## CS667/CO681/PH767 Quantum Information Processing (Fall 06)

Assignment 1
Due date: October 3, 2006

1. Distinguishing between quantum states. In each case, one of the states are selected according to the given probabilities and given to you. You do not know which one. Your goal is to guess which state was selected with as high a probability of success as possible. Describe your distinguishng procedure as a unitary operation followed by a measurement (in the computational basis).
(a) $\frac{1}{2}|0\rangle+\frac{\sqrt{3}}{2}|1\rangle$ with probability $\frac{1}{2}$, and $\frac{\sqrt{3}}{2}|0\rangle-\frac{1}{2}|1\rangle$ with probability $\frac{1}{2}$.
(b) $\frac{1}{2}|00\rangle+\frac{1}{2}|01\rangle+\frac{1}{2}|10\rangle-\frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$, $\frac{1}{2}|00\rangle+\frac{1}{2}|01\rangle-\frac{1}{2}|10\rangle+\frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$, $\frac{1}{2}|00\rangle-\frac{1}{2}|01\rangle+\frac{1}{2}|10\rangle+\frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$, and $-\frac{1}{2}|00\rangle+\frac{1}{2}|01\rangle+\frac{1}{2}|10\rangle+\frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$.
(c) $\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|01\rangle$ with probability $\frac{1}{3}$,
$\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|10\rangle$ with probability $\frac{1}{3}$, and $\frac{1}{\sqrt{2}}|01\rangle+\frac{1}{\sqrt{2}}|10\rangle$ with probability $\frac{1}{3}$.
2. Entangled states and product states. For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the states are entangled).
(a) $\frac{1}{2}|00\rangle+\frac{1}{2}|01\rangle+\frac{1}{2}|10\rangle-\frac{1}{2}|11\rangle$
(b) $\frac{1}{5}|00\rangle+\frac{2}{5}|01\rangle+\frac{2}{5}|10\rangle+\frac{4}{5}|11\rangle$
(c) $\frac{i}{2}|00\rangle-\frac{1}{2}|01\rangle+\frac{1}{2}|10\rangle-\frac{i}{2}|11\rangle \quad$ (where $i=\sqrt{-1}$ )
3. Operations on part of an entangled quantum state. Let

$$
|\psi\rangle=\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|11\rangle .
$$

Prove one of the following three statements.
(a) For any $2 \times 2$ unitary matrix $U$, applying $U$ to the first qubit of $|\psi\rangle$ has the same effect as applying $U$ to the second qubit of $|\psi\rangle$.
(b) For any $2 \times 2$ unitary matrix $U$, applying $U$ to the first qubit of $|\psi\rangle$ has the same effect as applying $U^{T}$ to the second qubit of $|\psi\rangle$ ( $U^{T}$ is the transpose of $U$ ).
(c) For any $2 \times 2$ unitary matrix $U$, applying $U$ to the first qubit of $|\psi\rangle$ has the same effect as applying $U^{\dagger}$ to the second qubit of $|\psi\rangle$ ( $U^{\dagger}$ is the conjugate transpose of $U$ ).

## 4. Constructing simple quantum circuits.

(a) Describe a two-qubit quantum circuit consisting of one CNOT gate and two Hadamard gates that computes the following unitary tranformation:

$$
\left(\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

(b) Describe a two-qubit quantum circuit consisting of just two onequbit gates (of your choosing) that computes the following unitary tranformation:

$$
\left(\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & i^{2} & 0 \\
0 & 0 & 0 & i^{3}
\end{array}\right)
$$

(c) Describe a two-qubit quantum circuit consisting of two CNOT gates plus any number of one-qubit gates that computes the following unitary transformation:

$$
M=\frac{1}{2}\left(\begin{array}{rrrr}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1
\end{array}\right)
$$

Hint: consider the relationship between $H \otimes H$ and $M$.
(d) [Optional for bonus credit] Same gate types as specified in part (c) except the unitary transformation computed should have the property that its square is $M$.

