Introduction to Quantum Information Processing

CS 667 / PH 767 / CO 681 / AM 871

Lecture 15 (2009)

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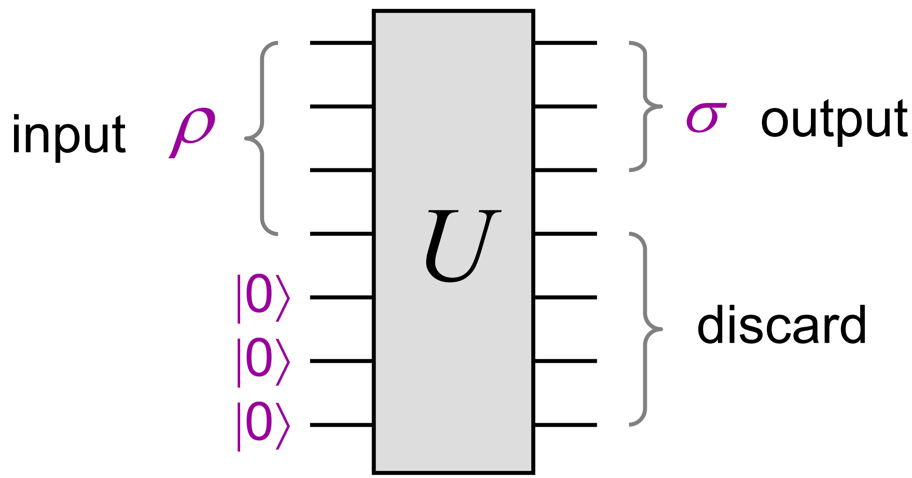
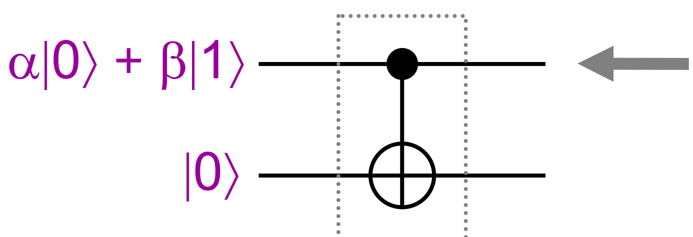
Simulations among operations

Simulations among operations (1)

Fact 1: any ***general quantum operation*** can be simulated by applying a unitary operation on a larger quantum system:



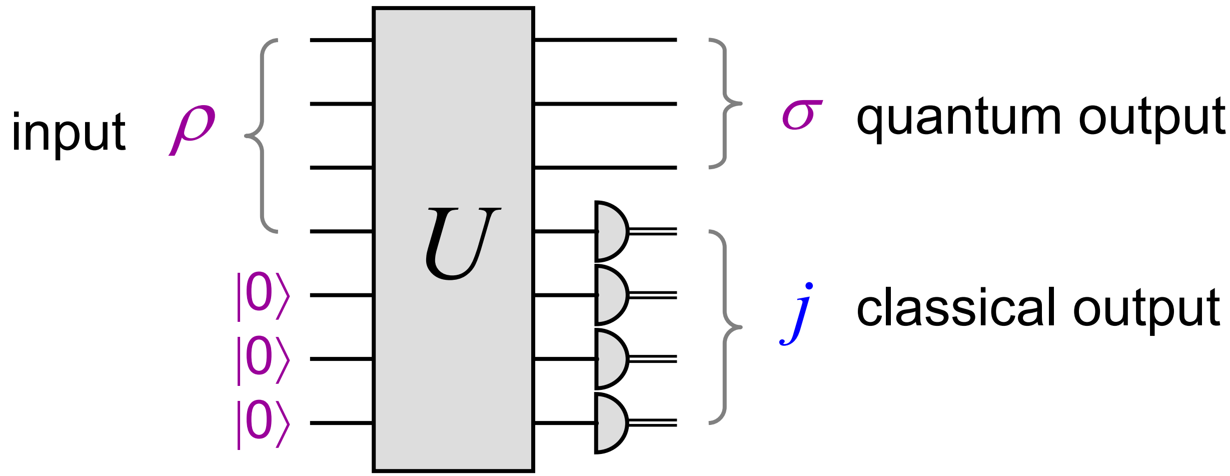
Example: decoherence



$$\rho = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

Simulations among operations (2)

Fact 2: any ***POVM measurement*** can also be simulated by applying a unitary operation on a larger quantum system and then measuring:



Separable states

Separable states

A bipartite (i.e. two register) state ρ is a:

- ***product state*** if $\rho = \sigma \otimes \xi$

- ***separable state*** if $\rho = \sum_{j=1}^{m} p_j \sigma_j \otimes \xi_j \quad (p_1, \ldots, p_m \geq 0)$

(i.e. a probabilistic mixture of product states)

Question: which of the following states are separable?

$$\rho_1 = \tfrac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$

$$\rho_2 = \tfrac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right) + \tfrac{1}{2}\left(|00\rangle - |11\rangle\right)\left(\langle 00| - \langle 11|\right)$$

Distance measures for quantum states

Distance measures

Some simple (and often useful) measures:

- **Euclidean distance:** $\| |\psi\rangle - |\varphi\rangle \|_2$

- **Fidelity:** $| \langle\varphi|\psi\rangle |$

Small Euclidean distance implies “closeness” but large Euclidean distance need not (for example, $|\psi\rangle$ vs $-|\psi\rangle$)

Not so clear how to extend these for mixed states …

… though fidelity does generalize, to $\mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$

Trace norm – preliminaries (1)

For a normal matrix M and a function $f : \mathbf{C} \to \mathbf{C}$, we define the matrix $f(M)$ as follows:

$M = U^\dagger D U$, where D is diagonal (i.e. unitarily diagonalizable)

Now, define $f(M) = U^\dagger f(D)\, U$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix} \qquad f(D) = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_d) \end{bmatrix}$$

Trace norm – preliminaries (2)

For a normal matrix $M = U^{\dagger}DU$, define $|M|$ in terms of replacing D with

$$|D| = \begin{bmatrix} |\lambda_1| & 0 & \cdots & 0 \\ 0 & |\lambda_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\lambda_d| \end{bmatrix}$$

This is the same as defining $|M| = \sqrt{M^{\dagger}M}$ and the latter definition extends to ***all*** matrices (since $M^{\dagger}M$ is positive definite)

Trace norm/distance – definition

The ***trace norm*** of M is $\|M\|_{\mathrm{tr}} = \mathrm{Tr}|M| = \mathrm{Tr}\sqrt{M^\dagger M}$

Intuitively, it's the 1-norm of the eigenvalues (or, in the non-normal case, the *singular values*) of M

The ***trace distance*** between ρ and σ is $\|\rho - \sigma\|_{\mathrm{tr}}$

Why is this a meaningful distance measure between quantum states?

Theorem: for any two quantum states ρ and σ, the optimal measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_{\mathrm{tr}}$