Introduction to Quantum Information Processing

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Lecture 11 (2005)

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Contents

- Continuation of density matrix formalism
- Taxonomy of various normal matrices
- Bloch sphere for qubits
- General quantum operations

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Recap: density matrices (I)

The ***density matrix*** of the mixed state

$((|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots, (|\psi_d\rangle, p_d))$ is: $\rho = \sum_{k=1}^{d} p_k |\psi_k\rangle\langle\psi_k|$

Examples (from previous lecture):

1. & 2. $|0\rangle + |1\rangle$ and $-|0\rangle - |1\rangle$ both have $\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{cases} |0\rangle \text{ with prob. } ½ \\ |1\rangle \text{ with prob. } ½ \end{cases}$

4. $\begin{cases} |0\rangle + |1\rangle \text{ with prob. } ½ \\ |0\rangle - |1\rangle \text{ with prob. } ½ \end{cases}$

6. $\begin{cases} |0\rangle \text{ with prob. } ¼ \\ |1\rangle \text{ with prob. } ¼ \\ |0\rangle + |1\rangle \text{ with prob. } ¼ \\ |0\rangle - |1\rangle \text{ with prob. } ¼ \end{cases}$

(3., 4. and 6. all have) $\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Recap: density matrices (II)

Examples (continued):

5. $\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$

has: $$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

7. The first qubit of $|01\rangle - |10\rangle$ **...? (later)**

Recap: density matrices (III)

Quantum operations in terms of density matrices:

- Applying U to ρ yields $U\rho U^{\dagger}$

- Measuring state ρ with respect to the basis $|\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_d\rangle$, yields: k^{th} outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$
  —and causes the state to collapse to $|\varphi_k\rangle\langle\varphi_k|$

Since these are expressible in terms of density matrices alone (independent of any specific probabilistic mixtures), states with identical density matrices are ***operationally indistinguishable***

Characterizing density matrices

$$\rho = \sum_{k=1}^{d} p_k |\psi_k\rangle\langle\psi_k|$$

Three properties of ρ :

- $\text{Tr}\rho = 1$ ($\text{Tr}M = M_{11} + M_{22} + ... + M_{dd}$)
- $\rho = \rho^\dagger$ (i.e. ρ is Hermitian)
- $\langle\varphi|\rho|\varphi\rangle \geq 0$, for all states $|\varphi\rangle$

Moreover, for ***any*** matrix ρ satisfying the above properties, there exists a probabilistic mixture whose density matrix is ρ

Exercise: show this

- Continuation of density matrix formalism
- Taxonomy of various normal matrices
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Normal matrices

Definition: A matrix M is ***normal*** if $M^\dagger M = MM^\dagger$

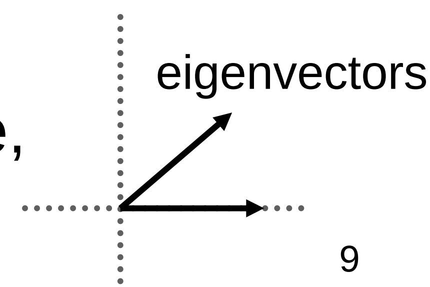
Theorem: M is normal iff there exists a unitary U such that $M = U^\dagger D U$, where D is diagonal (i.e. unitarily diagonalizable)

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$

Examples of ***ab***normal matrices:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not even diagonalizable

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable, but not unitarily



Unitary and Hermitian matrices

Normal: $$M = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$ with respect to some orthonormal basis

Unitary: $M^\dagger M = I$ which implies $|\lambda_k|^2 = 1$, for all k

Hermitian: $M = M^\dagger$ which implies $\lambda_k \in \mathbf{R}$, for all k

Question: which matrices are both unitary ***and*** Hermitian?

Answer: reflections ($\lambda_k \in \{+1, -1\}$, for all k)

Positive semidefinite

Positive semidefinite: Hermitian and $\lambda_k \geq 0$, for all k

Theorem: M is positive semidefinite iff M is Hermitian and, for all $|\varphi\rangle$, $\langle\varphi|M|\varphi\rangle \geq 0$

(**Positive *definite*:** $\lambda_k > 0$, for all k)

Projectors and density matrices

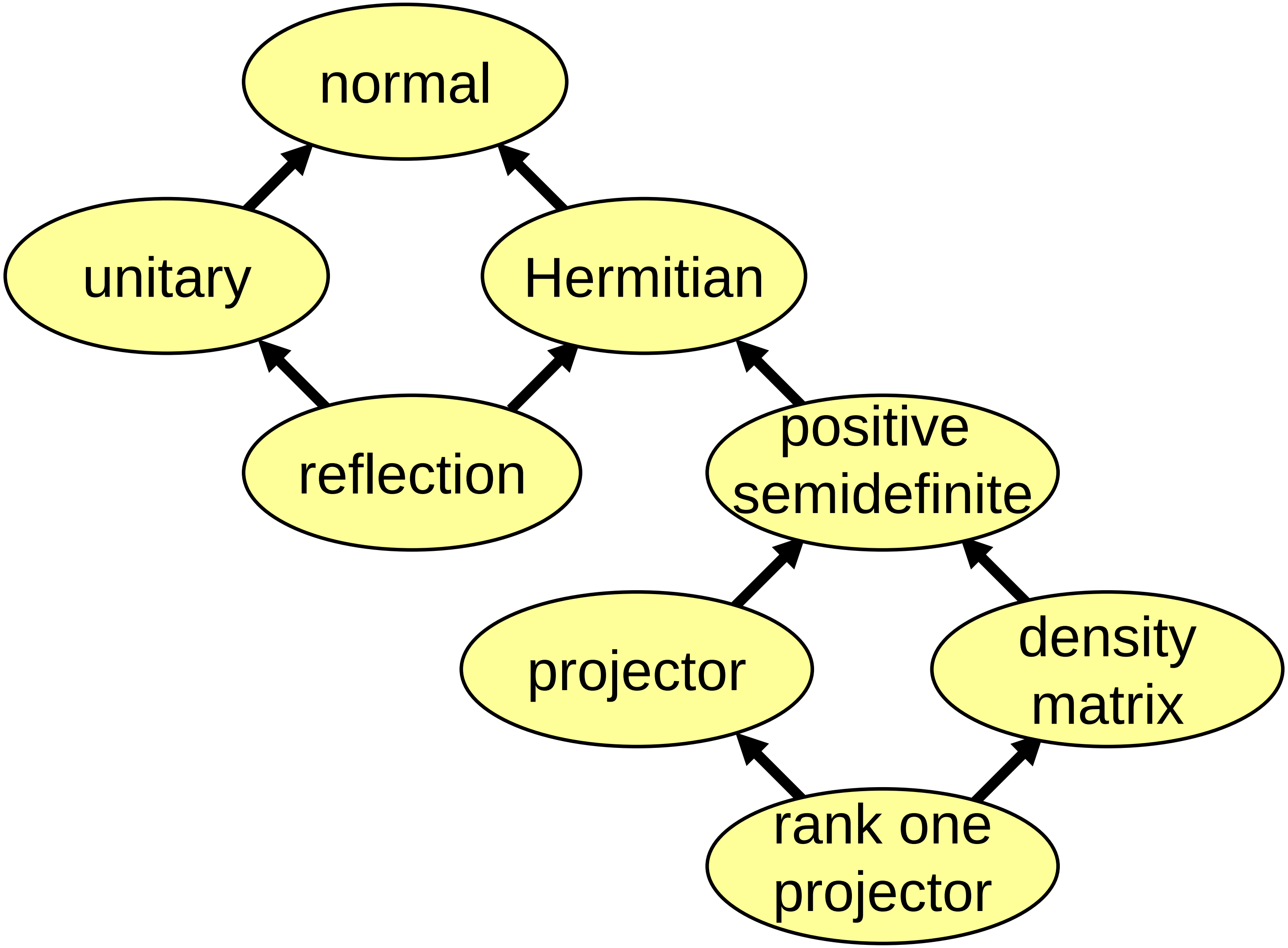
Projector: Hermitian and $M^2 = M$, which implies that M is positive semidefinite and $\lambda_k \in \{0,1\}$, for all k

Density matrix: positive semidefinite and Tr $M = 1$, so $\sum_{k=1}^{d} \lambda_k = 1$

Question: which matrices are both projectors ***and*** density matrices?

Answer: rank-1 projectors ($\lambda_k = 1$ if $k = j$; otherwise $\lambda_k = 0$)

Taxonomy of normal matrices



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Bloch sphere for qubits (I)

Consider the set of all 2x2 density matrices ρ

They have a nice representation in terms of the ***Pauli matrices*:**

$$\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Note that these matrices—combined with I—form a ***basis*** for the vector space of all 2x2 matrices

We will express density matrices ρ in this basis

Note that the coefficient of I is **½**, since X, Y, Y are traceless

Bloch sphere for qubits (II)

We will express $\rho = \dfrac{I + c_x X + c_y Y + c_z Z}{2}$

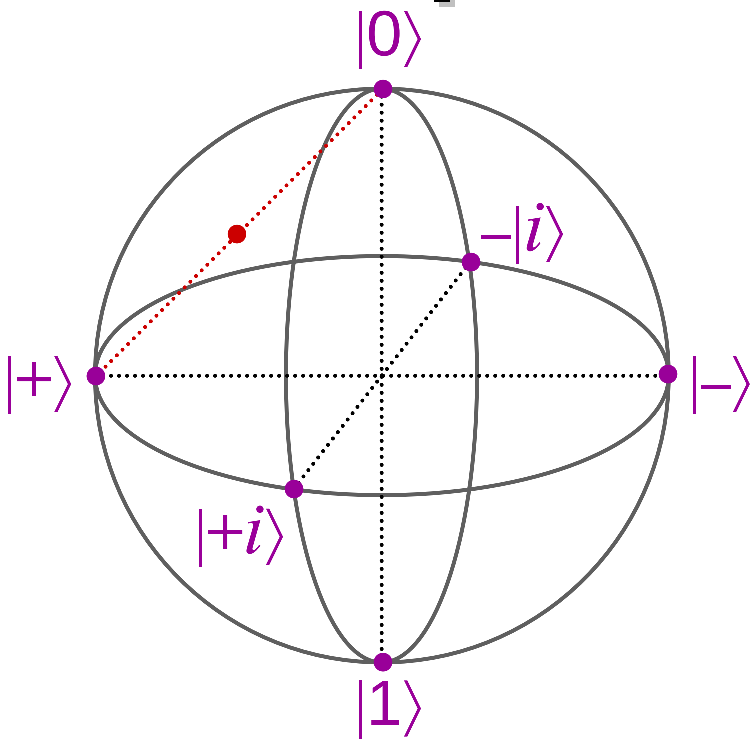
First consider the case of pure states $|\psi\rangle\langle\psi|$, where, without loss of generality, $|\psi\rangle = \cos(\theta)|0\rangle + e^{2i\phi}\sin(\theta)|1\rangle$ $(\theta, \phi \in \mathbf{R})$

$$\rho = \begin{bmatrix} \cos^2\theta & e^{-i2\varphi}\cos\theta\sin\theta \\ e^{i2\varphi}\cos\theta\sin\theta & \sin^2\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\cos(2\theta) & e^{-i2\varphi}\sin(2\theta) \\ e^{i2\varphi}\sin(2\theta) & 1-\cos(2\theta) \end{bmatrix}$$

Therefore $c_z = \cos(2\theta)$, $c_x = \cos(2\phi)\sin(2\theta)$, $c_y = \sin(2\phi)\sin(2\theta)$

These are ***polar coordinates*** of a unit vector $(c_x, c_y, c_z) \in \mathbf{R}^3$

Bloch sphere for qubits (III)



$|+\rangle = |0\rangle + |1\rangle$

$|-\rangle = |0\rangle - |1\rangle$

$|{+i}\rangle = |0\rangle + i|1\rangle$

$|{-i}\rangle = |0\rangle - i|1\rangle$

Note that ***orthogonal*** corresponds to ***antipodal*** here

Pure states are on the surface, and mixed states are inside (being weighted averages of pure states)

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General quantum operations (I)

General quantum operations (a.k.a. ***"completely positive trace preserving maps"***, ***"admissible operations"***)**:**

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^{\dagger} A_j = I$

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^{\dagger}$ is a general quantum op

Example 1 (unitary op): applying U to ρ yields $U \rho U^{\dagger}$

General quantum operations (II)

Example 2 (decoherence): let $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$

This quantum op maps ρ to $|0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$

For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

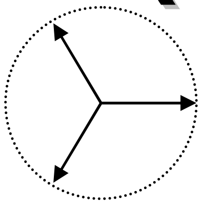
$$\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \mapsto \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

Corresponds to measuring ρ "without looking at the outcome"

After looking at the outcome, ρ becomes $\begin{cases} |0\rangle\langle 0| & \text{with prob. } |\alpha|^2 \\ |1\rangle\langle 1| & \text{with prob. } |\beta|^2 \end{cases}$

General quantum operations (III)

Example 3 (trine state "measurent"):



Let $|\varphi_0\rangle = |0\rangle$, $|\varphi_1\rangle = -1/2|0\rangle + \sqrt{3}/2|1\rangle$, $|\varphi_2\rangle = -1/2|0\rangle - \sqrt{3}/2|1\rangle$

Define $A_0 = 2/3|\varphi_0\rangle\langle\varphi_0| = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A_1 = 2/3|\varphi_1\rangle\langle\varphi_1| = \frac{1}{6}\begin{bmatrix} 1 & +\sqrt{3} \\ +\sqrt{3} & 3 \end{bmatrix}$ $A_2 = 2/3|\varphi_2\rangle\langle\varphi_2| = \frac{1}{6}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$

Then $A_0^t A_0 + A_1^t A_1 + A_2^t A_2 = I$

The probability that state $|\varphi_k\rangle$ results in "outcome" A_k is 4/9, and this can be adapted to actually yield the value of k with this success probability

General quantum operations (IV)

Example 4 (discarding the second of two qubits):

Let $A_0 = I \otimes \langle 0| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $A_1 = I \otimes \langle 1| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

State $\rho \otimes \sigma$ becomes ρ

State $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\langle 00| + \frac{1}{\sqrt{2}}\langle 11|\right)$ becomes $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note 1: it's the same density matrix as for $\big((½, |0\rangle), (½, |1\rangle)\big)$

Note 2: the operation is the ***partial trace*** $\mathrm{Tr}_2\, \rho$

THE END