# Introduction to Quantum Information Processing

## CS 467 / CS 667
## Phys 467 / Phys 767
## C&O 481 / C&O 681

## Lecture 12 (2005)

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- Correction to Lecture 11: 2/3 → $\sqrt{2}/3$
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- Simulations among operations
- Separable states
- Continuous-time evolution

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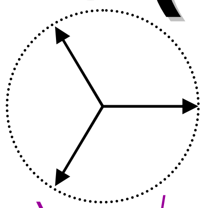
Correction to Lecture 11: √2/3 instead of 2/3

# General quantum operations (III)

**Example 3 (trine state "measurent"):**



Let $|\varphi_0\rangle = |0\rangle$, $|\varphi_1\rangle = -1/2|0\rangle + \sqrt{3}/2|1\rangle$, $|\varphi_2\rangle = -1/2|0\rangle - \sqrt{3}/2|1\rangle$

Define $A_0 = \sqrt{2/3}|\varphi_0\rangle\langle\varphi_0| = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A_1 = \sqrt{2/3}|\varphi_1\rangle\langle\varphi_1| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & +\sqrt{2} \\ +\sqrt{2} & \sqrt{6} \end{bmatrix}$ $\quad A_2 = \sqrt{2/3}|\varphi_2\rangle\langle\varphi_2| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{bmatrix}$

Then $A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = I$

The probability that state $|\varphi_k\rangle$ results in "outcome" $A_k$ is 2/3, and this can be adapted to actually yield the value of $k$ with this success probability

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# Distinguishing mixed states (I)

What's the best distinguishing strategy between these two mixed states?

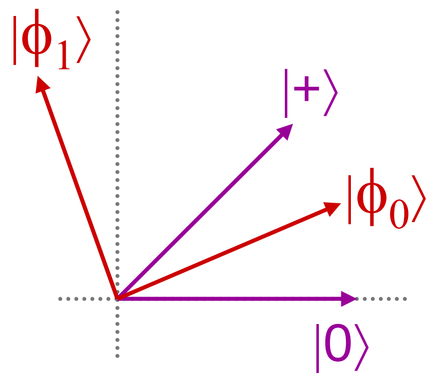
$$\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$$

$$\begin{cases} |0\rangle & \text{with prob. } ½ \\ |1\rangle & \text{with prob. } ½ \end{cases}$$

$$\rho_1 = \begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

$$\rho_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rho_1$ also arises from this orthogonal mixture:




… as does $\rho_2$ from:

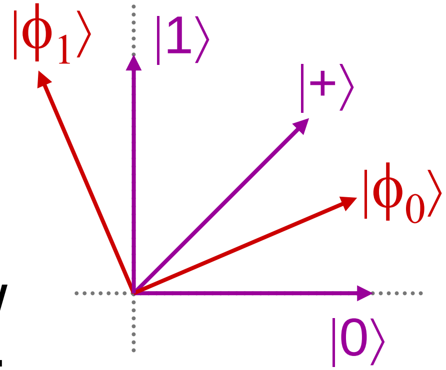
$$\begin{cases} |\phi_0\rangle & \text{with prob. } \cos^2(\pi/8) \\ |\phi_1\rangle & \text{with prob. } \sin^2(\pi/8) \end{cases}$$

$$\begin{cases} |\phi_0\rangle & \text{with prob. } ½ \\ |\phi_1\rangle & \text{with prob. } ½ \end{cases}$$

# Distinguishing mixed states (II)

We've effectively found an orthonormal basis $|\phi_0\rangle$, $|\phi_1\rangle$ in which both density matrices are diagonal:

$$\rho_2' = \begin{bmatrix} \cos^2(\pi/8) & 0 \\ 0 & \sin^2(\pi/8) \end{bmatrix} \qquad \rho_1' = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$




Rotating $|\phi_0\rangle$, $|\phi_1\rangle$ to $|0\rangle$, $|1\rangle$ the scenario can now be examined using classical probability theory:

Distinguish between two ***classical*** coins, whose probabilities of "heads" are $\cos^2(\pi/8)$ and ½ respectively (details: exercise)

**Question:** what do we do if we aren't so lucky to get two density matrices that are simultaneously diagonalizable?

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# Basic properties of the trace

The ***trace*** of a square matrix is defined as $\mathrm{Tr}\,M = \sum_{k=1}^{d} M_{k,k}$

It is easy to check that

$\mathrm{Tr}(M+N) = \mathrm{Tr}\,M + \mathrm{Tr}\,N$ and $\mathrm{Tr}(M\,N) = \mathrm{Tr}(N\,M)$

The second property implies $\mathrm{Tr}(M) = \mathrm{Tr}(U^{-1}MU) = \sum_{k=1}^{d} \lambda_k$

Calculation maneuvers worth remembering are:

$\mathrm{Tr}(|a\rangle\langle b|M) = \langle b|M|a\rangle$ and $\mathrm{Tr}(|a\rangle\langle b|c\rangle\langle d|) = \langle b|c\rangle\langle d|a\rangle$

Also, keep in mind that, in general, $\mathrm{Tr}(M\,N) \neq \mathrm{Tr}\,M\,\mathrm{Tr}\,N$

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# Recap: general quantum ops

***General quantum operations*** (a.k.a. ***"completely positive trace preserving maps"***)**:**

Let $A_1$, $A_2$, …, $A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^{\dagger} A_j = I$

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^{\dagger}$ is a general quantum op

**Example:** applying $U$ to $\rho$ yields $U \rho U^{\dagger}$

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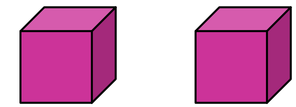
# Partial trace (I)



Two quantum registers (e.g. two qubits) in states $\sigma$ and $\mu$ (respectively) are ***independent*** if then the combined system is in state $\rho = \sigma \otimes \mu$

In such circumstances, if the second register (say) is discarded then the state of the first register remains $\sigma$

In general, the state of a two-register system may not be of the form $\sigma \otimes \mu$ (it may contain ***entanglement*** or ***correlations***)

We can define the ***partial trace***, $\mathrm{Tr}_2\, \rho$, as the unique linear operator satisfying the identity $\mathrm{Tr}_2(\sigma \otimes \mu) = \sigma$

**index means 2nd system traced out**

For example, it turns out that

$$\mathrm{Tr}_2\Big(\Big(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\Big) \otimes \Big(\tfrac{1}{\sqrt{2}}\langle 00| + \tfrac{1}{\sqrt{2}}\langle 11|\Big)\Big) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Partial trace (II)

We've already seen this defined in the case of 2-qubit systems: discarding the second of two qubits

Let $A_0 = I \otimes \langle 0| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $A_1 = I \otimes \langle 1| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

For the resulting quantum operation, state $\sigma \otimes \mu$ becomes $\sigma$

For $d$-dimensional registers, the operators are $A_k = I \otimes \langle \phi_k|$, where $|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{d-1}\rangle$ are an orthonormal basis

# Partial trace (III)

For 2-qubit systems, the partial trace is explicitly

$$\mathrm{Tr}_2\begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00}+\rho_{01,01} & \rho_{00,10}+\rho_{01,11} \\ \rho_{10,00}+\rho_{11,01} & \rho_{10,10}+\rho_{11,11} \end{bmatrix}$$

and

$$\mathrm{Tr}_1\begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00}+\rho_{10,10} & \rho_{00,01}+\rho_{10,11} \\ \rho_{01,00}+\rho_{11,10} & \rho_{01,01}+\rho_{11,11} \end{bmatrix}$$

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# POVMs (I)

***Positive operator valued measurement (POVM)***:

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Then the corresponding POVM is a stochastic operation on $\rho$ that, with probability $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)$ produces the outcome:

$$\begin{cases} j & \text{(classical information)} \\ \dfrac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} & \text{(the collapsed quantum state)} \end{cases}$$

**Example 1:** $A_j = |\phi_j\rangle\langle\phi_j|$ (orthogonal projectors)

This reduces to our previously defined measurements …

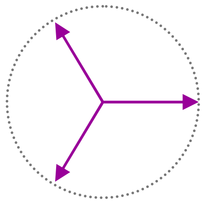
# POVMs (II)

When $A_j = |\phi_j\rangle\langle\phi_j|$ are orthogonal projectors and $\rho = |\psi\rangle\langle\psi|$,

$$\begin{aligned}\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) &= \mathrm{Tr}|\phi_j\rangle\langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle\langle\phi_j| \\ &= \langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle\langle\phi_j|\phi_j\rangle \\ &= |\langle\phi_j|\psi\rangle|^2\end{aligned}$$

Moreover, $$\frac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} = \frac{|\varphi_j\rangle\langle\varphi_j|\psi\rangle\langle\psi|\varphi_j\rangle\langle\varphi_j|}{\left|\langle\varphi_j|\psi\rangle\right|^2} = |\varphi_j\rangle\langle\varphi_j|$$

# POVMs (III)



**Example 3 (trine state "measurent"):**

Let $|\varphi_0\rangle = |0\rangle$, $|\varphi_1\rangle = -1/2|0\rangle + \sqrt{3}/2|1\rangle$, $|\varphi_2\rangle = -1/2|0\rangle - \sqrt{3}/2|1\rangle$

Define $A_0 = \sqrt{2/3}|\varphi_0\rangle\langle\varphi_0| = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A_1 = \sqrt{2/3}|\varphi_1\rangle\langle\varphi_1| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & +\sqrt{2} \\ +\sqrt{2} & \sqrt{6} \end{bmatrix}$ $\quad A_2 = \sqrt{2/3}|\varphi_2\rangle\langle\varphi_2| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{bmatrix}$

Then $A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = I$

If the input itself is an unknown trine state, $|\varphi_k\rangle\langle\varphi_k|$, then the probability that classical outcome is $k$ is 2/3 = 0.6666…

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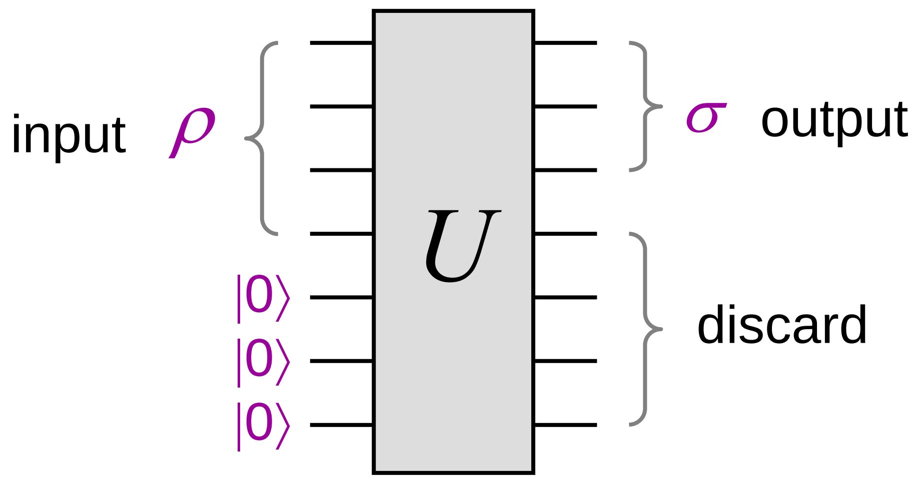
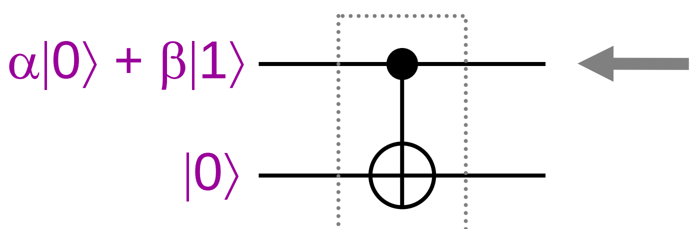
# Simulations among operations (I)

**Fact 1:** any ***general quantum operation*** can be simulated by applying a unitary operation on a larger quantum system:



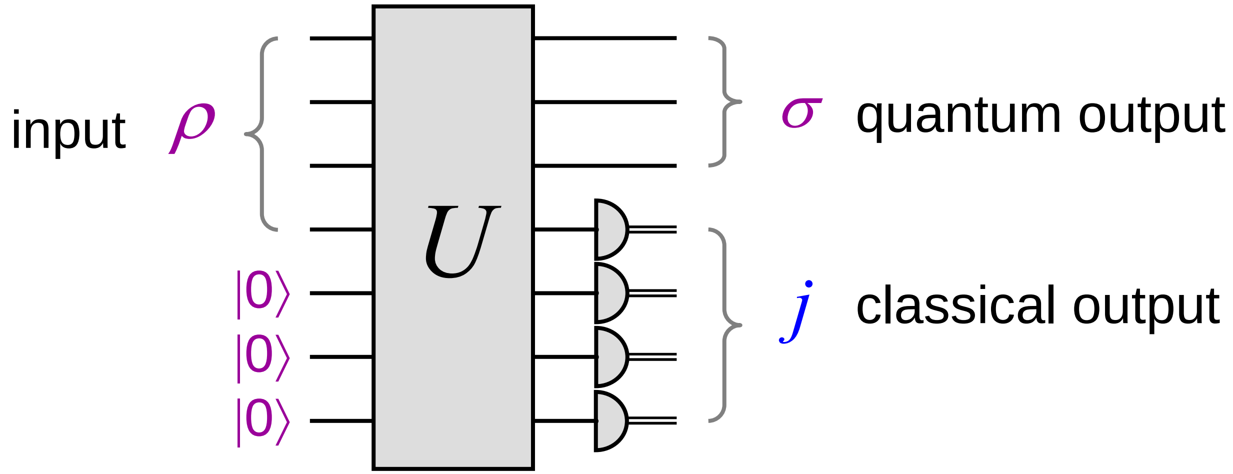

**Example:** decoherence




$$\rho = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

# Simulations among operations (II)

**Fact 2:** any ***POVM*** can also be simulated by applying a unitary operation on a larger quantum system and then measuring:



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# Separable states

A bipartite (i.e. two register) state $\rho$ is a:

- ***product state*** if $\rho = \sigma \otimes \xi$
- ***separable state*** if $\rho = \sum_{j=1}^{m} p_j \sigma_j \otimes \xi_j \quad (p_1, \ldots, p_m \geq 0)$

(i.e. a probabilistic mixture of product states)

**Question:** which of the following states are separable?

$$\rho_1 = \tfrac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$

$$\rho_2 = \tfrac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right) + \tfrac{1}{2}\left(|00\rangle - |11\rangle\right)\left(\langle 00| - \langle 11|\right)$$

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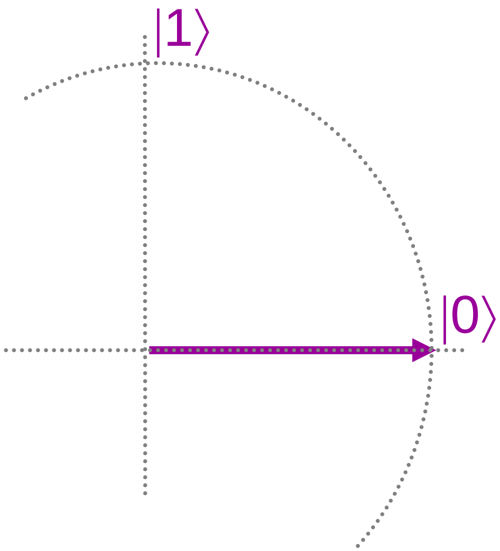
# Continuous-time evolution

Although we've expressed quantum operations in discrete terms, in real physical systems, the evolution is continuous

Let $H$ be any ***Hermitian*** matrix and $t \in \mathbf{R}$

Then $e^{iHt}$ is ***unitary*** – why?




$H = U^\dagger D U$, where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}$

Therefore $e^{iHt} = U^\dagger e^{iDt} U = U^\dagger \begin{bmatrix} e^{i\lambda_1 t} & & \\ & \ddots & \\ & & e^{i\lambda_d t} \end{bmatrix} U$ (unitary)

# THE END