Introduction to Quantum Information Processing
CS 467 / CS 667
Phys 467 / Phys 767
C&O 481 / C&O 681

Lecture 16 (2005)

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- The GHZ "paradox"

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- The magic square game

- **Preliminary remarks about quantum communication**

- The GHZ "paradox"

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Quantum information can apparently be used to substantially reduce **computation** costs for a number of interesting problems

How does quantum information affect the **communication costs** of information processing tasks?

We explore this issue ...

Entanglement and signaling

Recall that Entangled states, such as $\frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle$,

qubit   qubit
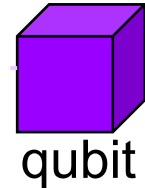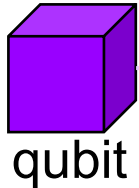
can be used to perform some intriguing feats, such as *teleportation* and *superdense coding*

—but they *cannot* be used to "signal instantaneously"

Any operation performed on one system has no affect on the state of the other system (its reduced density matrix)

Basic communication scenario

Goal: convey n bits from Alice to Bob

$x_1 x_2 \ldots x_n$

Alice

Resources

Bob

$x_1 x_2 \ldots x_n$

Basic communication scenario
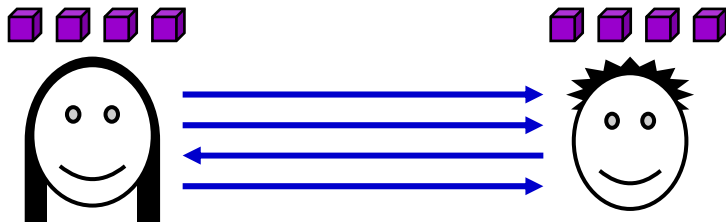
Bit communication:

Cost: n

Qubit communication:

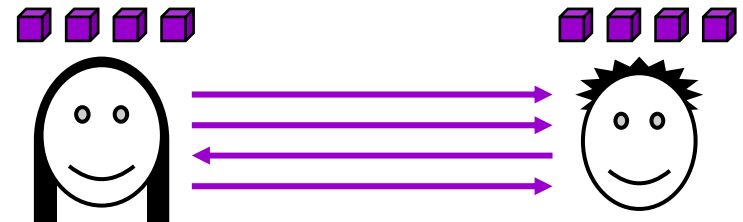**Cost:** n [Holevo's Theorem, 1973]

**Bit communication
& prior entanglement:**

Cost: n (can be deduced)

**Qubit communication
& prior entanglement:**
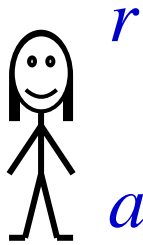

Cost: $n/2$ superdense coding

[Bennett & Wiesner, 1992]

GHZ scenario

[Greenberger, Horne, Zeilinger, 1980]

Input: r s t

Output: $a \leftarrow r$ $b \leftarrow \neg s$ $c \leftarrow 1$

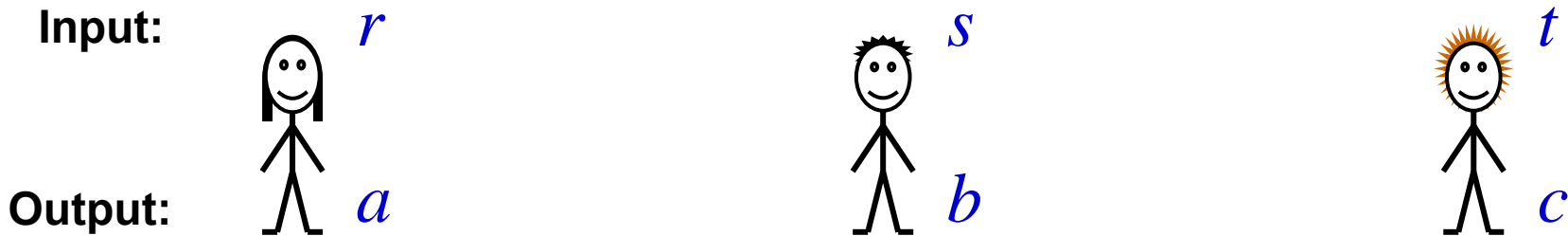**Alice**   **Bob**   **Carol**

Rules of the game:

1. It is promised that $r \oplus s \oplus t = 0$

2. No communication after inputs received

3. They **win** if $a \oplus b \oplus c = r \vee s \vee t$

rst	$a \oplus b \oplus c$	abc
000	0 ☺	011
011	1 ☺	001
101	1 ☺	111
110	1 ☹	101

No perfect strategy for GHZ

Input:

r s t

Output:

a b c

rst	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

General deterministic strategy:

$a_0, a_1, b_0, b_1, c_0, c_1$

Winning conditions:

Has no solution, thus no perfect strategy exists

$$\begin{cases} a_0 \oplus b_0 \oplus c_0 = 0 \\ a_0 \oplus b_1 \oplus c_1 = 1 \\ a_1 \oplus b_0 \oplus c_1 = 1 \\ a_1 \oplus b_1 \oplus c_0 = 1 \end{cases}$$

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GHZ: preventing communication

Input: *r* $\longrightarrow$ *s* $\longrightarrow$ *t*
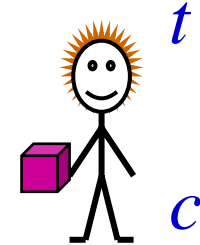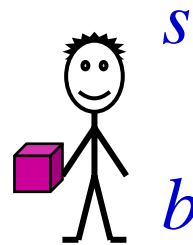$\longleftarrow$ $\longleftarrow$
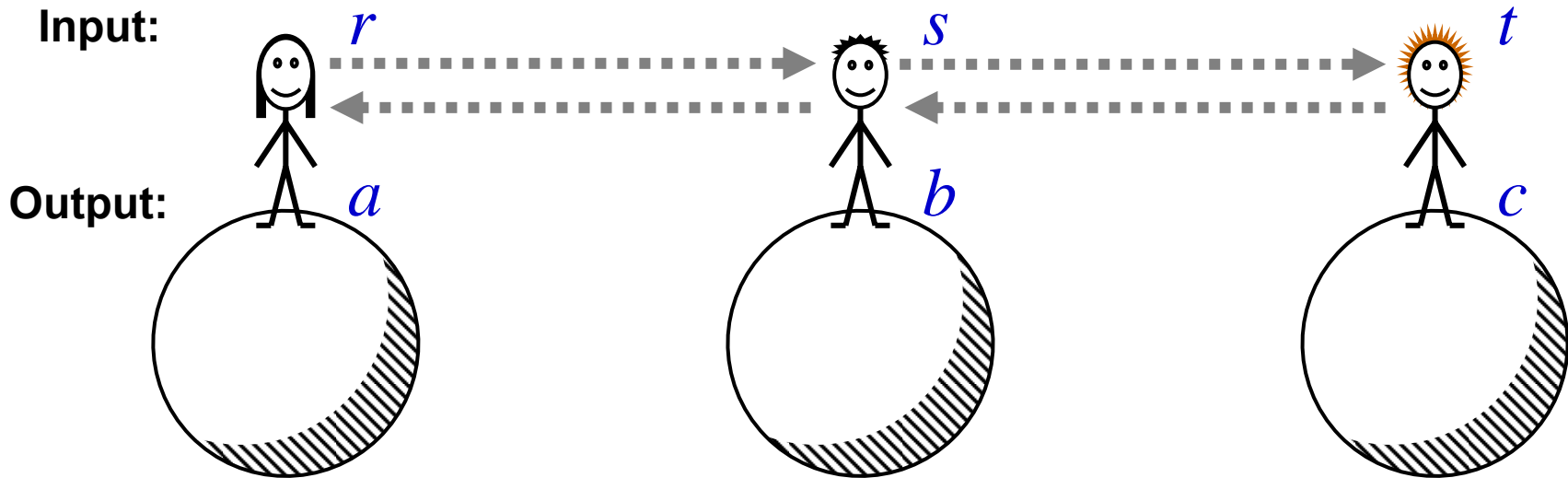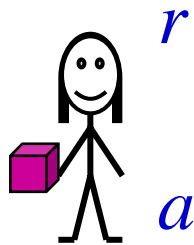
Output: *a* *b* *c*

Input and output events can be ***space-like*** separated:
so signals at the speed of light are not fast enough for cheating

What if Alice, Bob, and Carol ***still*** keep on winning?

"GHZ Paradox" explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$

r a s b t c

Alice's strategy:

1. if $r = 1$ then apply H to qubit

2. measure qubit and set a to result

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Bob's & Carol's strategies: similar

Case 2 ($rst = 000$): state measured directly $|001\rangle + |010\rangle + |100\rangle - |111\rangle$ ☺

Case 1 ($rst = 011$): new state

Cases 3 & 4 ($rst = 101$ & 110): similar by symmetry ☺

GHZ: conclusions

- For the GHZ game, any *classical* team succeeds with probability at most ¾

- Allowing the players to communicate would enable them to succeed with probability 1

- Entanglement cannot be used to communicate

- Nevertheless, allowing the players to have entanglement enables them to succeed with probability 1

- Thus, entanglement is a useful resource for the task of *winning the GHZ game*
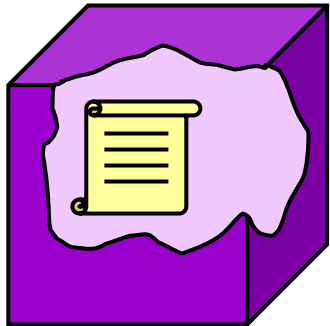
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- The magic square game

Bell's Inequality and its violation

Part I: physicist's view:

Can a quantum state have **pre-determined** outcomes for each possible measurement that can be applied to it?

qubit:

where the "manuscript" is something like this:

called **hidden variables**

if $\{|0\rangle,|1\rangle\}$ measurement then output **0**

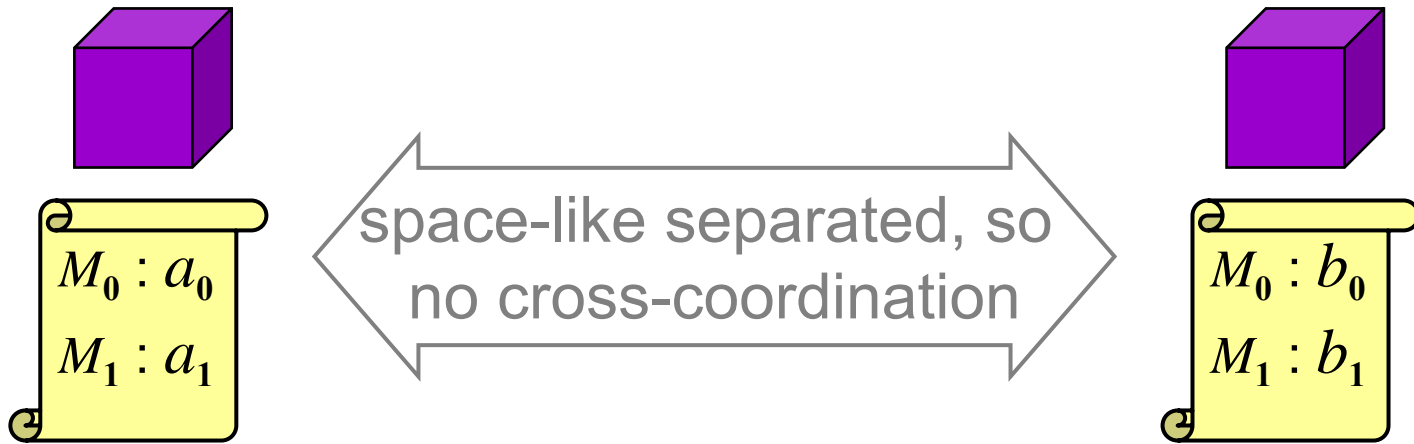if $\{|+\rangle,|-\rangle\}$ measurement then output **1**

if **...** (etc)

table could be implicitly given by some formula

[Bell, 1964]
[Clauser, Horne, Shimony, Holt, 1969]

Bell Inequality

Imagine a two-qubit system, where one of two measurements, called M_0 and M_1, will be applied to each qubit:

$M_0 : a_0$
$M_1 : a_1$

space-like separated, so
no cross-coordination

$M_0 : b_0$
$M_1 : b_1$

Define:
$A_0 = (-1)^{a_0}$
$A_1 = (-1)^{a_1}$
$B_0 = (-1)^{b_0}$
$B_1 = (-1)^{b_1}$

Claim: $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

Proof: $A_0 (B_0 + B_1) + A_1 (B_0 - B_1) \leq 2$

one is ± 2 and the other is 0

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Bell Inequality

$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$ is called a ***Bell Inequality****

Question: could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?
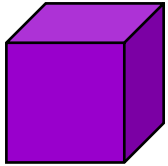
Answer 1: *no, not directly*, because A_0, A_1, B_0, B_1 cannot all be measured (only ***one*** $A_s B_t$ term can be measured)

Answer 2: *yes, indirectly*, by making many runs of this experiment: pick a random $st \in \{00, 01, 10, 11\}$ and then measure with M_s and M_t to get the value of $A_s B_t$

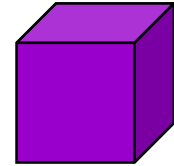The ***average*** of $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ should be $\leq \frac{1}{2}$

* also called CHSH Inequality

Violating the Bell Inequality

Two-qubit system in state
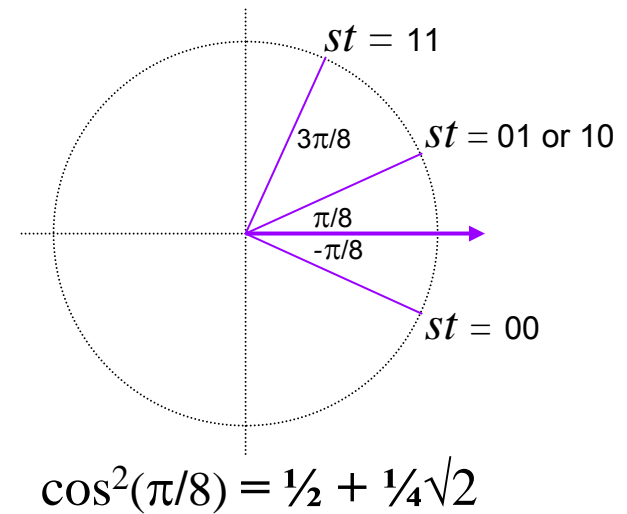$$|\phi\rangle = |00\rangle - |11\rangle$$

Applying rotations θ_A and θ_B yields:

$$\cos(\theta_A + \theta_B)\,(|00\rangle - |11\rangle) + \sin(\theta_A + \theta_B)\,(|01\rangle + |10\rangle)$$

$$\underbrace{}_{A\,B\ =\ +1} \qquad \underbrace{}_{A\,B\ =\ -1}$$

Define

M_0: rotate by $-\pi/16$ then measure
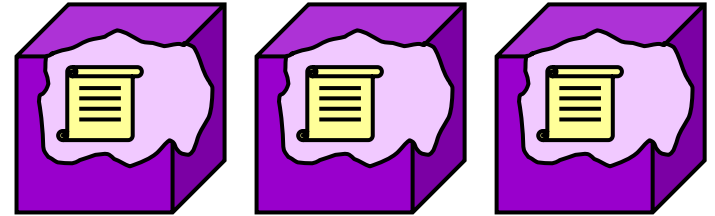
M_1: rotate by $+3\pi/16$ then measure

Then $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ all have expected value $\tfrac{1}{2}\sqrt{2}$, which ***contradicts*** the upper bound of $\tfrac{1}{2}$

$st = 11$

$3\pi/8$ $st = 01$ or 10

$\pi/8$
$-\pi/8$

$st = 00$

$$\cos^2(\pi/8) = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2}$$

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Bell Inequality violation: summary

Assuming that quantum systems are governed by **local hidden variables** leads to the Bell inequality

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$$

But this is **violated** in the case of Bell states (by a factor of $\sqrt{2}$)

Therefore, no such hidden variables exist

This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted
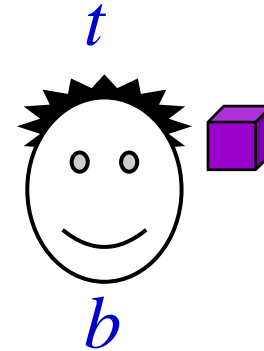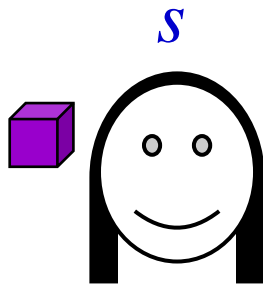
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Bell's Inequality and its violation

Part II: computer scientist's view:

input: s t

output: a b

Rules: 1. No communication after inputs received

              2. They ***win*** if $a \oplus b = s \wedge t$

st	$a \oplus b$
00	0
01	0
10	0
11	1

With classical resources, $\Pr[a \oplus b = s \wedge t] \leq 0.75$

But, with prior entanglement state $|00\rangle - |11\rangle$,

$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2} = 0.853\ldots$
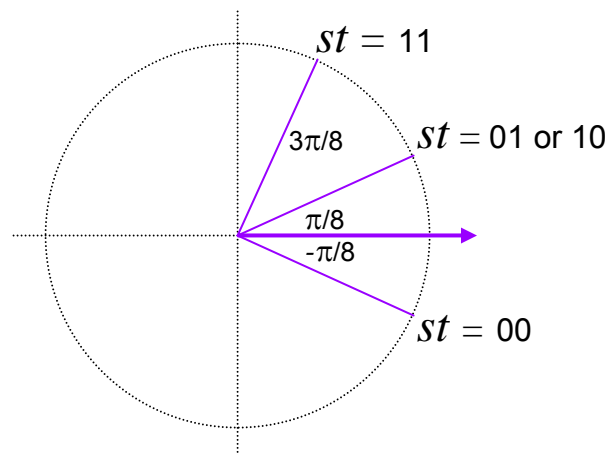
The quantum strategy

- Alice and Bob start with entanglement
$|\phi\rangle = |00\rangle - |11\rangle$

- **Alice:** if $s = 0$ then rotate by $\theta_A = -\pi/16$
  else rotate by $\theta_A = +3\pi/16$ and measure

- **Bob:** if $t = 0$ then rotate by $\theta_B = -\pi/16$
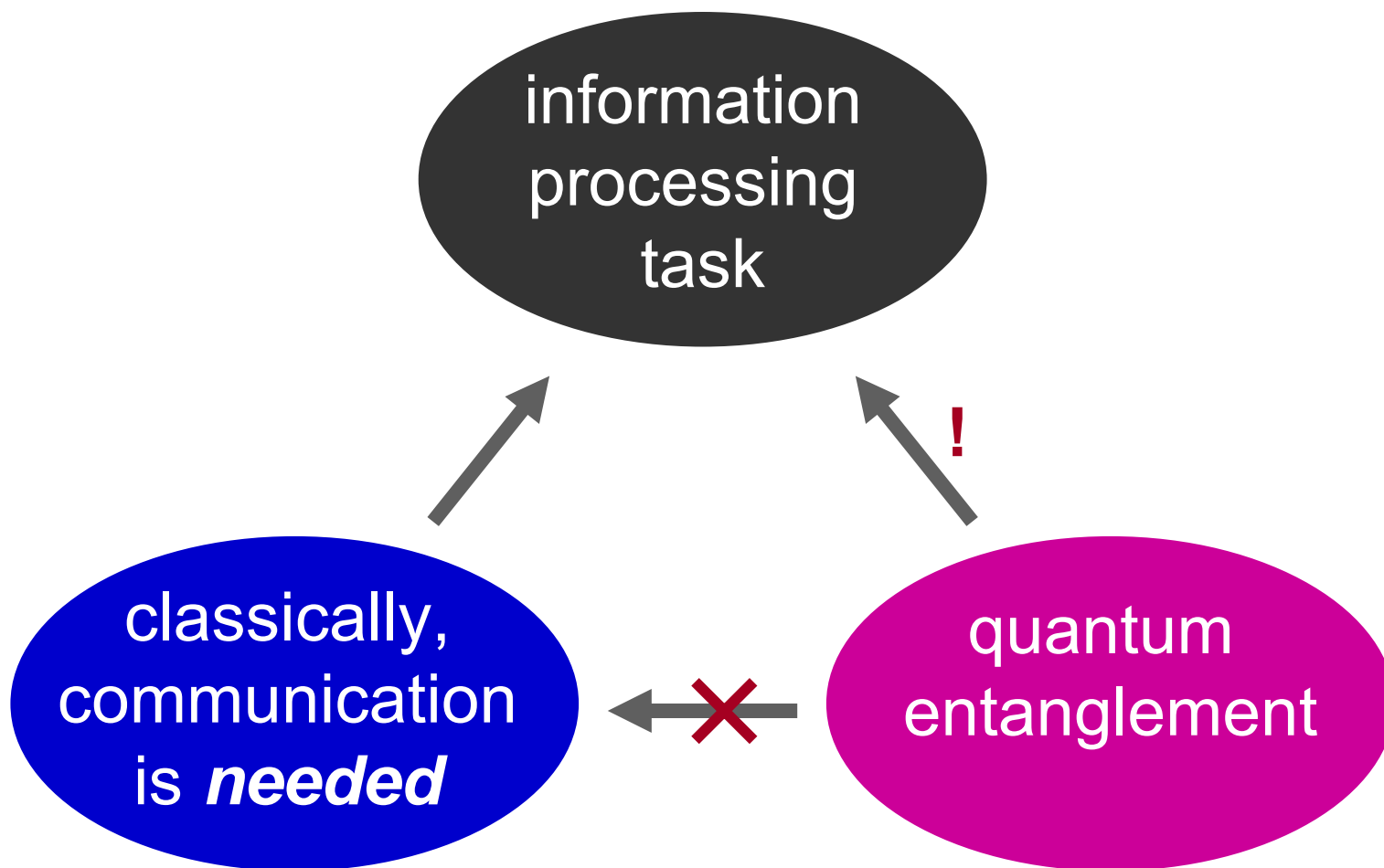 else rotate by $\theta_B = +3\pi/16$ and measure

$st = 11$
$3\pi/8$
$st = 01$ or 10
$\pi/8$
$-\pi/8$
$st = 00$

$$\cos(\theta_A - \theta_B)\,(|00\rangle - |11\rangle) + \sin(\theta_A - \theta_B)\,(|01\rangle + |10\rangle)$$

Success probability:

$$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2} = 0.853\ldots$$

Nonlocality in operational terms

information processing task

classically, communication is *needed*

quantum entanglement

!

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- **The magic square game**

Magic square game

Problem: fill in the matrix with bits such that each row has even parity and each column has odd parity

$$
\begin{array}{|c|c|c|}
\hline
a_{11} & a_{12} & a_{13} \\
\hline
a_{21} & a_{22} & a_{23} \\
\hline
a_{31} & a_{32} & a_{33} \\
\hline
\end{array}
$$

even
even
even

odd odd odd

IMPOSSIBLE

Game: ask Alice to fill in one row and Bob to fill in one column

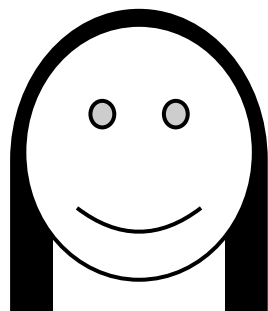They **win** iff parities are correct and bits agree at intersection

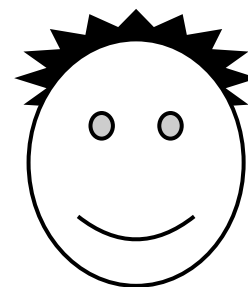**Success probabilities:** $8/9$ classical and 1 quantum

[Aravind, 2002]   (details omitted here)

Preview of communication complexity

Classical Communication Complexity

[Yao, 1979]

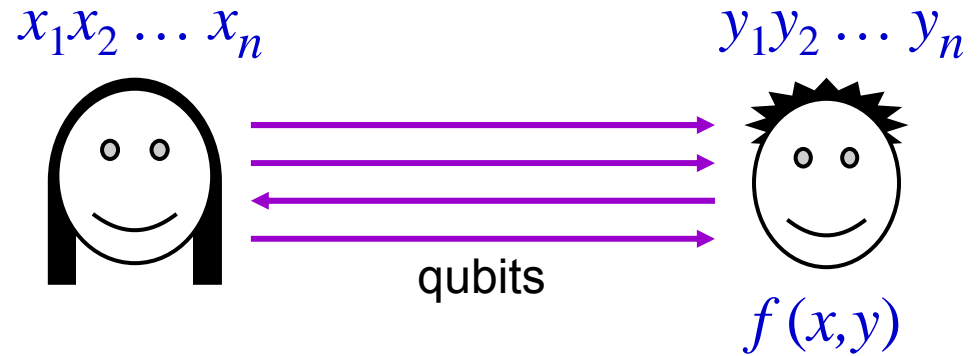$$x_1 x_2 \ldots x_n \qquad\qquad y_1 y_2 \ldots y_n$$

$$f(x, y)$$

E.g. equality function: $f(x,y) = 1$ if $x = y$, and 0 if $x \neq y$

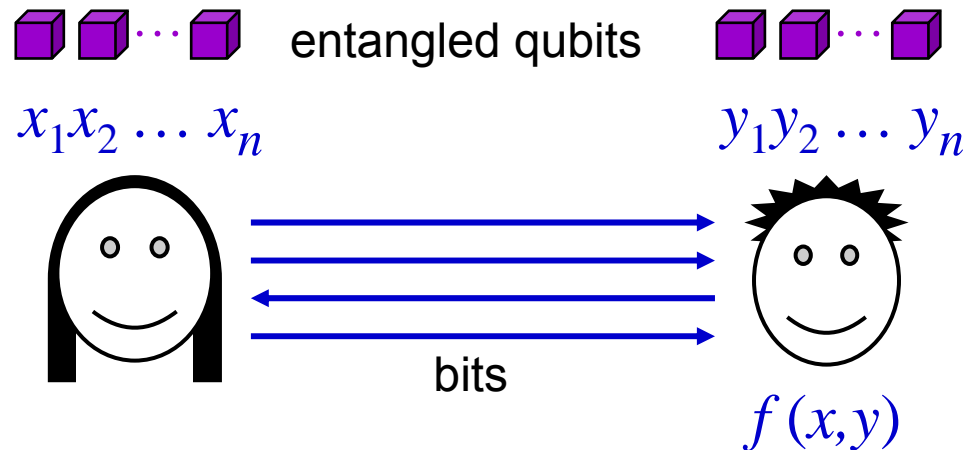Any ***deterministic*** protocol requires n bits communication

Probabilistic protocols can solve with only $O(\log(n/\varepsilon))$ bits communication (error probability ε)

Quantum Communication Complexity

Qubit communication

$x_1 x_2 \dots x_n$ $y_1 y_2 \dots y_n$

qubits

$f(x,y)$

Prior entanglement

entangled qubits

$x_1 x_2 \dots x_n$ $y_1 y_2 \dots y_n$

bits

$f(x,y)$

Question: can quantum beast classical in this context?

THE END