CS667/CO681/PH767 Quantum Information Processing (Fall 06)

Assignment 4

Due date: November 23, 2006

1. **Unitary implementation of trine measurement.** Let $|\phi_0\rangle = |0\rangle$, $|\phi_1\rangle = -1/2|0\rangle + \sqrt{3}/2|1\rangle$, and $|\phi_2\rangle = -1/2|0\rangle - \sqrt{3}/2|1\rangle$. We have seen that a good POVM measurement for distinguishing among these states is the one whose elements are $A_0 = \sqrt{2/3}|\phi_0\rangle\langle\phi_0|$, $A_1 = \sqrt{2/3}|\phi_1\rangle\langle\phi_1|$, and $A_2 = \sqrt{2/3}|\phi_2\rangle\langle\phi_2|$. Suppose that you don't have any device the performs such POVMs; rather, you can perform arbitrary unitary operation and you can measure in the computational basis. Show how to simulate the above POVM by adding some number of qubits in state $|0\rangle$ to the input state, performing some unitary operation on the entire system, and then performing a measurement in the computational basis (and interpreting the outcome as an element of $\{0, 1, 2\}$).

2. **Measurements that never err.** Recall that there is no measurement that perfectly distinguishes between the states $|\psi_0\rangle = |0\rangle$ and $|\psi_1\rangle = |+\rangle$. The "best" measurement that we've seen outputs a bit which is correct with probability $\cos^2(\pi/8) \approx 0.853$. Suppose that we're in a scenario where we cannot afford to make a wrong guess, but we can sometimes decline to make a guess: the output can be 0, 1, or "inconclusive". Describe a measurement procedure that, for any input state $|\psi_k\rangle$ ($k \in \{0, 1\}$), outputs one of the three answers such that: (a) whenever it outputs a bit, the bit is *guaranteed* to be correct; and (b) the probability of it outputting a bit is at least p (making p as large as you can). You may use the POVM formalism or describe your procedure in terms of unitary operations and measurements in the computational basis.

3. **Converting from unitary form to Krauss form.** In each case below, consider the one-qubit to one-qubit quantum operation that results from the following process. First, the input quantum state is extended by a one-qubit "ancilla" in state $|\psi\rangle$ (as given below), then a CNOT gate is applied to the two-qubit system (with the ancilla as target), and then the ancilla is discarded (i.e., traced out).

For each case below, give a set of 2×2 matrices $A_1, \ldots, A_m$ satisfying $\sum_{k=1}^{m} A_k^\dagger A_k = I$ such that, for any input qubit with density matrix ρ, the density matrix of the corresponding output qubit is $\sum_{k=1}^{m} A_k \rho A_k^\dagger$.

(a) $|\psi\rangle = |0\rangle$.

(b) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(c) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

4. **Converting from Krauss form to unitary form.** Since the matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfy $A_1^\dagger A_1 + A_2^\dagger A_2 = I$, they define a two-qubit to two-qubit quantum operation that maps the state with density matrix ρ to the state with density matrix $A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger$. We explore this quantum operation here.

(a) For each of the computational basis states, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, give the corresponding output state of the operation.

Based on your results so far, can you deduce whether or not the operation is unitary?

(b) What is the output state for input state $\frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle)$? What about the output state for input state $\frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle)$? (Hint: in one case you should get a pure state and in the other case a mixed state.)

(c) If you did part (a) correctly, the output for each computational basis state is also a computational basis state (possibly a different one). For computational basis states, give a simple boolean expression for each bit of the output state $|a'b'\rangle$ (where $a', b' \in \{0, 1\}$) in terms of the bits of the input state $|ab\rangle$ (where $a, b \in \{0, 1\}$).

(d) Describe a unitary operation U acting on three qubits such that the quantum operation $\rho \mapsto A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger$ is equivalent to first extending the input state by an ancilla qubit in state $|0\rangle$, and then applying U to the three qubit system, followed by tracing out the third qubit. You may specify U in terms of its quantum circuit (for which a simple solution exists), or as an 8×8 matrix.

5. **Four states as close to orthogonal as possible.** Give four *one-qubit* quantum states $|\phi_0\rangle$, $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, such that, for all $j \neq k$, $|\langle\phi_j|\phi_k\rangle| \leq r$, for as small an r as possible. Note that, using states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, works for $r = 1/\sqrt{2}$, but a smaller r is achievable.

   (Hint: it might be easier to reason on the Bloch sphere; it is acceptable to give your answer in terms of four points on the Bloch sphere.)

6. **Grover's search algorithm when $N/4$ items are marked.** Recall Grover's algorithm that searches for a marked item among $N = 2^n$ possibilities. In general, the expected number of queries that the algorithm performs is $O(\sqrt{N})$. Show that, in the special case where there are exactly $N/4$ marked items, the algorithm succeeds in finding a marked item after a single iteration (thus, one query).

   Bonus to question 6: Suppose that there are exactly $3N/8$ marked items (but we don't know where they are). Then one iteration of Grover's algorithm no longer finds a marked item (at least not for sure). Explain how to *modify* Grover's algorithm so that it finds the marked item making a single query.