2 Quantum Error Correction

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2.1 Error models

2.1.1 Generic 1 qubit error

A generic qubit has the state

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

but qubits might not be isolated (and now we know that there can be information hidden in quantum correlation between systems) so the most general evolution which include an environment (with state $|\epsilon\rangle$) takes the form

$$|0\rangle|\epsilon\rangle \longrightarrow |0\rangle|\epsilon_0^0\rangle + |1\rangle|\epsilon_0^1\rangle$$
$$|1\rangle|\epsilon\rangle \longrightarrow |0\rangle|\epsilon_1^0\rangle + |1\rangle|\epsilon_1^1\rangle$$

and thus

$$
\begin{aligned}
(\alpha|0\rangle + & \ \beta|1\rangle)|\epsilon\rangle \rightarrow \\
& (\alpha|0\rangle + \beta|1\rangle)\frac{1}{2}(|\epsilon_0^0\rangle + |\epsilon_1^1\rangle) \quad (\Rightarrow \mathbb{1}|\Psi\rangle) \\
& +(\alpha|0\rangle - \beta|1\rangle)\frac{1}{2}(|\epsilon_0^0\rangle - |\epsilon_1^1\rangle) \quad (\Rightarrow \mathbf{Z}|\Psi\rangle) \\
& +(\alpha|1\rangle + \beta|0\rangle)\frac{1}{2}(|\epsilon_0^1\rangle + |\epsilon_1^0\rangle) \quad (\Rightarrow \mathbf{X}|\Psi\rangle) \\
& +(\alpha|1\rangle - \beta|0\rangle)\frac{1}{2}(|\epsilon_0^1\rangle - |\epsilon_1^0\rangle) \quad (\Rightarrow \mathbf{iY}|\Psi\rangle)
\end{aligned}
$$

The effect of the noise is to apply the error operators $\mathbb{1}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ to the state $|\Psi\rangle$ depending on what the state of the environment is.

Note that these four operator form an operator basis in the acting on the 2 dimensional Hilbert space of one qubit. For n qubits we have 4^n possible operators, obtained by the tensor product of each one-qubit operator, i.e.. for two qubits we would have $\mathbb{1} \otimes \mathbb{1}, \boldsymbol{X} \otimes \mathbb{1}, \ldots, \boldsymbol{X} \otimes \boldsymbol{X}, \ldots \boldsymbol{Z} \otimes \boldsymbol{Z}$.

2.1.2 Phase shift

Let’s look at some simple examples of noise operators in physical systems such as decoherence:

$$
\begin{aligned}
|0\rangle|\epsilon\rangle &\longrightarrow |0\rangle|\epsilon_0\rangle = |0\rangle|\epsilon\rangle \\
|1\rangle|\epsilon\rangle &\longrightarrow |1\rangle|\epsilon_1\rangle = \mathrm{e}^{\mathrm{i}\theta}|1\rangle|\epsilon\rangle
\end{aligned}
$$

Thus

$$
(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \rightarrow (\alpha|0\rangle + \mathrm{e}^{\mathrm{i}\theta}\beta|1\rangle)|\epsilon\rangle
$$

and which can be rewritten as

$$
\begin{aligned}
(\alpha|0\rangle + e^{i\theta}\beta|1\rangle)|\epsilon\rangle &= \frac{1+e^{i\theta}}{2}(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \\
&+ \frac{1-e^{i\theta}}{2}(\alpha|0\rangle - \beta|1\rangle)|\epsilon\rangle \\
&= \frac{1+e^{i\theta}}{2}\mathbb{1}(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \\
&+ \frac{1-e^{i\theta}}{2}Z(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle
\end{aligned}
$$

Here we have a certain amplitude ($\frac{1+e^{i\theta}}{2}$) of nothing happening ($\mathbb{1}$) and ($\frac{1+e^{i\theta}}{2}$) of a Z error happening.

2.1.3 Unwanted interaction with another system

Another example is a system which interacts with an environment (qubit 2) (with a coupling $U = e^{-i\theta Z_1 Z_2/2}$ previously encountered). If the second qubit starts in the state $(|0_2\rangle + |1_2\rangle)/2$, we will end up in a state

$$|0\rangle(|0_2\rangle + |1_2\rangle)/\sqrt{2} \rightarrow |0\rangle \underbrace{(e^{-i\theta/2}|0_2\rangle + e^{i\theta/2}|1_2\rangle)/\sqrt{2}}_{|\epsilon_1\rangle}$$

$$|1\rangle(|0_2\rangle + |1_2\rangle)/\sqrt{2} \rightarrow |1\rangle \underbrace{(e^{i\theta/2}|0_2\rangle + e^{-i\theta/2}|1_2\rangle)/\sqrt{2}}_{|\epsilon_2\rangle}$$

the overlap between the two states of the environment is then

$$\begin{aligned}\langle\epsilon_1||\epsilon_2\rangle &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta/2}\langle 0_2| + \mathrm{e}^{-\mathrm{i}\theta/2}\langle 1_2|)(\mathrm{e}^{\mathrm{i}\theta/2}|0_2\rangle + \mathrm{e}^{-\mathrm{i}\theta/2}|1_2\rangle) \\ &= \cos\theta\end{aligned}$$

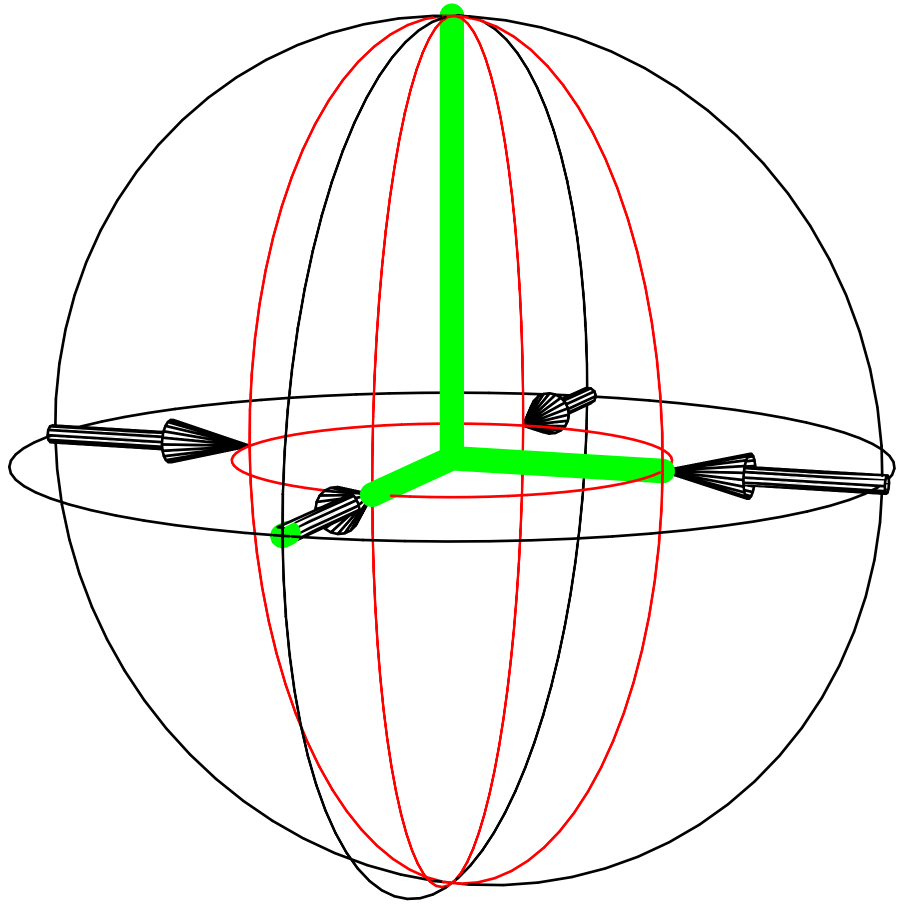
so as the interaction increases, the overlap between the environment states decrease up to $\theta = \pi/2$ when the overlap is zero.

With the state

$$|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle)$$

the density matrix for the first qubit becomes

$$\begin{aligned}\rho_1 &= \frac{1}{2}\mathrm{Tr}[\mathrm{U}|\Psi\rangle(|0_2\rangle + |1_2\rangle)(\langle 0_2| + \langle 1_2|)\langle\Psi|\mathrm{U}^{\dagger}] \\ &= \begin{pmatrix} \alpha\alpha^* & \alpha\beta^*\cos\theta \\ \alpha^*\beta\cos\theta & \beta\beta^* \end{pmatrix}\end{aligned}$$



The density matrix for the first qubit is:

$$\begin{aligned}
\rho_1 &= \frac{1}{2}\mathrm{Tr}_2[\mathrm{U}|\Psi\rangle(|0_2\rangle + |1_2\rangle)(\langle 0_2| + \langle 1_2|)\langle\Psi|\mathrm{U}^\dagger] \\
&= \begin{pmatrix} \alpha\alpha^* & \alpha\beta^*\cos\theta \\ \alpha^*\beta\cos\theta & \beta\beta^* \end{pmatrix} \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta \mathrm{Z}_1/2} \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{pmatrix} \mathrm{e}^{\mathrm{i}\theta \mathrm{Z}_1/2} \\
&\quad + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta \mathrm{Z}_1/2} \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{pmatrix} \mathrm{e}^{-\mathrm{i}\theta \mathrm{Z}_1/2} \\
&= \sum_i A_i \rho A_i^\dagger
\end{aligned}$$

The A_i are called Krauss operators.

2.1.4 The depolarization error model

Let's look at the depolarizing error model which consists in applying any one of the Pauli matrices (chosen at random) to the state. So the quantum operation is given by the family of operators:

$$\{\mathcal{A}_i\} = \{(1 - \frac{3p}{4})\mathbb{1}, \frac{p}{4}X, \frac{p}{4}Y, \frac{p}{4}Z\}$$

or

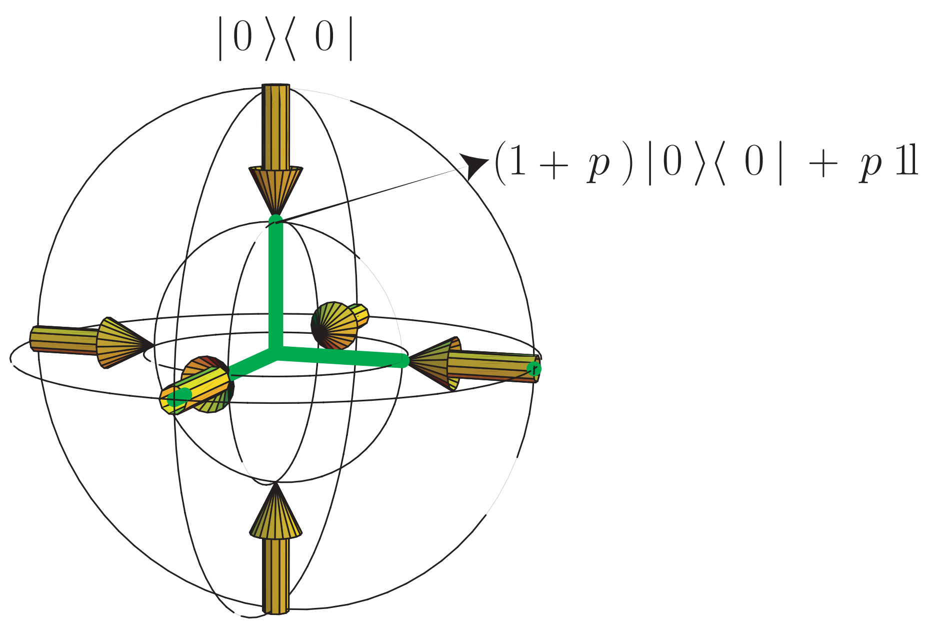
$$\rho \longrightarrow (1-p)\rho + \frac{p}{4}(\mathbb{1}\rho\mathbb{1} + X\rho X + Y\rho Y + Z\rho Z)$$

for

$$\rho = \begin{pmatrix} \alpha & \beta \\ \beta^* & 1-\alpha \end{pmatrix}$$

$$\rho \to (1-p)\rho + \frac{p}{4}\Big[\begin{pmatrix} \alpha & \beta \\ \beta^* & 1-\alpha \end{pmatrix} + \begin{pmatrix} 1-\alpha & \beta^* \\ \beta & \alpha \end{pmatrix} + \begin{pmatrix} 1-\alpha & -\beta^* \\ -\beta & \alpha \end{pmatrix} + \begin{pmatrix} \alpha & -\beta \\ -\beta^* & 1-\alpha \end{pmatrix} \Big]$$

$$= (1-p)\rho + p\mathbb{1}$$



2.1.5 Krauss operators

Let's suppose that a system and its environment start in a separable state and for simplicity that they are both in pure states ($|\Psi\rangle = |\Psi_s\rangle \otimes |\Psi_e\rangle$).

$$\begin{aligned}
\rho_s &= Tr_e U|\Psi_e\rangle \otimes |\Psi_s\rangle\langle\Psi_s| \otimes \langle\Psi_e|U^\dagger \\
&= \sum_i \underbrace{\langle\Psi_e^i|U|\Psi_e\rangle}_{A_i} \otimes |\Psi_s\rangle\langle\Psi_s| \otimes \underbrace{\langle\Psi_e|U^\dagger|\Psi_e^i\rangle}_{A_i^\dagger}
\end{aligned}$$

The set of operators $\{A_i\}$ are called Krauss operators. They are not unique, as we can use another basis for the trace over the environment, but up to this freedom they are uniquely defined. The unitarity of the whole system-environment implies that

$$\sum_{\mathbf{i}} \mathbf{A}_{\mathbf{i}}^{\dagger} \mathbf{A}_{\mathbf{i}} = \mathbb{1} \tag{1}$$

The $\{A_i\}$ described the non-unitary evolution (when we look only at the first system and the initial state factorizes) and describe the noise influencing the device which we want to use for quantum information processing.

2.2 Quantum encoding

2.2.1 Code

A code $\mathcal{C}$ is a subspace of a Hilbert space. For k qubits it would have dimension 2^k.

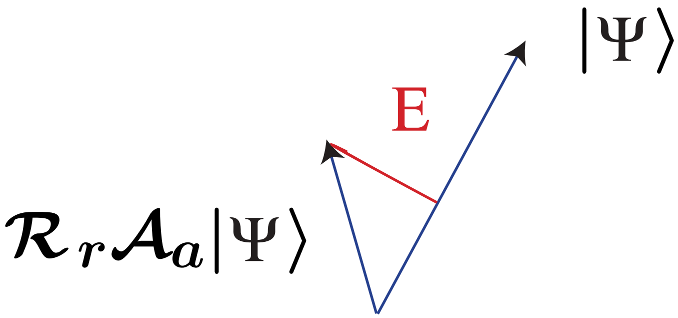
We can reach this subspace using an encoding, i.e. a unitary transformation which maps a state with the quantum information and extra qubits (called ancillae) into a new state which is protected against some form of noise.

We also define $\{\mathcal{R}_r\}$ as a recovery operator. This will be the operator which bring the state back to its original after a set of errors $\{\mathcal{A}_a\}$ occurred.

2.2.2 Error

In order to define an error correcting code we need to define a notion of error

$$\boldsymbol{E} = |(\mathcal{R}_r\mathcal{A}_a - \langle\Psi|\mathcal{R}_r\mathcal{A}_a|\Psi\rangle)|\Psi\rangle|^2$$



2.2.3 Quantum error correcting code

An error correcting code is a triple $(\mathcal{C}, \mathcal{A}, \mathcal{R})$such that $E(\mathcal{C}, \mathcal{A}, \mathcal{R}$
0. This implies that

$$\mathcal{R}_r \mathcal{A}_a = \lambda_{ra} \mathbb{1}$$

on the code $\mathcal{C}$.

An equivalent definition for a quantum error correcting code in term of properties of a basis state $(|i_L\rangle)$ of the code $\mathcal{C}$

$$\langle i_L | \mathcal{A}_a^\dagger \mathcal{A}_b | j_L \rangle = \delta_{ij} c_{ab}$$

- If $i \neq j$ $\longrightarrow$ basis states are mapped to orthogonal states
- If $i = j$ $\longrightarrow$ that coherence is preserved (relative length of the basis vectors).

Note if $\{\mathcal{A}_a\}$ is a correctable set of errors, any other set obtained from a linear combination of the these errors also form a correctable set.

2.2.4 Example of Quantum Encoding:Collective errors

As an example,let's look at a quantum version of the classical case we have given before (where the errors are collective bit flips $\{A_i\} = \{\sqrt{(1-p)}\mathbb{1}, \sqrt{p}\, X X\}$) would be given by

$$\underbrace{(\alpha|0\rangle + \beta|1\rangle)}_{Q.Info} \;\; \underbrace{|0\rangle}_{ancilla} \rightarrow$$

$$(1-p)(\alpha|00\rangle + \beta|10\rangle)(\alpha^*\langle 00| + \beta^*\langle 00|)$$
$$+p(\alpha|11\rangle + \beta|01\rangle)(\alpha^*\langle 11| + \beta^*\langle 01|)$$

In this example the encoding is trivial (the unit operator). The recovery operator consist in measuring the parity, which is achieved by a control not with the control on the second bit i.e. in Dirac notation $\mathbb{1}|0\rangle\langle 0| + \boldsymbol{X}|1\rangle\langle 1|$ or in matrix notation

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2}$$

2.2.5 The 3-qubit phase error QEC code

Let's look at slightly more complex quantum error correction code, we mention before the model of decoherence. Lets assume that the error are independent from one bit to another. For one qubit the error model is $\{\sqrt{1-p}\mathbb{1}, \sqrt{p}Z\}$. For 3 qubits we get the quantum operation defined by the Krauss operators

$$\{\mathcal{A}_a\} = \{(1-p)^{3/2}\mathbb{1},$$

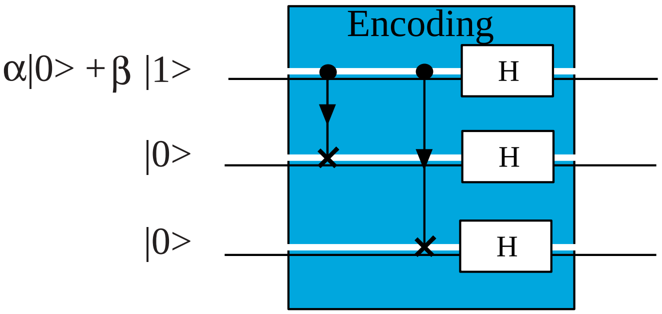
$$(1-p)\sqrt{p}Z_1, (1-p)\sqrt{p}Z_2, (1-p)\sqrt{p}Z_3,$$

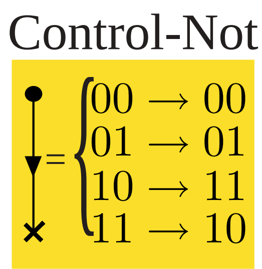
$$p\sqrt{1-p}Z_1Z_2, p\sqrt{1-p}Z_2Z_3, p\sqrt{1-p}Z_1Z_3,$$

$$p^{3/2}Z_1Z_2Z_3\}.$$

Unfortunately we cannot find a code which protects for all these errors but if $p << 1$, the dominant error term is the one error term Z_i term and we can neglect the other ones (as $p^2 << p$).

To protect for at most one Z error, we can get use the encoding from the following quantum circuit





Hadamard

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

which transform the state into

$$
\begin{aligned}
(\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle &\longrightarrow (\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle)|0\rangle \\
&\longrightarrow (\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle) \\
&\longrightarrow (\alpha|+\rangle|+\rangle|+\rangle + \beta|-\rangle|-\rangle|-\rangle).
\end{aligned}
$$

$$\{\mathcal{A}_a\} \approx \{(1-3p/2)\mathbb{1},$$

$$\sqrt{p}Z_1, \sqrt{p}Z_2, \sqrt{p}Z_3, + \text{ higher order in p}$$

and remember that

$$\rho_f = \sum_a \mathcal{A}_a |\Psi\rangle\langle\Psi| \mathcal{A}_a^\dagger$$

And thus the state becomes for each operator

$$(\alpha \; |+\rangle|+\rangle|+\rangle + \beta|-\rangle|-\rangle|-\rangle) \rightarrow$$

$$(\alpha|+\rangle|+\rangle|+\rangle + \beta|-\rangle|-\rangle|-\rangle) \text{with prob. (1-3p/2)}$$

$$(\alpha|-\rangle|+\rangle|+\rangle + \beta|+\rangle|-\rangle|-\rangle) \text{with prob. p}$$

$$(\alpha|+\rangle|-\rangle|+\rangle + \beta|-\rangle|+\rangle|-\rangle) \text{with prob. p}$$

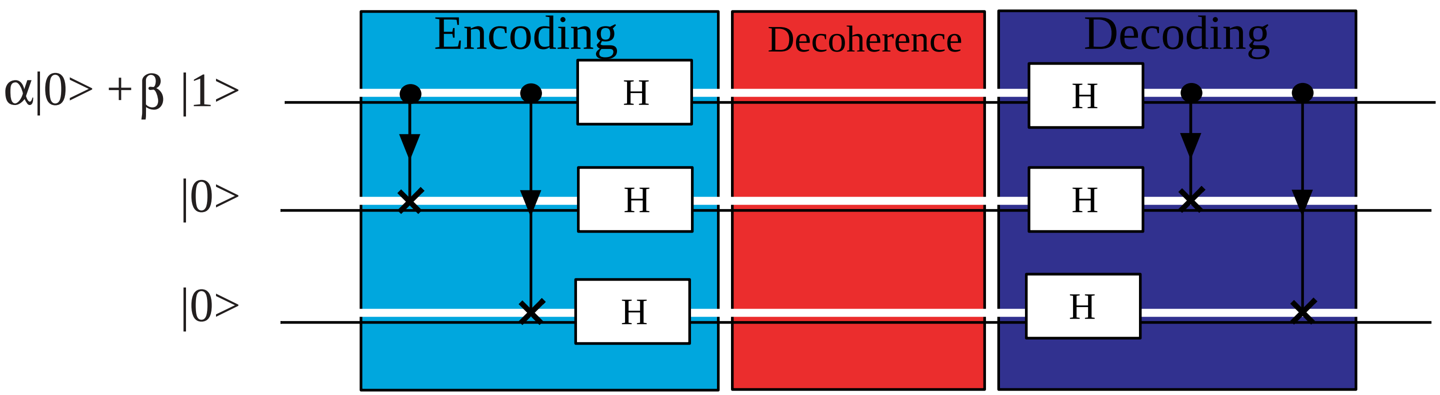
$$(\alpha|+\rangle|+\rangle|-\rangle + \beta|-\rangle|-\rangle|+\rangle) \text{with prob. p}$$

Note: the initial state and its corrupted version are orthogonal and have kept relative coherence

After this circuit we get the states

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle)\text{with prob. (1-3p/2)}$$
$$(\alpha|1\rangle + \beta|0\rangle)|1\rangle|1\rangle)\text{with prob. p}$$
$$(\alpha|0\rangle + \beta|1\rangle)|1\rangle|0\rangle\text{with prob. p}$$
$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle|1\rangle\text{with prob. p}$$

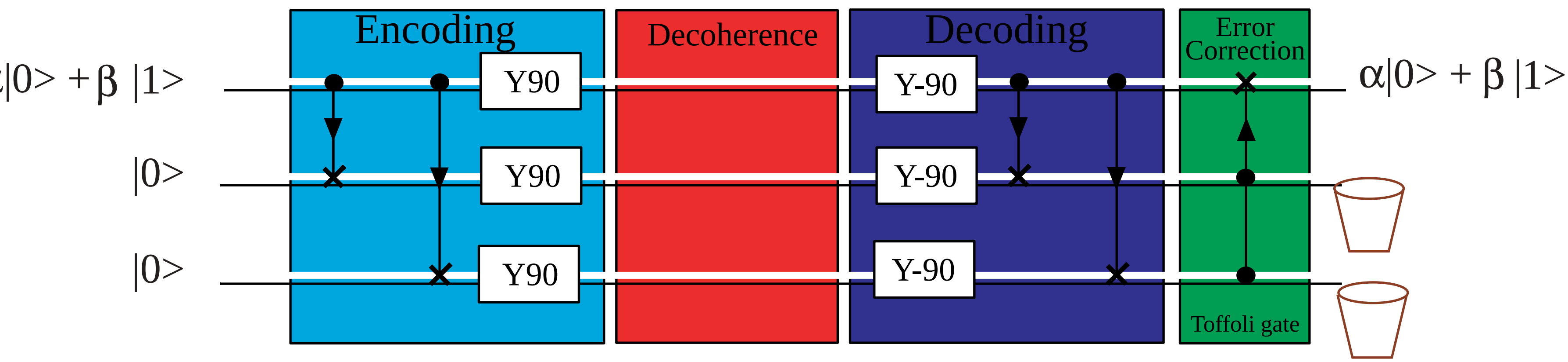
The last two qubits identify which error has occurred. It is called the syndrome.



To get the original state on the first qubit we just need to flip the first bit if and only if the two ancilla bits are in the state $|1\rangle$ (that is called a Toffoli gate) and get

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle) \text{with prob. (1-3p/2)}$$
$$(\alpha|0\rangle + \beta|1\rangle)|1\rangle|1\rangle) \text{with prob. p}$$
$$(\alpha|0\rangle + \beta|1\rangle)|1\rangle|0\rangle \text{with prob. p}$$
$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle|1\rangle \text{with prob. p}$$

Thus the whole circuit is given by:



2.2.6 Error analysis:

What is achieved by this code and its error correction procedure is that the no or one error part of the quantum operation ($\mathbb{1}$ or Z_1, Z_2, Z_3), are not going to lead to corrupted quantum information.

In the quantum operation consisting of independent 1-qubit errors, they corresponded to the linear term in p. However in this error model there were amplitudes $p^2(1-p)$ to get the errors Z_1Z_2, Z_2Z_3, Z_1Z_3 and p^3 to get the error $Z_1Z_2Z_3$. These errors are detectable because they lead to state outside the codes but they are not correctable because they lead to states which are not orthogonal to the ones with errors Z_1, Z_2, Z_3.

We are thus left with a probability $3p^2(1-p) - p^3$ of still having a corrupted state. This should be compared to the probability p of having a single corrupted qubit and not doing any quantum error correction. The quantum error correction procedure is useful if

$$3p^2(1-p) - p^3 < p \Rightarrow p < 0.5 \tag{3}$$

2.2.7 Shor's code

We have shown that we can correct phase Z errors using the code mentioned above, but generic (independent) errors are linear combination of X, Y, Z, so how can we generalize the above code?

In turns out that if we correct for X and Z errors independently we will also correct for Y. Realizing this, Peter Shor gave an explicit code for correcting a generic one qubit error using 8 extra qubits (ancilla):

$$(\alpha|0\rangle+\beta|1\rangle)|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle \Rightarrow$$

$$\begin{aligned}
&\alpha(|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle)\\
&\qquad(|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle)\\
&\qquad\qquad(|0\rangle|0\rangle|0\rangle+|1\rangle|1\rangle|1\rangle)\\
+&\\
&\beta(|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle)\\
&\qquad(|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle)\\
&\qquad\qquad(|0\rangle|0\rangle|0\rangle-|1\rangle|1\rangle|1\rangle)
\end{aligned}$$

$$\mathcal{A}=\{\mathbb{1},X_1,Y_1,Z_1,\ldots,Y_9,Z_9\}$$

We have seen how we can take quantum information and encoded it in a new state so that is more robust against corruption. This is a big step towards having robust quantum information processing. But there some of the questions remaining:

- How do we find codes?
- How do we protect information during information processing?
- How do we encode so that a given algorithm with “N” gates is performed robustly?

3 Fault tolerant quantum computing

Contents

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- How do we find codes?
- How do we protect information during information processing?
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3.1 Encoded operations and error propagation

Everything we have done now has assumed that we wanted to keep a state intact, but in quantum computation we need to manipulate states, i.e we need to make transformation

$$\alpha|0_L\rangle + \beta|1_L\rangle \rightarrow \alpha'|0_L\rangle + \beta'|1_L\rangle$$
$$\alpha|+++\rangle + \beta|---\rangle \rightarrow \alpha'|+++\rangle + \beta'|---\rangle$$

We could decode, then do an operation on the qubit and reencode, but this would leave the qubit unprotected from noise. So we need to do gates in such a way that they remain protected.

A crucial element for understanding how to implement gates in a fault tolerant way on encoded states is to see how errors propagates through a circuit. In particular there are gates organized in such a way that one error will propagate to more than one error. These are bad as, if we use 1 error correcting codes, these gates will destroy the advantage of error correction.

A useful set of operations are the normalizer operations. They are operation which preserve the Pauli operators. Example are given by The Pauli matrices themselves:

$$\boldsymbol{X} \longrightarrow \boldsymbol{ZXZ} = -\boldsymbol{X}$$

but these one either give you the same operator or minus this operator. More interesting is the Hadamard gate

$$\boldsymbol{H} = \boldsymbol{H}^{\dagger} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

$$\boldsymbol{X} \longrightarrow \boldsymbol{HXH} = \boldsymbol{Z} \text{ and } \boldsymbol{Z} \longrightarrow \boldsymbol{HZH} = \boldsymbol{X}$$

Even more interesting are the controlled-gate is also in the

normalizer

$$X\mathbb{1} \longrightarrow \text{CNOT}\, X\mathbb{1}\, \text{CNOT} = X X$$
$$Z\mathbb{1} \longrightarrow \text{CNOT}\, Z\mathbb{1}\, \text{CNOT} = Z\mathbb{1}$$
$$\mathbb{1}X \longrightarrow \text{CNOT}\, \mathbb{1}X\, \text{CNOT} = \mathbb{1}X$$
$$\mathbb{1}Z \longrightarrow \text{CNOT}\, \mathbb{1}Z\, \text{CNOT} = Z Z$$

The normalizer can be generated by by the gates

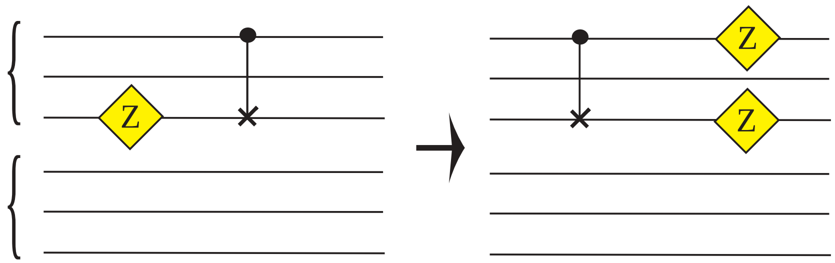
$$\text{Hadamard:} H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\text{Phase gate:} P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \text{ and}$$

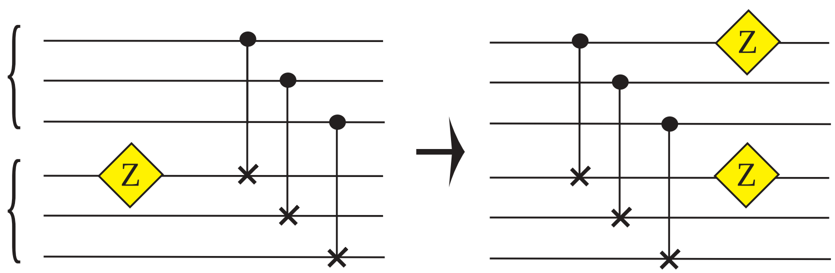
$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3.1.1 Transversal gates

Bad error:



Good Error:



This latter gate is called transversal, i.e. one qubit of an encoded quite affect acts on at most one qubit of another encoded qubit. Stabilizer operations can be implemented through transversal gates.

3.1.2 Other gates

It turns out that the normalizer gates are not a universal set of gates when they act on the Pauli matrices. So we need to be able to make other gates, it turns out that the preparation of the states $|\mathbf{0}\rangle$, $|\mathbf{1}\rangle$and $|\boldsymbol{\pi}/\mathbf{8}\rangle$

$$|\boldsymbol{\pi}/\mathbf{8}\rangle = \mathbf{cos}[\boldsymbol{\pi}/\mathbf{8}]|\mathbf{0}\rangle + \mathbf{sin}[\boldsymbol{\pi}/\mathbf{8}]|\mathbf{1}\rangle$$

is sufficient to make a universal set of gates. To be complete we should show that we can reliably check that we go the state $|\boldsymbol{\pi}/\mathbf{8}\rangle$ but this will be left as an exercise.

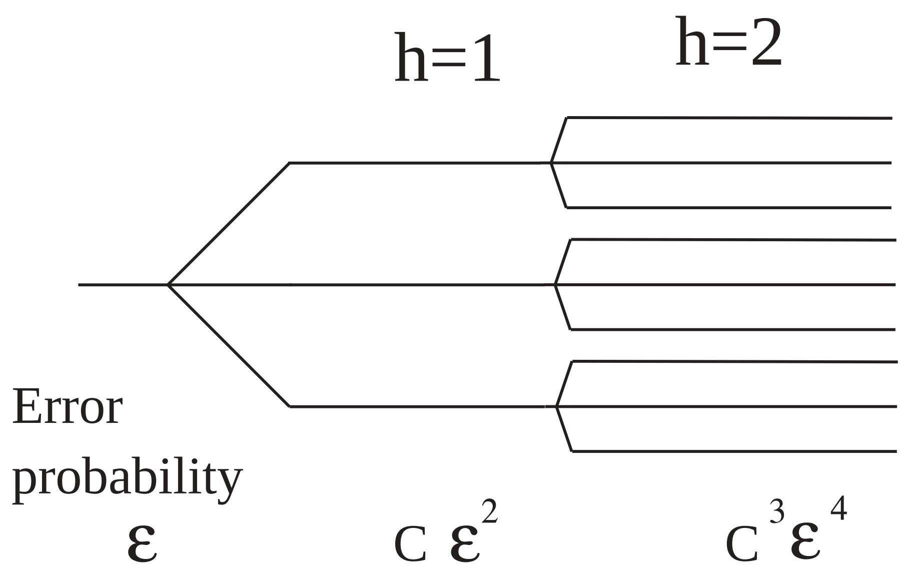
3.2 Error correcting codes and fault tolerant operations

The elements of the previous page show that we can we can protect and manipulate information in fault tolerant way: i.e. that if the bare error rate is ϵ, the error rate after the information has been encoded becomes $c\epsilon^2$ (where we have assumed that a one error correcting code has been used.

Thus we can increase the number of operation we do reliably from $1/\epsilon$ to $1/c\epsilon^2$.

The next question is how to increase this number of operations so we can do an algorithm with a larger number of gates?

The idea is to increase the number of error which can be corrected. This can be done in different ways: either by looking at better codes or another possibility os to use concatenation. This idea of the latter methods is to reencode encoded qubits in a hierarchical way. The advantage of this method is that it is possible to arrange the gates so that the error model is the same at all level of the hierarchy.



3.3 Accuracy threshold theorem:

A quantum computation can be as long as required with any desired accuracy as long as the noise level is below a threshold value:

$$P_{error} < P_{threshold}$$

The threshold can be calculated to be around 10^{-2} with the following assumptions:

- Operations can be done in parallel
- Errors are independent from one qubit to qubit
- Any two qubits can interact in one operation
- There are no lost of qubits
- Classical computing comes for free
- There is a supply of fresh qubits on demand at no cost