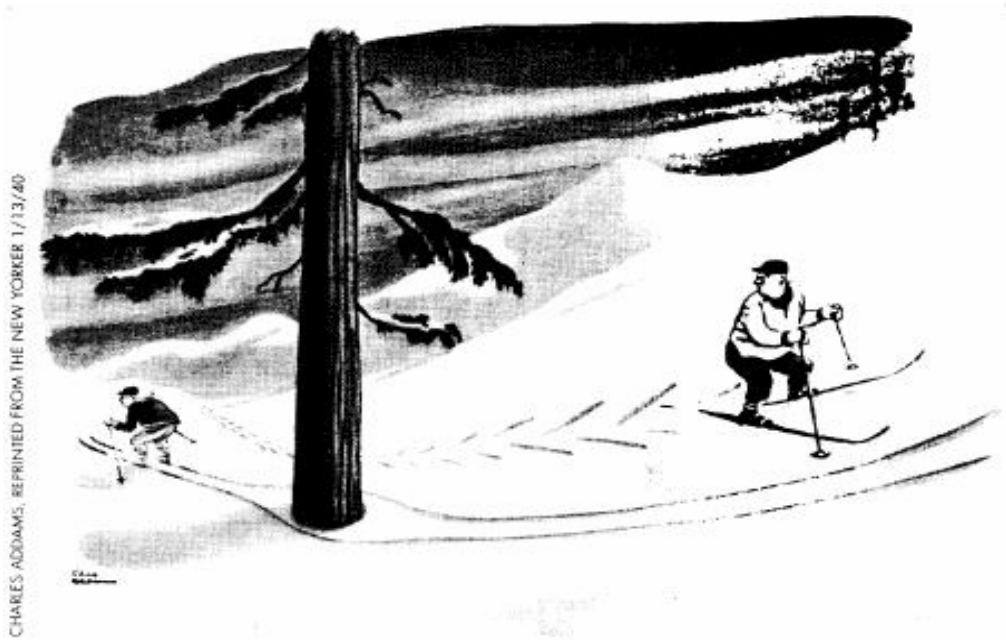
CS 497 Frontiers of Computer Science

Introduction to Quantum Computing

Lecture 2 of 2



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October 1 & 3, 2007

Recap of previous lecture

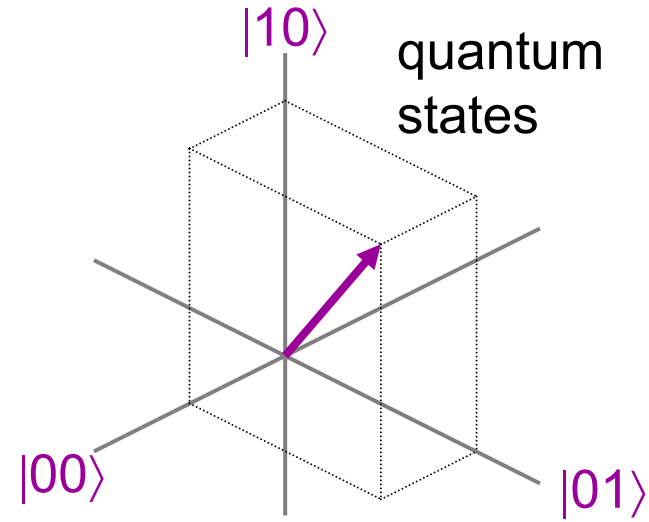
Quantum states on n qubits are 2^n-dimensional unit vectors

The basic operations on them are:

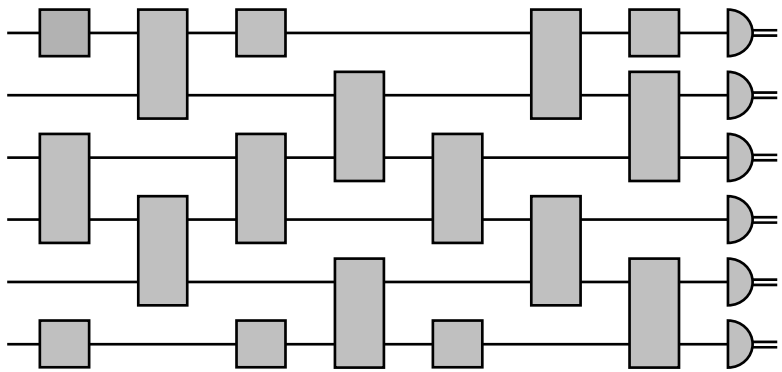
- ***unitary operations*** (rotations)
- ***measurements***, that project on to the basis states

Quantum algorithms so far:

- $f(0) \oplus f(1)$ [Deutsch]
- one-out-of-four search



quantum circuits



one-out-of-N search?

Natural question: what about search problems in spaces larger than ***four*** (and without uniqueness conditions)?

For spaces of size ***eight*** (say), the previous method breaks down—the state vectors will not be orthogonal

Later on, we'll see how to search a space of size N with $O(\sqrt{N})$ queries ...

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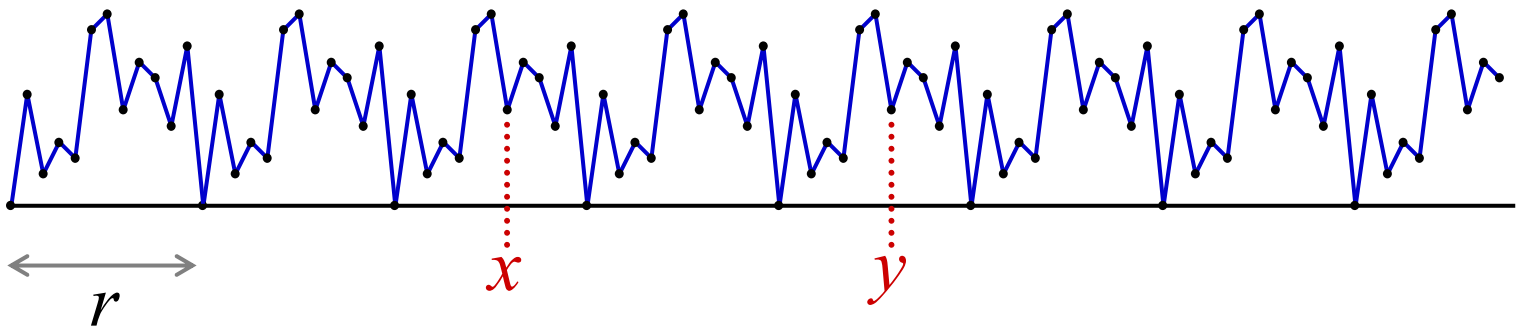
Period-finding

Given: $f:\{0,1\}^n \to T$ such that f is (strictly) r-periodic, with unknown period r



Classically this is very hard in the general case—essentially it is as hard as finding a collision, which costs $2^{O(n)}$ queries

Yet ***Quantum*** algorithms can determine r very efficiently: with only $O(1)$ queries to f

This is the basis of Shor's factoring algorithm ...

Application of period-finding algorithm

<u>Order-finding problem</u> (≈ factoring)

Input: m (an n-bit integer) and $a < m$ such that $\gcd(x, m) = 1$

Output: the minimum $r > 0$ such that $a^r = 1 \pmod{m}$

Example: let $a = 4$ and $m = 35$
(note that $\gcd(4,35) = 1$)

Question: what is r in this case?

Answer: $r = 6$

The sequence is cyclic because the set
$\mathbf{Z}_m^* = \{a \in \{1,2,\ldots, m-1\} : \gcd(x, m) = 1\}$
is a ***group*** under multiplication $\bmod m$

$4^1 \bmod 35 = 4$
$4^2 \bmod 35 = 16$
$4^3 \bmod 35 = 29$
$4^4 \bmod 35 = 11$
$4^5 \bmod 35 = 9$
$4^6 \bmod 35 = 1$
$4^7 \bmod 35 = 4$
$4^8 \bmod 35 = 16$
$\vdots$

Application of period-finding algorithm

<u>Order-finding problem</u> (≈ factoring)

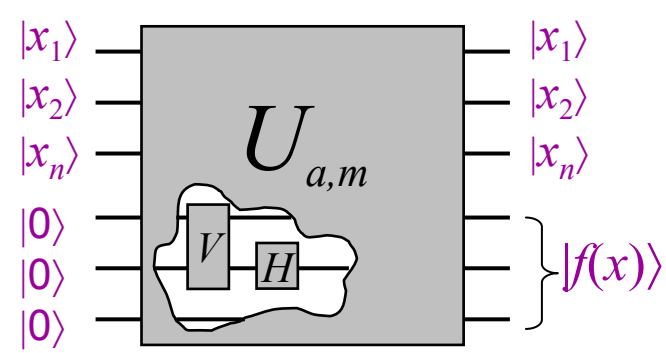
Input: m (an n-bit integer) and $a < m$ such that $\gcd(x, m) = 1$

Output: the minimum $r > 0$ such that $a^r = 1 \pmod{m}$

No classical polynomial-time algorithm is known for this problem—***in fact, the factoring problem reduces to it***

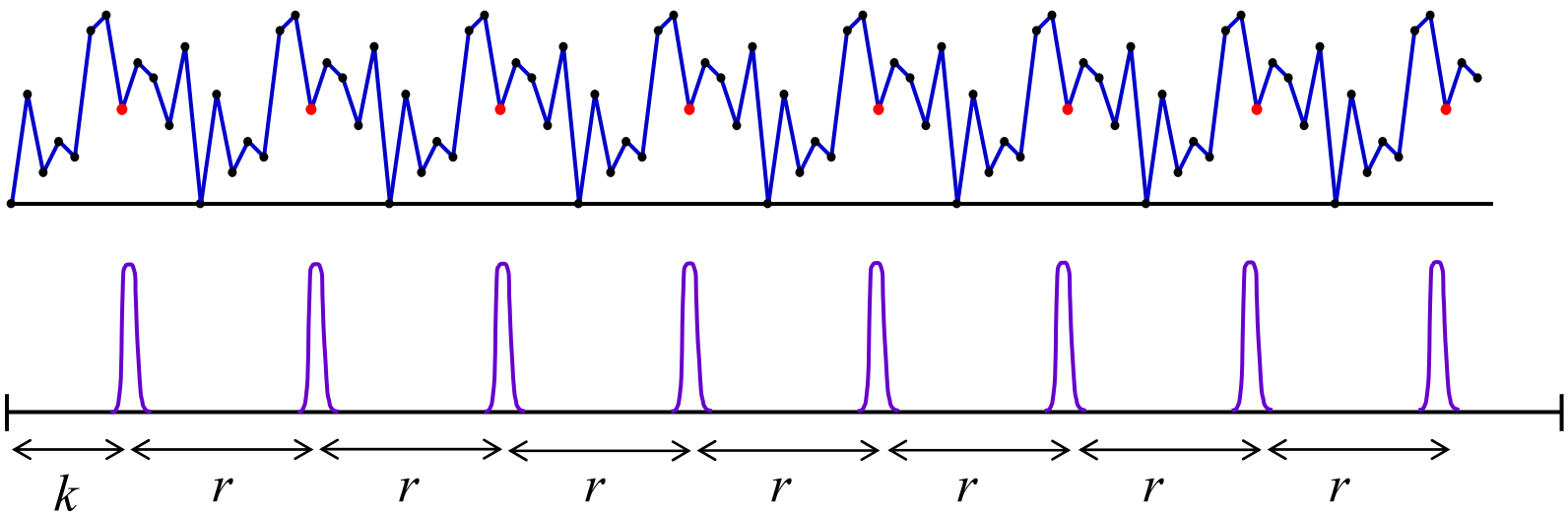
Order-finding reduces to finding the period of the function $f(x) = a^x \bmod m$, which can be computed in polynomial time

A circuit computing the function f is substituted into the black-box:



Sketch of period-finding algorithm

Construct $\Sigma_x |x, f(x)\rangle$ and then measure ***second*** register to get the following state:



(k random)

$$|k\rangle + |k+r\rangle + |k+2r\rangle + |k+3r\rangle + \ldots + |k+(s-1)r\rangle$$

Measuring this state yields just a uniformly random value

More is needed to extract r, a global property of f ...

Quantum Fourier transform

$$F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix}$$

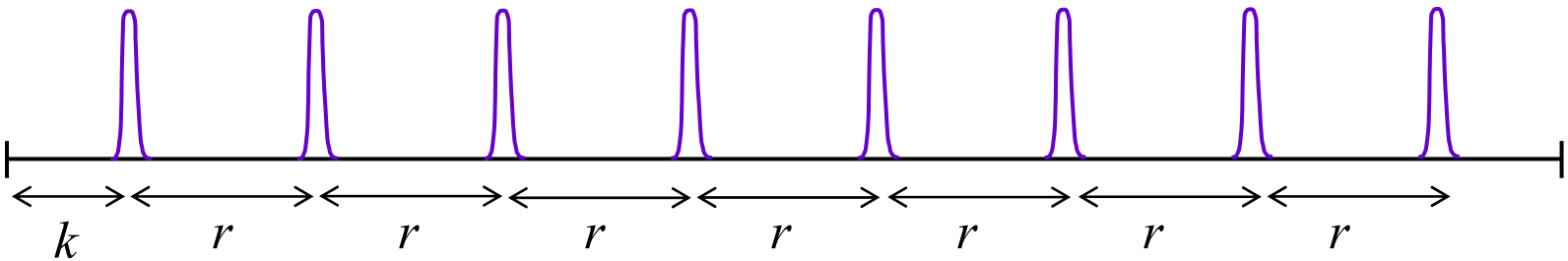
where $\omega = e^{2\pi i/N}$

It's a unitary operation on n qubits (an $N \times N$ matrix, where $N = 2^n$)

Period inversion property

Applying the quantum Fourier transform to the state

$$|k\rangle + |k+r\rangle + |k+2r\rangle + |k+3r\rangle + \ldots + |k+(s-1)r\rangle$$

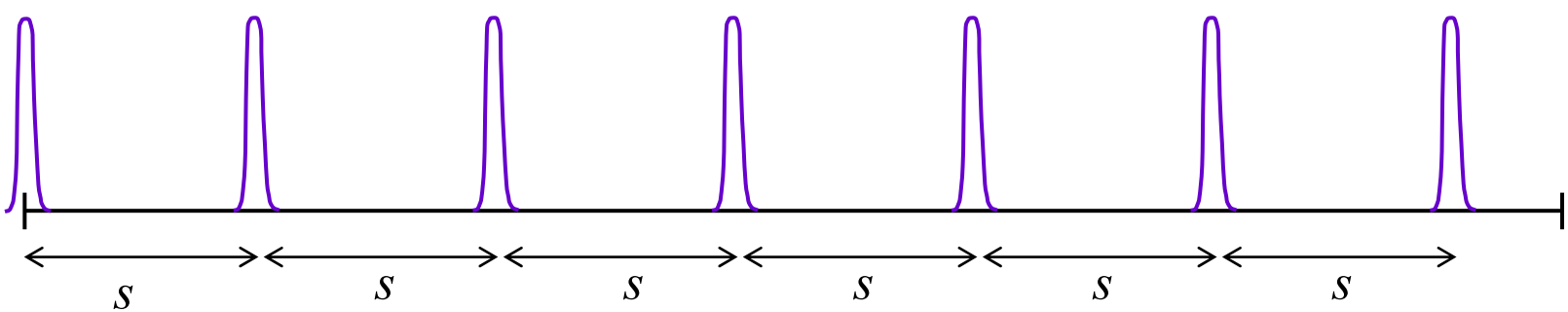


yields the state

$$|0\rangle + \omega^k |s\rangle + \omega^{2k} |2s\rangle + \omega^{3k} |3s\rangle + \ldots + \omega^{(r-1)k} |(r-1)s\rangle$$

$$\omega = e^{2\pi i/r}$$

$$s = r^{-1}$$

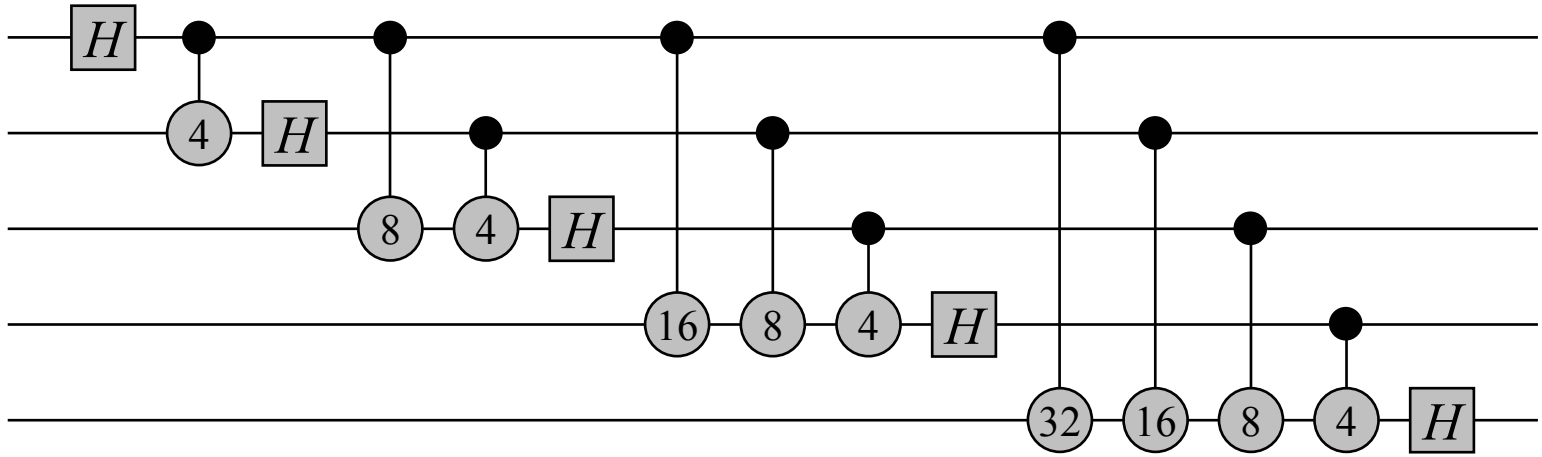


Note: there is no longer an offset k (it's now part of the "phase")

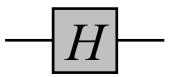
Measure to get multiple of s, from which r can be deduced

Computing the QFT

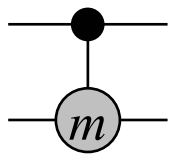
Quantum circuit for F_{32}:



reverse order

Gates: 

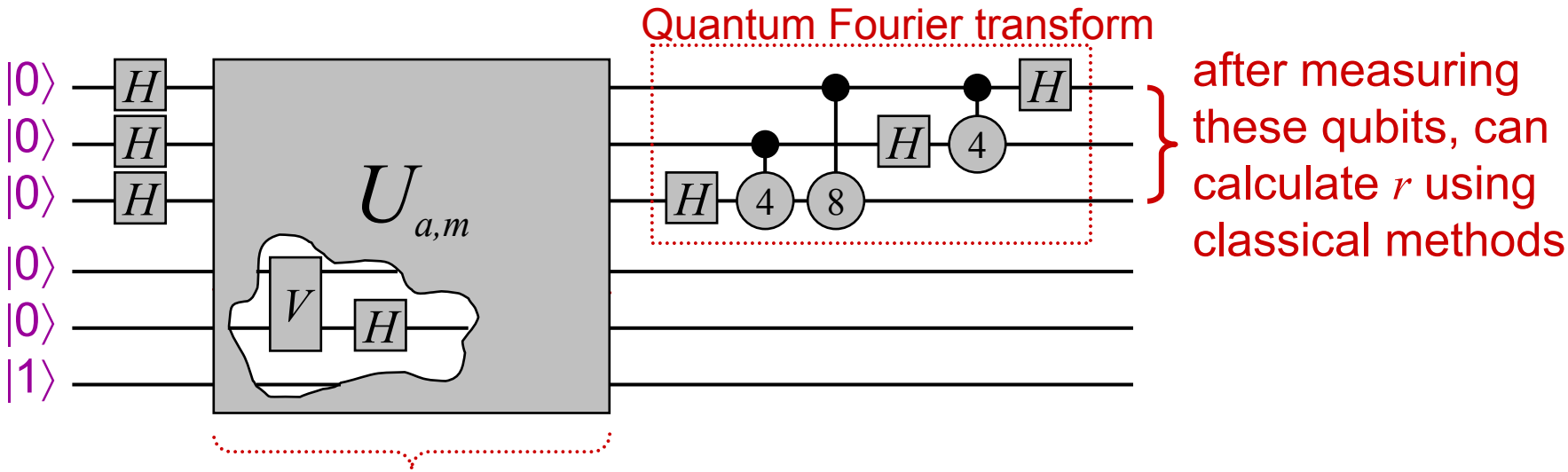
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/m} \end{bmatrix}$$

For F_{2^n} costs $O(n^2)$ gates

Quantum algorithm for order-finding



$$U_{a,M}|x,y\rangle = |x, a^x y \bmod m\rangle$$ (poly-size circuit)

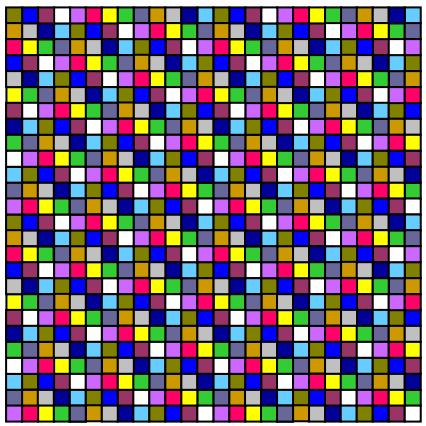
Number of gates for a constant success probability is:

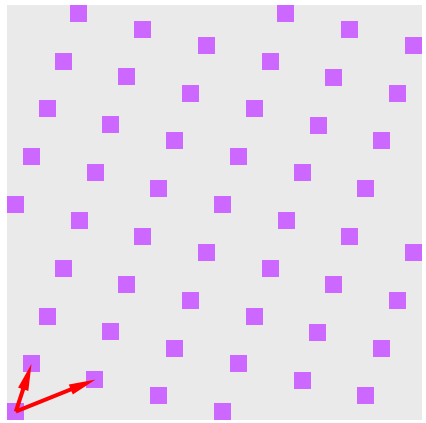
$O(n^2 \log n \log\log n)$

Two-dimensional periodicity

Given:

$f:\{0,1\}^n \times \{0,1\}^n \rightarrow T$ with a two-dimensional repeating pattern





Goal: find a simple description of this periodic strucuture

Quantum algorithms can also solve this very efficiently, and this is the basis of Shor's discrete logarithm algorithm [1994]

Hidden subgroup problem

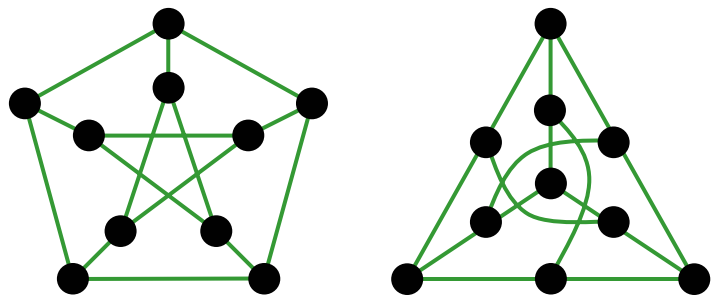
Let G be a known ***group*** and H be an unknown subgroup of G

Let $f: G \to T$ have the property $f(x) = f(y)$ iff $xy^{-1} \in H$
(i.e., x and y are in the same ***right coset*** of H)

Problem: given a method for computing f, determine H

Example: $G = S_n$, the symmetric group (permutations of $\{1,2,\ldots,n\}$)

Interesting fact: a fast algorithm for this leads to a fast algorithm for the ***graph isomorphism problem***



… alas no efficient quantum has been found for this version of HSP, despite significant effort by many people

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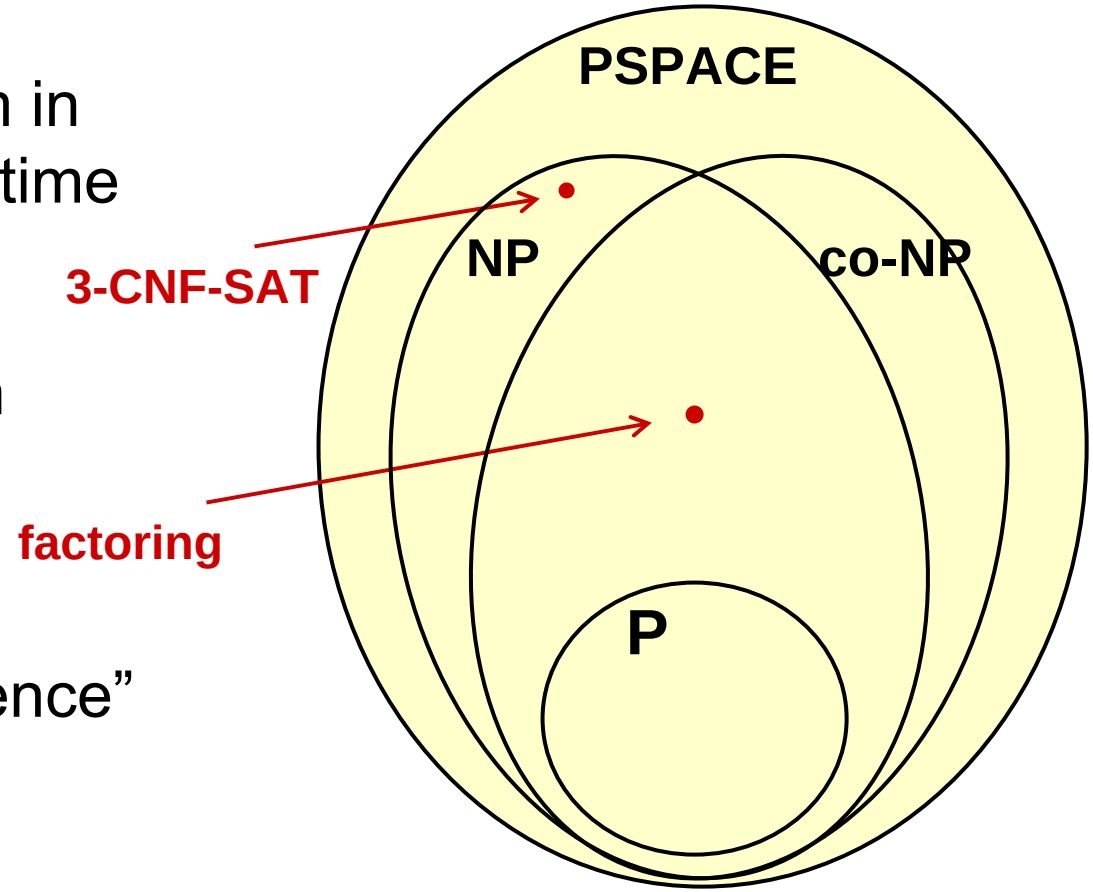
Prelude: factoring vs. NP

Is factoring an **NP**-hard problem?

If so, then ***every*** problem in **NP** is solvable by a poly-time quantum algorithm!

But factoring hasn't been shown to be **NP**-hard



Moreover, there is "evidence" that it is not **NP**-hard:
factoring $\in$ **NP**$\cap$**co-NP**
If factoring is **NP**-hard then **NP** = **co-NP**

Quantum search problem

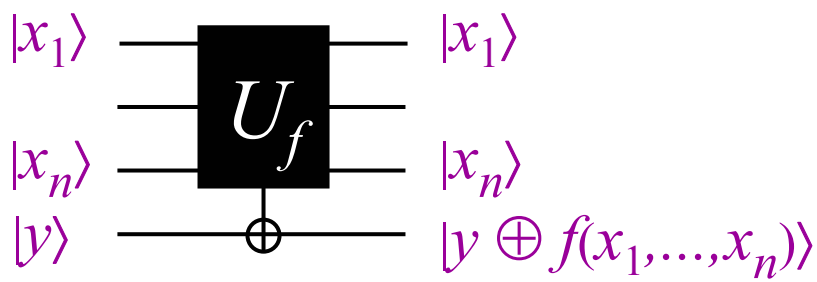
Given: a black box computing $f: \{0,1\}^n \rightarrow \{0,1\}$

Goal: find $x \in \{0,1\}^n$ such that $f(x) = 1$

Classically, using probabilistic procedures, order 2^n queries are necessary to succeed—even with probability ¾ (say)

Grover's ***quantum*** algorithm makes only $O(\sqrt{2^n})$ queries

Query:



[Grover '96]

Applications of quantum search

The function f could be realized as a **3-CNF formula**:

$$f(x_1,\ldots,x_n) = (x_1 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_5) \wedge \cdots \wedge (\bar{x}_1 \vee x_5 \vee \bar{x}_n)$$

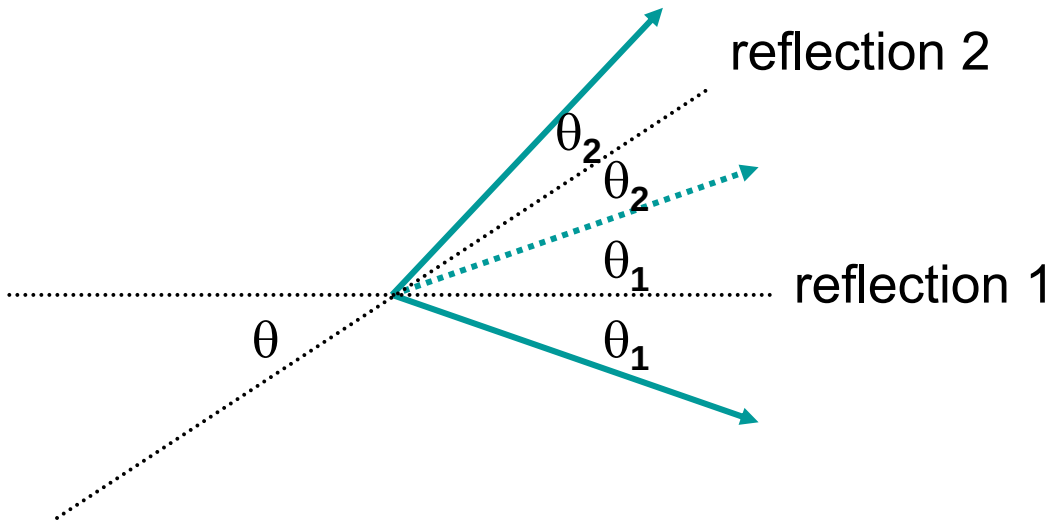
In fact, the search could be for a certificate for any problem in **NP**

The resulting quantum algorithms appear to be ***quadratically*** more efficient than the best classical algorithms known

Prelude to Grover's algorithm:

two reflections = a rotation

Consider two lines with intersection angle θ:

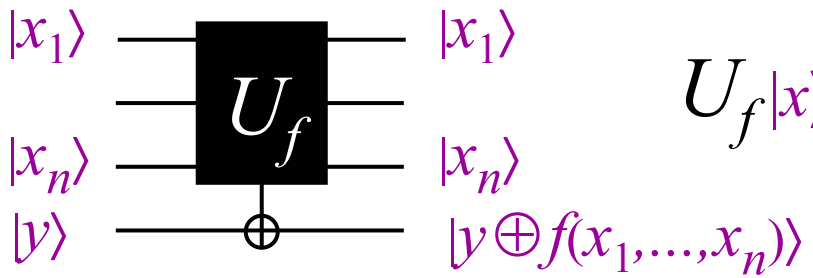


Net effect: rotation by angle 2θ, ***regardless of starting vector***

Grover's algorithm I

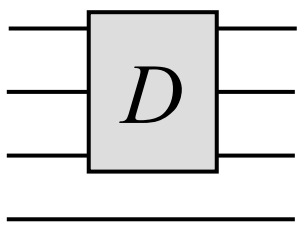
Basic operations used:

Query:



$$U_f|x\rangle(|0\rangle-|1\rangle) = (-1)^{f(x)}|x\rangle(|0\rangle-|1\rangle)$$

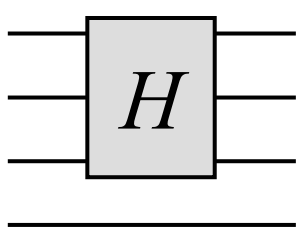
Diffusion:

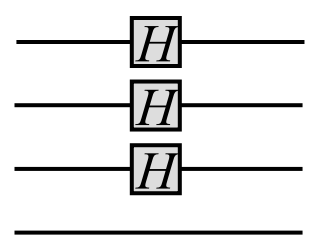


$$D = \begin{bmatrix} -1+2/2^n & 2/2^n & \cdots & 2/2^n \\ 2/2^n & -1+2/2^n & \cdots & 2/2^n \\ \vdots & \vdots & \ddots & \vdots \\ 2/2^n & 2/2^n & \cdots & -1+2/2^n \end{bmatrix}$$

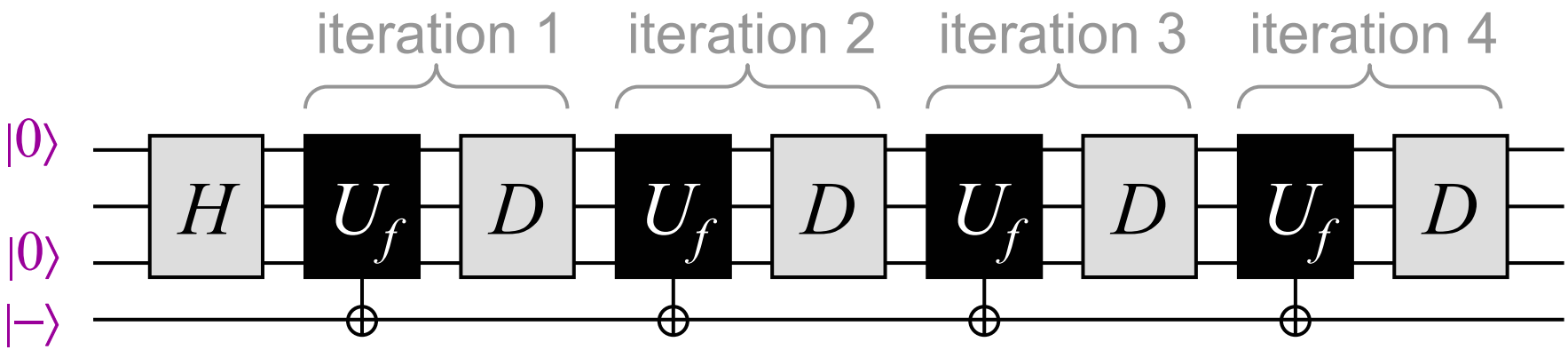
Costs only $\mathrm{O}(n)$ gates

Hadamard:





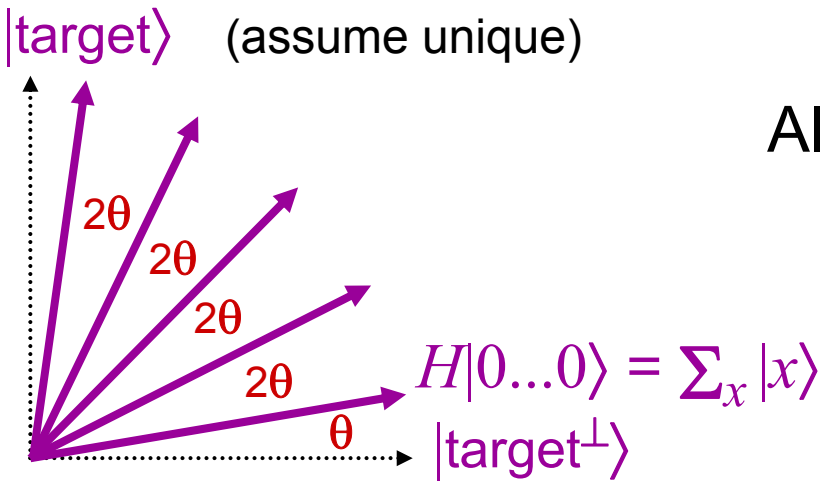
Grover's algorithm II



1. construct state $H|0...0\rangle|-\rangle$
2. <u>repeat</u> k <u>times</u>:

   apply DU_f to state
3. measure state, to get $x \in \{0,1\}^n$, and check if $f(x)=1$

Grover's algorithm III



Algorithm: $(D\,U_f)^k H|0...0\rangle$

Since $D\,U_f$ is a composition of two reflections, it is a rotation by 2θ, where $\sin(\theta) \approx 1/\sqrt{N}$

We want $(2k+1)(1/\sqrt{N}) \approx \pi/2$, so set $k \approx (\pi/4)\sqrt{N}$

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Conclusion

Extended Chruch-Turing Thesis: any polynomial-time algorithm can be simulated by a probabilistic polynomial-time Turing machine

Quantum computing ***challenges*** this thesis

Either:

- The Extended Chruch-Turing Thesis is false
- Quantum mechanics as we understand it is false
- There is a classical poly-time factoring algorithm

There are many curious properties of quantum information in the context of computation, communication, and cryptography

Waterloo has many people in the faculties of Mathematics, Science, and Engineering working in quantum computing (please see www.iqc.ca more information)

Some possible project topics

1. Efficient “quantum proofs”: an example is the group non-membership problem
2. Nonlocal effects: apparently paradoxical tricks that can be performed with entangled states
3. Quantum error-correcting codes: important for physical implementations of quantum information processing
4. Alternate models of quantum computation: an example is the measurement-based model
5. Quantum walks: the quantum analogues of random walks, and their algorithmic applications
6. Quantum cryptography: information-theoretical security in a public-key setting

THE END