Introduction to Quantum Information Processing

CS 467 / CS 667
Phys 667 / Phys 767
C&O 481 / C&O 681

Lecture 2 (2005)

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Superdense coding

How much classical information in n qubits?

$2^n - 1$ complex numbers apparently needed to describe an arbitrary n-qubit pure quantum state:

$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \ldots + \alpha_{111}|111\rangle$

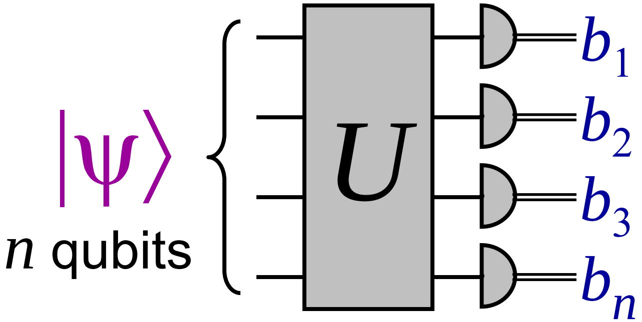
Does this mean that an exponential amount of classical information is somehow stored in n qubits?

Not in an operational sense ...

For example, Holevo's Theorem (from 1973) implies: one cannot convey more than n classical bits of information in n qubits

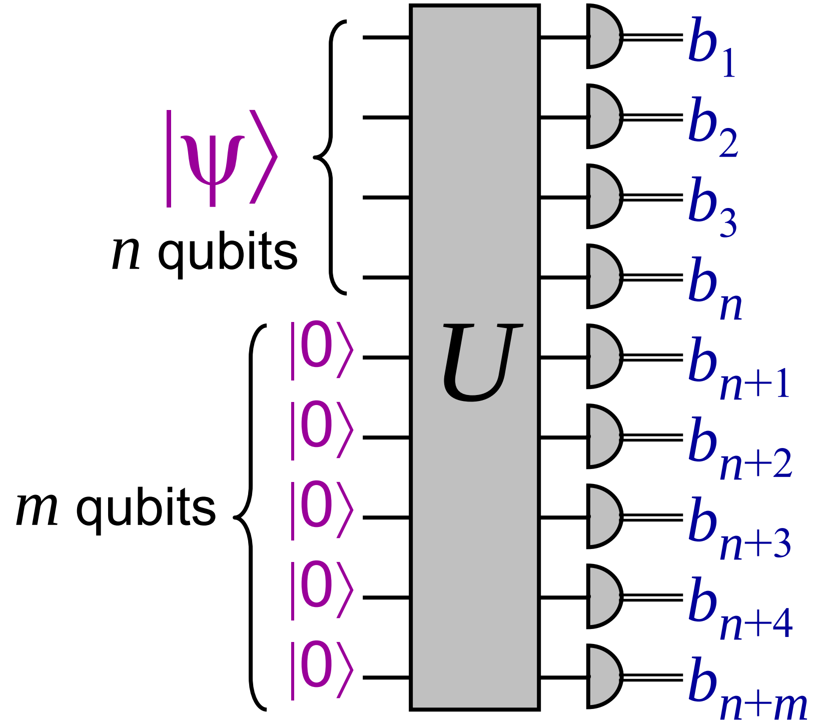
Holevo's Theorem

Easy case:



$b_1 b_2 \ldots b_n$ certainly cannot convey more than n bits!

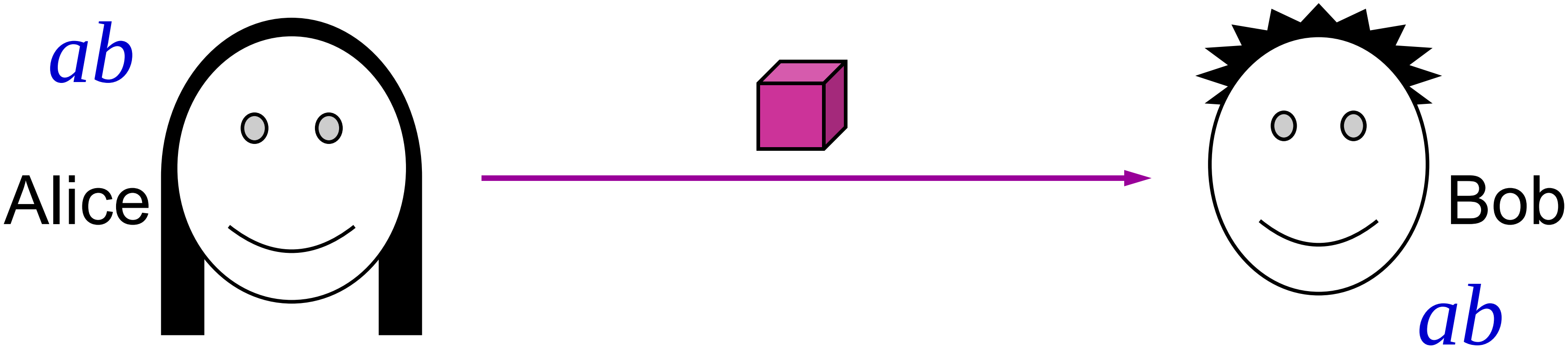
Hard case (the general case):



The difficult proof is beyond the scope of this course

Superdense coding (prelude)

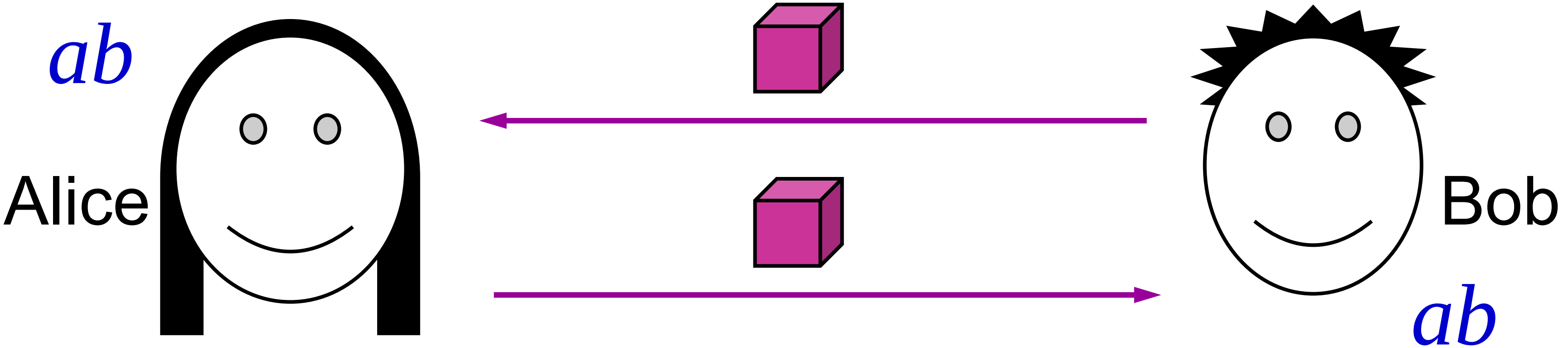
Suppose that Alice wants to convey ***two*** classical bits to Bob sending just ***one*** qubit



By Holevo's Theorem, this is ***impossible***

Superdense coding

In ***superdense coding***, Bob is allowed to send a qubit to Alice first



How can this help?

How superdense coding works

1. Bob creates the state $|00\rangle + |11\rangle$ and sends the ***first*** qubit to Alice

2. Alice: if $a = 1$ then apply X to qubit
   if $b = 1$ then apply Z to qubit
   send the qubit back to Bob

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

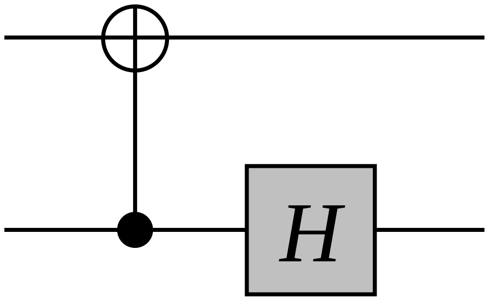
ab	state
00	$\lvert 00\rangle + \lvert 11\rangle$
01	$\lvert 00\rangle - \lvert 11\rangle$
10	$\lvert 01\rangle + \lvert 10\rangle$
11	$\lvert 01\rangle - \lvert 10\rangle$

} Bell basis

3. Bob measures the two qubits in the ***Bell basis***

Measurement in the Bell basis

Specifically, Bob applies



input	output
$\|00\rangle + \|11\rangle$	$\|00\rangle$
$\|01\rangle + \|10\rangle$	$\|01\rangle$
$\|00\rangle - \|11\rangle$	$\|10\rangle$
$\|01\rangle - \|10\rangle$	$\|11\rangle$

to his two qubits ...

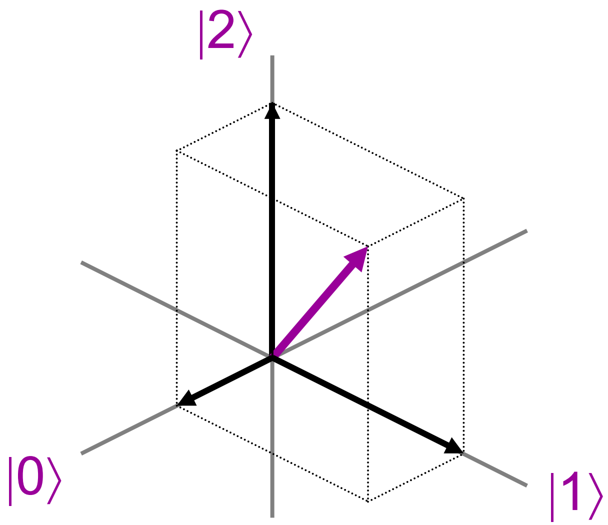
and then measures them, yielding ab

This concludes superdense coding

Teleportation

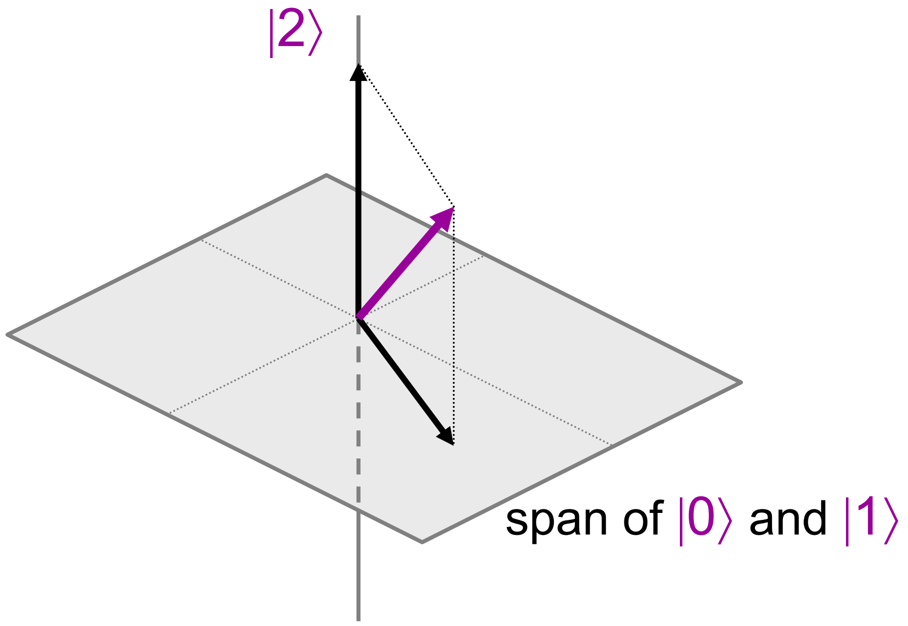
Incomplete measurements (I)

Measurements up until now are with respect to orthogonal one-dimensional subspaces:



(qutrit)

The orthogonal subspaces can have other dimensions:



Incomplete measurements (II)

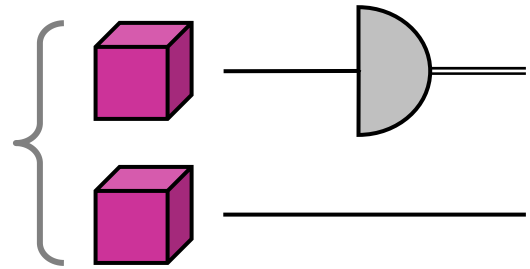
Such a measurement on $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle$

(renormalized)

results in
$$\begin{cases} \alpha_0|0\rangle + \alpha_1|1\rangle & \text{with prob } |\alpha_0|^2 + |\alpha_1|^2 \\ |2\rangle & \text{with prob } |\alpha_2|^2 \end{cases}$$

Measuring the first qubit of a two-qubit system

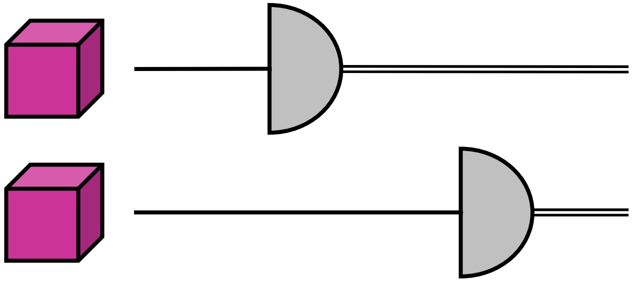
$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$



Defined as the incomplete measurement with respect to the two dimensional subspaces:

- span of $|00\rangle$ & $|01\rangle$ (all states with first qubit 0), and
- span of $|10\rangle$ & $|11\rangle$ (all states with first qubit 1)

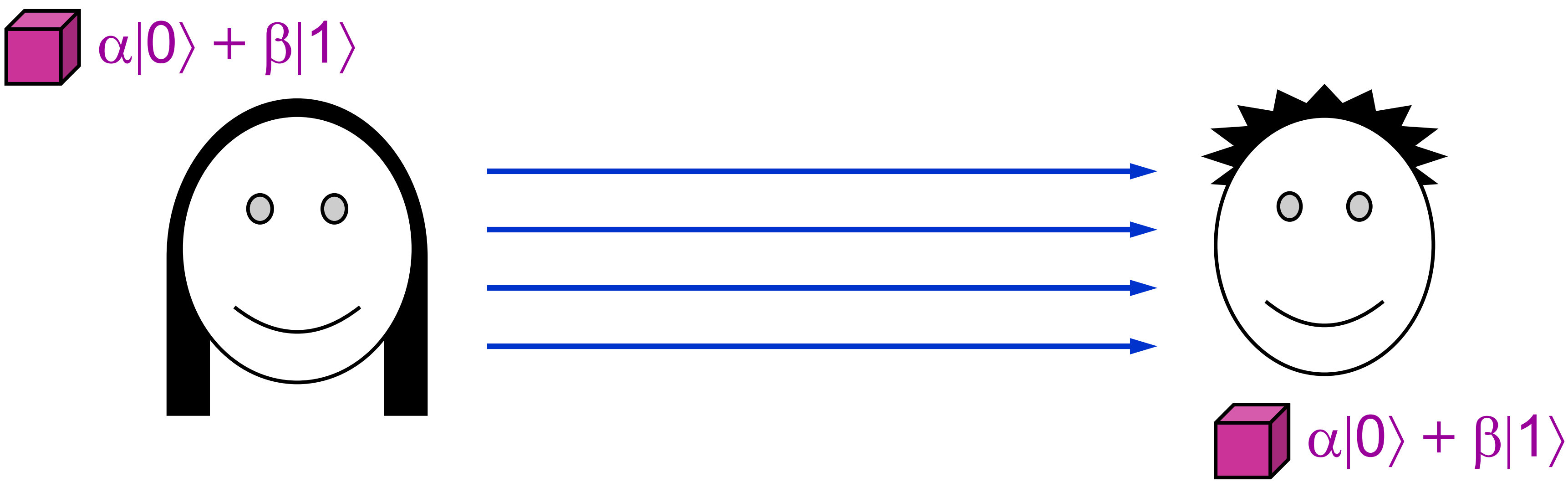
Result is the mixture
$$\begin{cases} \alpha_{00}|00\rangle + \alpha_{01}|01\rangle \text{ with prob } |\alpha_{00}|^2 + |\alpha_{01}|^2 \\ \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \text{ with prob } |\alpha_{10}|^2 + |\alpha_{11}|^2 \end{cases}$$



Easy exercise: show that measuring the first qubit and *then* measuring the second qubit gives the same result as measuring both qubits at once

Teleportation (prelude)

Suppose Alice wishes to convey a qubit to Bob by sending just classical bits

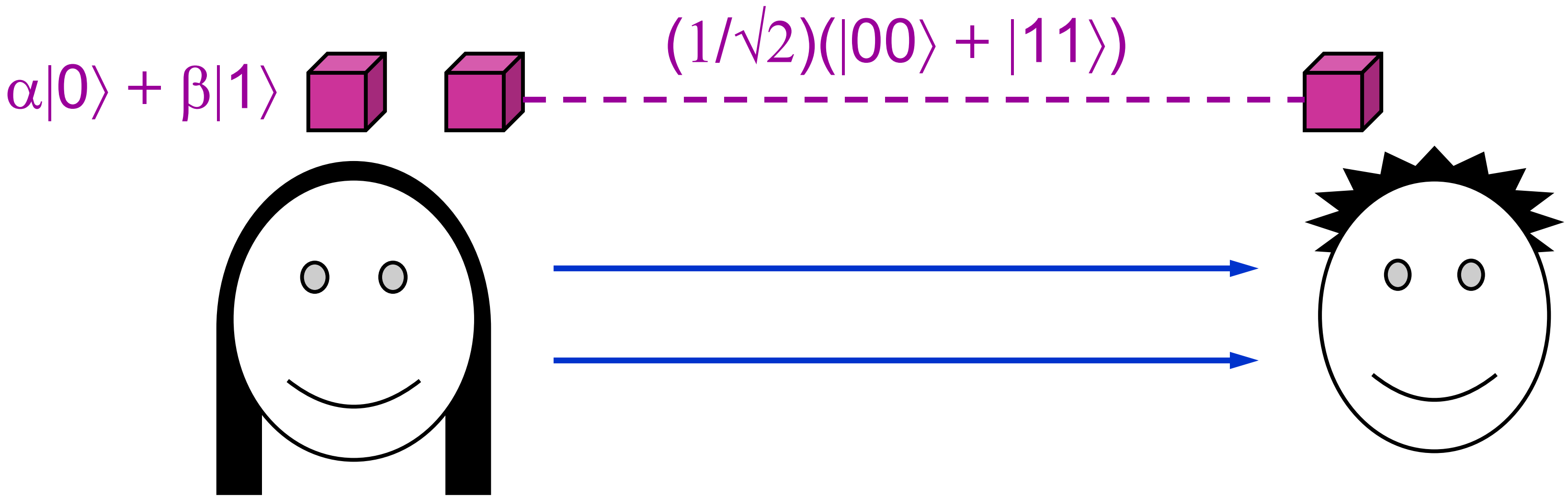


If Alice ***knows*** α and β, she can send approximations of them —but this still requires infinitely many bits for perfect precision

Moreover, if Alice does ***not*** know α or β, she can at best acquire ***one bit*** about them by a measurement

Teleportation scenario

In teleportation, Alice and Bob also start with a Bell state



and Alice can send two classical bits to Bob

Note that the initial state of the three qubit system is:

$$(1/\sqrt{2})(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle)$$
$$= (1/\sqrt{2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

How teleportation works

Initial state: $(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle)$ (omitting the $1/\sqrt{2}$ factor)

$$= \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle$$

$$= \tfrac{1}{2}(|00\rangle + |11\rangle)(\alpha|0\rangle + \beta|1\rangle)$$

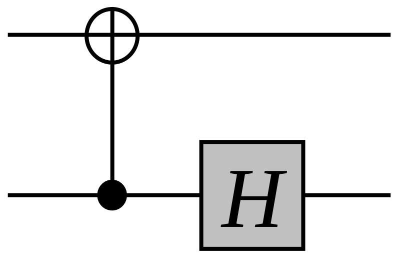
$$+ \tfrac{1}{2}(|01\rangle + |10\rangle)(\alpha|1\rangle + \beta|0\rangle)$$

$$+ \tfrac{1}{2}(|00\rangle - |11\rangle)(\alpha|0\rangle - \beta|1\rangle)$$

$$+ \tfrac{1}{2}(|01\rangle - |10\rangle)(\alpha|1\rangle - \beta|0\rangle)$$

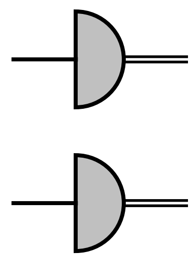
Protocol: Alice measures her two qubits ***in the Bell basis*** and sends the result to Bob (who then "corrects" his state)

What Alice does specifically

Alice applies 

to her two qubits, yielding:

$$\left\{\begin{array}{l} \tfrac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) \\ +\, \tfrac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) \\ +\, \tfrac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) \\ +\, \tfrac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle) \end{array}\right.$$



$$\left\{\begin{array}{ll} (00,\ \alpha|0\rangle + \beta|1\rangle) & \text{with prob. } \tfrac{1}{4} \\ (01,\ \alpha|1\rangle + \beta|0\rangle) & \text{with prob. } \tfrac{1}{4} \\ (10,\ \alpha|0\rangle - \beta|1\rangle) & \text{with prob. } \tfrac{1}{4} \\ (11,\ \alpha|1\rangle - \beta|0\rangle) & \text{with prob. } \tfrac{1}{4} \end{array}\right.$$

Then Alice sends her two classical bits to Bob, who then adjusts his qubit to be $\alpha|0\rangle + \beta|1\rangle$ whatever case occurs

Bob's adjustment procedure

Bob receives two classical bits a, b from Alice, and:

if $b = 1$ he applies X to qubit

if $a = 1$ he applies Z to qubit

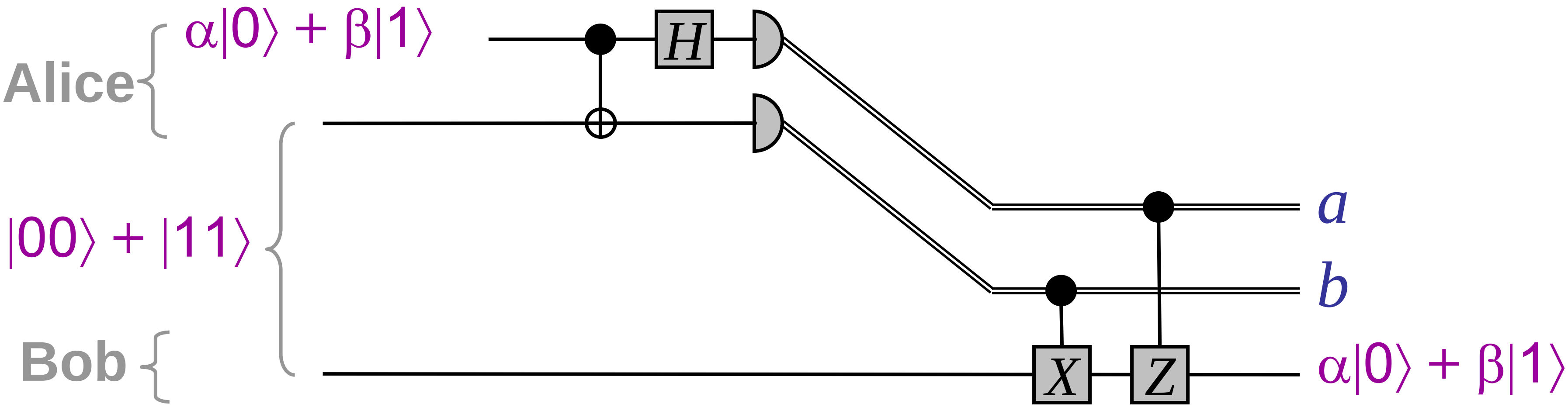
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

yielding:

$$\begin{cases} 00, & \alpha|0\rangle + \beta|1\rangle \\ 01, & X(\alpha|1\rangle + \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \\ 10, & Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle \\ 11, & ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \end{cases}$$

Note that Bob acquires the correct state in each case

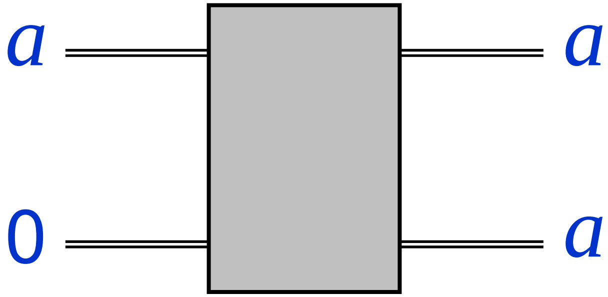
Summary of teleportation

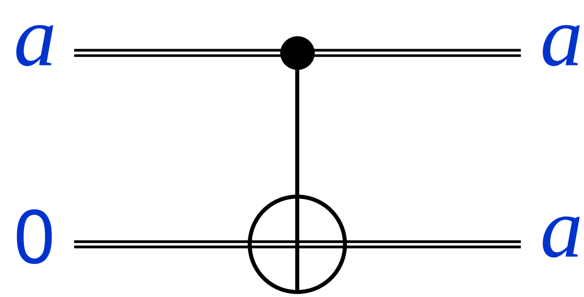


Quantum circuit exercise: try to work through the details of the analysis of this teleportation protocol

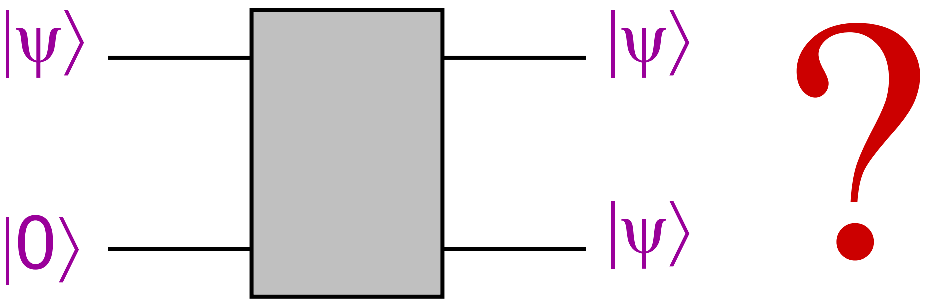
No-cloning theorem

Classical information can be copied

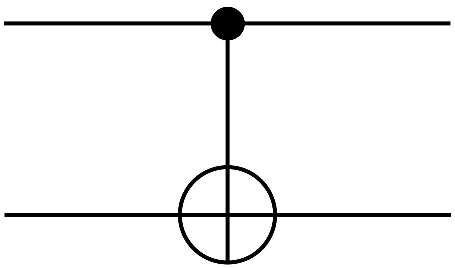




What about quantum information?



Candidate:



works fine for $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$

... but it fails for $|\psi\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ ...

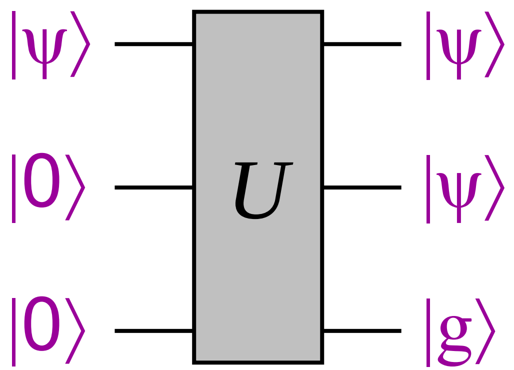
... where it yields output $(1/\sqrt{2})(|00\rangle + |11\rangle)$

instead of $|\psi\rangle|\psi\rangle = (1/4)(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

No-cloning theorem

Theorem: there is ***no*** valid quantum operation that maps an arbitrary state $|\psi\rangle$ to $|\psi\rangle|\psi\rangle$

Proof:



Let $|\psi\rangle$ and $|\psi'\rangle$ be two input states, yielding outputs $|\psi\rangle|\psi\rangle|g\rangle$ and $|\psi'\rangle|\psi'\rangle|g'\rangle$ respectively

Since U preserves inner products:

$\langle\psi|\psi'\rangle = \langle\psi|\psi'\rangle\langle\psi|\psi'\rangle\langle g|g'\rangle$ so

$\langle\psi|\psi'\rangle(1 - \langle\psi|\psi'\rangle\langle g|g'\rangle) = 0$ so

$|\langle\psi|\psi'\rangle| = 0$ or 1

THE END