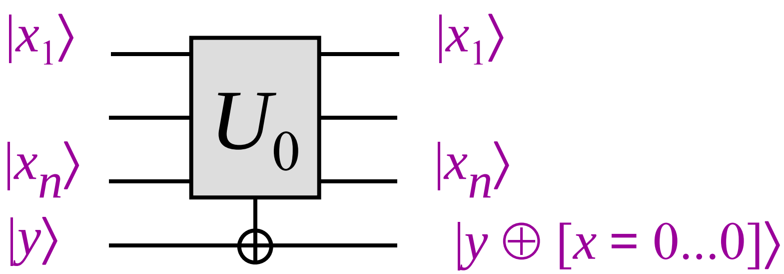
Net effect: rotation by angle 2θ, ***regardless of starting vector***

Grover's algorithm: description I

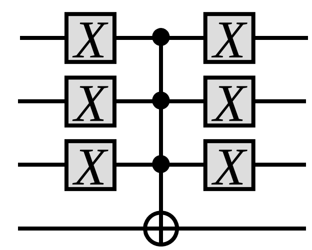
Basic operations used:



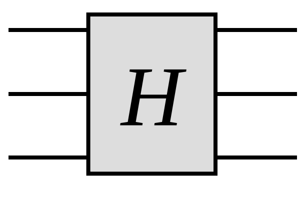
$$U_f\,|x\rangle|-\rangle = (-1)^{f(x)}\,|x\rangle|-\rangle$$



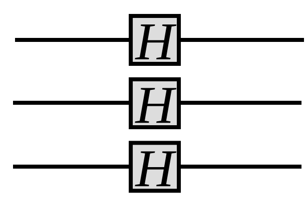
Implementation?



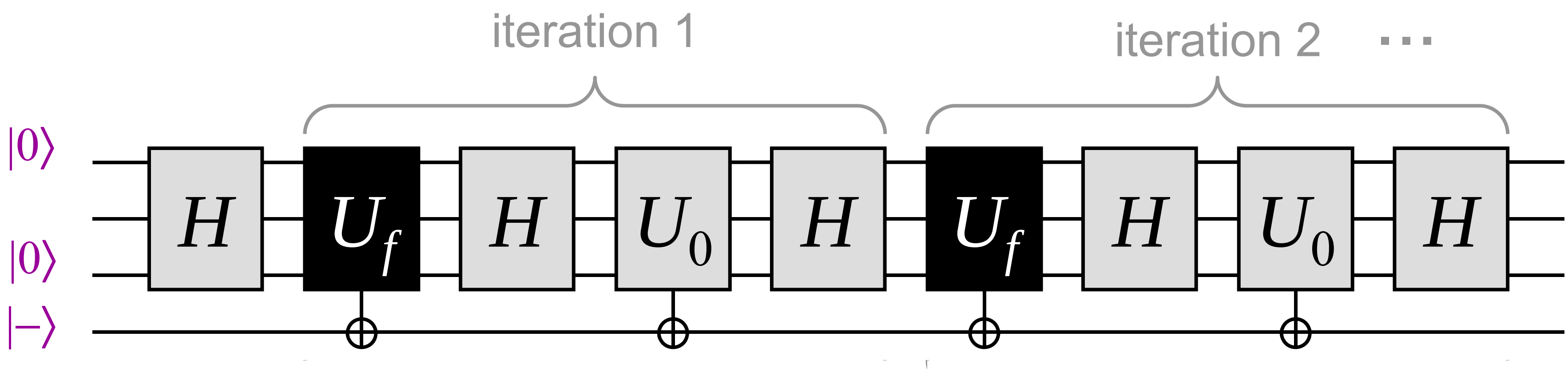
$$U_0\,|x\rangle|-\rangle = (-1)^{[x\,=\,0\ldots0]}|x\rangle|-\rangle$$



Hadamard



Grover's algorithm: description II



1. construct state $H|0\ldots0\rangle|-\rangle$
2. <u>repeat</u> k <u>times</u>:
   apply $-HU_0HU_f$ to state
3. measure state, to get $x \in \{0,1\}^n$, and check if $f(x) = 1$

(The setting of k will be determined later)

Grover's algorithm: analysis I

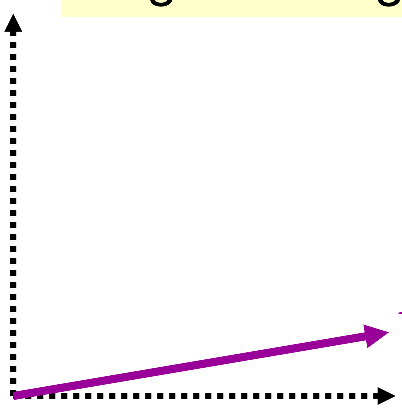
Let $A = \{x \in \{0,1\}^n : f(x) = 1\}$ and $B = \{x \in \{0,1\}^n : f(x) = 0\}$

and $N = 2^n$ and $a = |A|$ and $b = |B|$

Let $|A\rangle = \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle$ and $|B\rangle = \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle$

Consider the space spanned by $|A\rangle$ and $|B\rangle$

$|A\rangle$ ← goal is to get close to this state

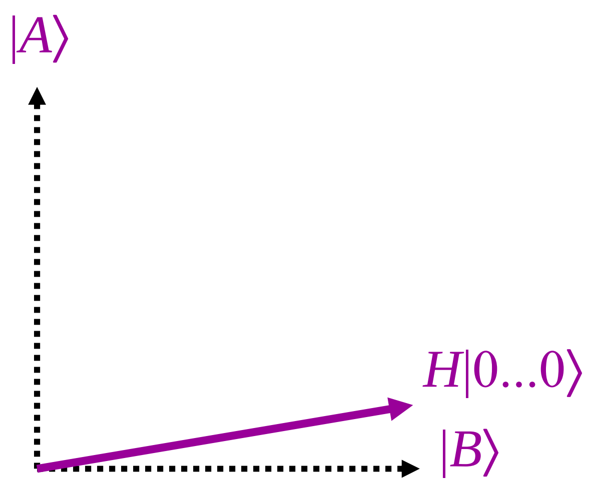


$H|0\ldots0\rangle$ $= \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle = \sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle$

$|B\rangle$

Interesting case: $a << N$

Grover's algorithm: analysis II



Algorithm: $(-HU_0HU_f)^k H|0...0\rangle$

Observation:

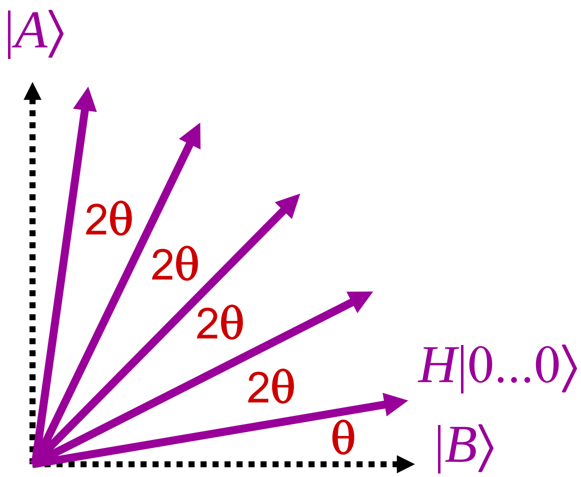
U_f is a reflection about $|B\rangle$: $U_f|A\rangle = -|A\rangle$ and $U_f|B\rangle = |B\rangle$

Question: what is $-HU_0H$? U_0 is a reflection about $H|0...0\rangle$

Partial proof:

$-HU_0HH|0...0\rangle = -HU_0|0...0\rangle = -H(-|0...0\rangle) = H|0...0\rangle$

Grover's algorithm: analysis III



Algorithm: $(-HU_0HU_f)^k H|0...0\rangle$

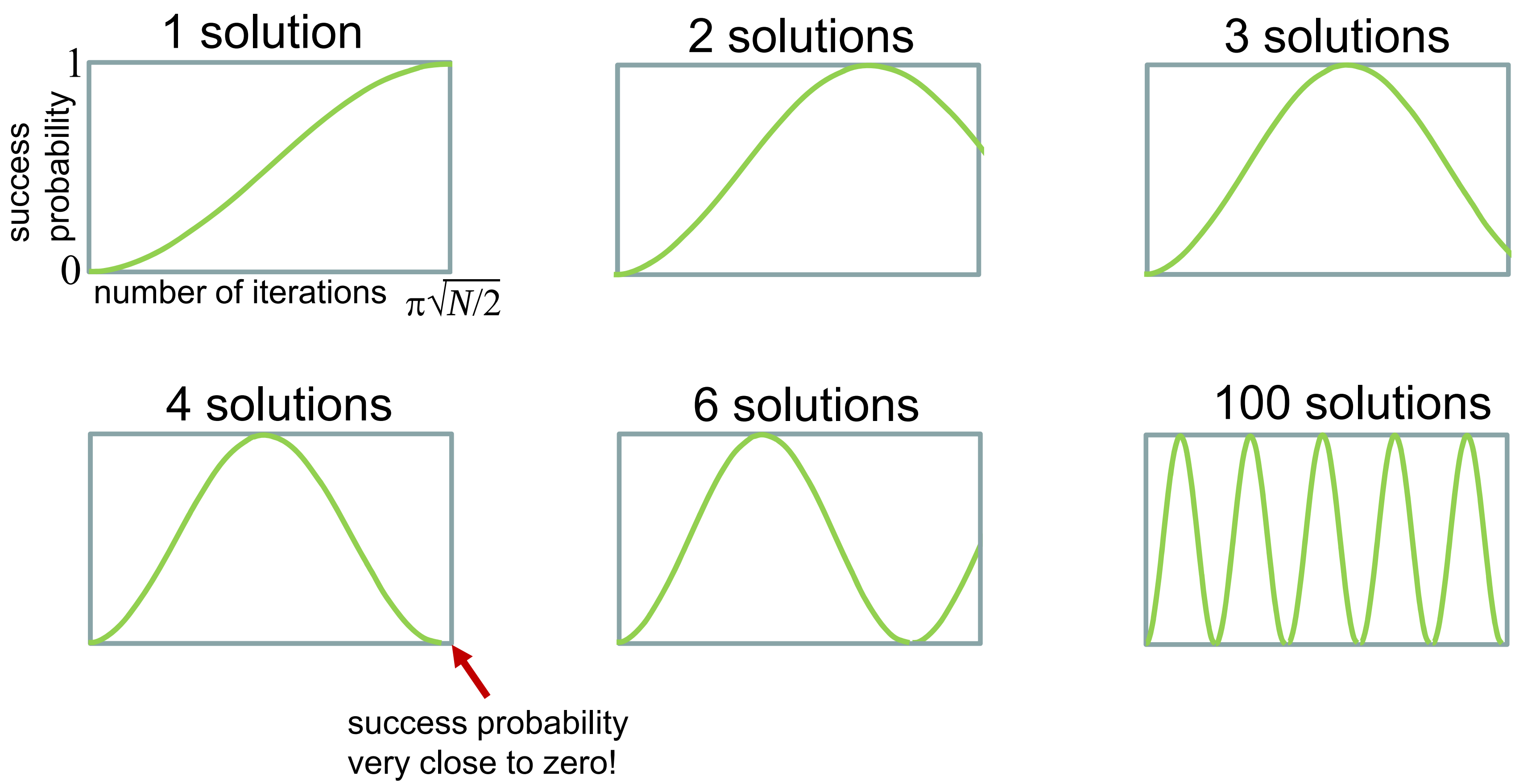
Since $-HU_0HU_f$ is a composition of two reflections, it is a rotation by 2θ, where $\sin(\theta)=\sqrt{a/N} \approx \sqrt{a/N}$

When $a = 1$, we want $(2k+1)(1/\sqrt{N}) \approx \pi/2$, so $k \approx (\pi/4)\sqrt{N}$

More generally, it suffices to set $k \approx (\pi/4)\sqrt{N/a}$

Question: what if a is not known in advance?

Unknown number of solutions



Choose a ***random*** k in the range to get success probability > 0.43

Optimality of Grover's algorithm

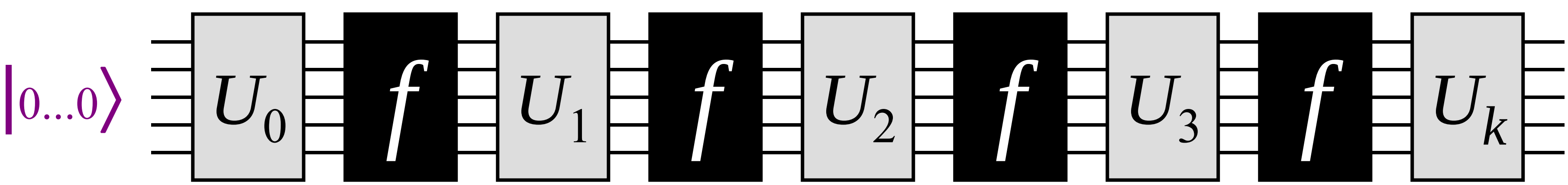
Optimality of Grover's algorithm I

Theorem: any quantum search algorithm for $f: \{0,1\}^n \rightarrow \{0,1\}$ must make $\Omega\left(\sqrt{2^n}\right)$ queries to f (if f is used as a black-box)

Proof (of a slightly simplified version)**:**

Assume queries are of the form

and that a k-query algorithm is of the form

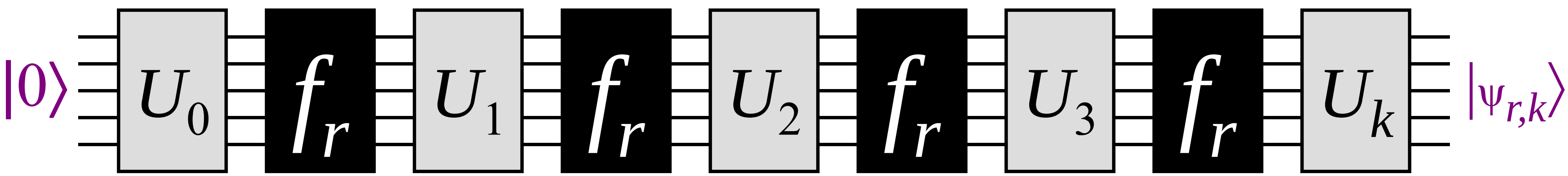


where U_0, U_1, U_2, ..., U_k, are arbitrary unitary operations

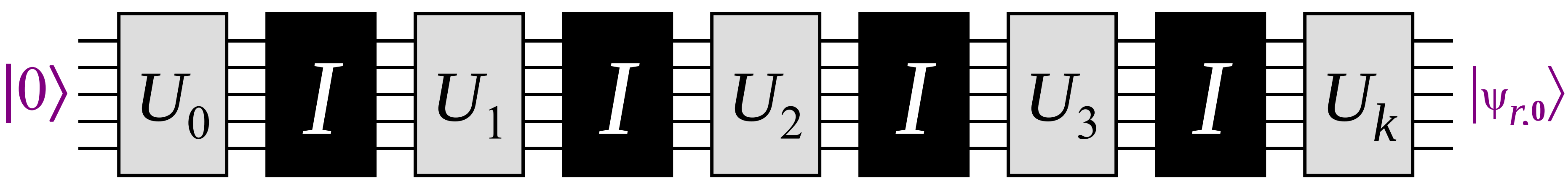
Optimality of Grover's algorithm II

Define $f_r : \{0,1\}^n \rightarrow \{0,1\}$ as $f_r(x) = 1$ iff $x = r$

Consider



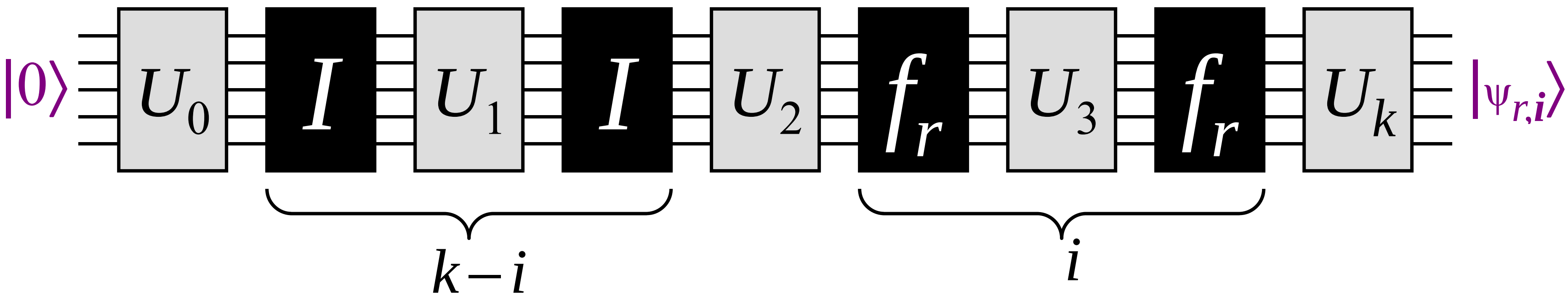
versus



We'll show that, averaging over all $r \in \{0,1\}^n$, $\| \, |\psi_{r,k}\rangle - |\psi_{r,0}\rangle \, \| \leq 2k/\sqrt{2^n}$

Optimality of Grover's algorithm III

Consider



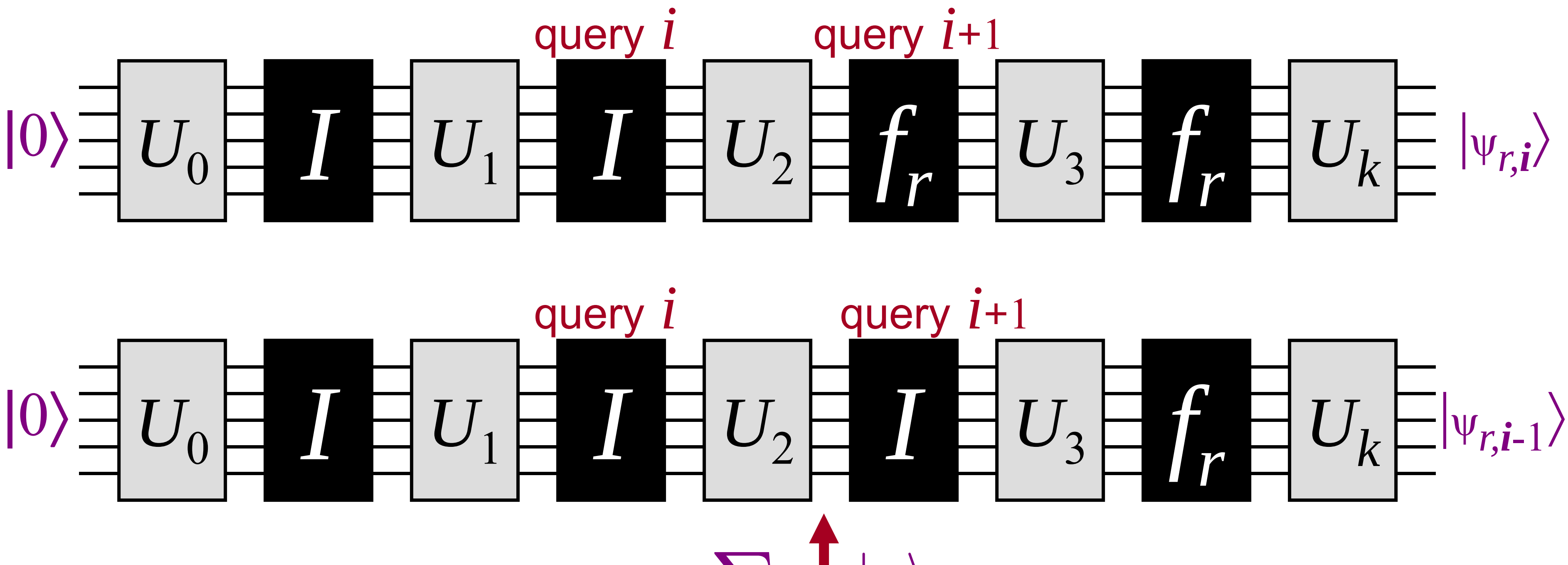
Note that

$$|\psi_{r,k}\rangle - |\psi_{r,\mathbf{0}}\rangle = \big(|\psi_{r,k}\rangle - |\psi_{r,k-1}\rangle\big) + \big(|\psi_{r,k-1}\rangle - |\psi_{r,k-2}\rangle\big) + \ldots + \big(|\psi_{r,1}\rangle - |\psi_{r,0}\rangle\big)$$

which implies

$$\|\, |\psi_{r,k}\rangle - |\psi_{r,\mathbf{0}}\rangle \,\| \leq \|\, |\psi_{r,k}\rangle - |\psi_{r,k-1}\rangle \,\| + \ldots + \|\, |\psi_{r,1}\rangle - |\psi_{r,0}\rangle \,\|$$

Optimality of Grover's algorithm IV



$$\sum_x \alpha_{i,x} |x\rangle$$

$\| \, |\psi_{r,\boldsymbol{i}}\rangle - |\psi_{r,\boldsymbol{i}-1}\rangle \, \| = |2\alpha_{i,r}|$, since query only negates $|r\rangle$

Therefore, $\| \, |\psi_{r,k}\rangle - |\psi_{r,\mathbf{0}}\rangle \, \| \leq \displaystyle\sum_{i=0}^{k-1} 2|\alpha_{i,r}|$

Optimality of Grover's algorithm V

Now, averaging over all $r \in \{0,1\}^n$,

$$\frac{1}{2^n}\sum_r \left\| \left|\psi_{r,k}\right\rangle - \left|\psi_{r,0}\right\rangle \right\| \leq \frac{1}{2^n}\sum_r \left(\sum_{i=0}^{k-1} 2\left|\alpha_{i,r}\right| \right)$$

$$= \frac{1}{2^n}\sum_{i=0}^{k-1} 2\left(\sum_r \left|\alpha_{i,r}\right| \right)$$

$$\leq \frac{1}{2^n}\sum_{i=0}^{k-1} 2\left(\sqrt{2^n}\right) \qquad \text{(By Cauchy-Schwarz)}$$

$$= \frac{2k}{\sqrt{2^n}}$$

Therefore, for ***some*** $r \in \{0,1\}^n$, the number of queries k must be $\Omega\left(\sqrt{2^n}\right)$, in order to distinguish f_r from the all-zero function

This completes the proof