CS 497 Frontiers of Computer Science (Fall 07)

Assignment (Lecturer: R. Cleve)

Due date: October 22, 2007

Note: Please refer to the lecture note slides for background and definitions.

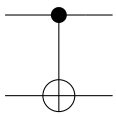
1. **Distinguishing between quantum states.** In each case, one of the states is selected according to the given probabilities and given to you. You do not know which one. Your goal is to apply unitary operations and measurements in order to guess which state it is with as high a probability of success as possible.

   (a) Give a perfect distinguishing procedure for:
   $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ with probability $\frac{1}{2}$, and
   $\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ with probability $\frac{1}{2}$.
   (Hint: consider a rotation.)

   (b) Give a perfect distinguishing procedure for:
   $+\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$,
   $-\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$,
   $-\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$, and
   $+\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$ with probability $\frac{1}{4}$.
   (Hint: consider the pairwise inner products between the vectors.)

   (c) Give a distinguishing procedure that succeeds with probability 7/12 (or greater) for:
   $|0\rangle$ with probability $\frac{1}{3}$,
   $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ with probability $\frac{1}{3}$, and
   $-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ with probability $\frac{1}{3}$.
   For *bonus* credit, achieve success probability $\frac{2}{3}\cos^2(\pi/12) \approx 0.622$, which is more that $7/12 \approx 0.583$. For *extra bonus* credit, achieve success probability 2/3, which is the best possible—but be aware that achieving 2/3 is tricky.

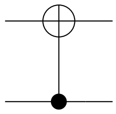
2. **Quantum circuits.** The *controlled-NOT* gate is denoted as



and is the two-qubit unitary gate that, for all $x, y \in \{0, 1\}$, maps $|x, y\rangle$ to $|x, x \oplus y\rangle$.

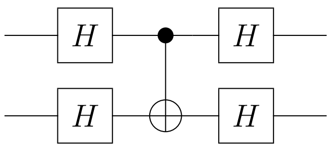
(a) Give the 4×4 matrix that corresponds to the controlled-NOT operation.

(b) Give the 4×4 matrix that corresponds to the controlled-NOT operation when it is oriented this way



which means that the order of the qubits going in and out of the gate are swapped. (This can be derived by first swapping the two qubits, then applying the previous controlled-NOT, and then swapping again.)

(c) For all $x, y \in \{0, 1\}$, give the output state when the following circuit is executed with input $|x, y\rangle$.



(The gate labeled H is the *Hadamard* gate.) In view of your answer, give a simpler circuit (with a single controlled-NOT gate) that has the same effect as the one above.