Introduction to Quantum Information Processing

(CS 467/667, C&0 481/681, PHYS 467/767)

Fall 2005

Assignment 1

Due date: October 14, 2005

1. **Distinguishing between quantum states.** In (a) and (b) below, suppose that one of the two quantum states is chosen randomly, according to a fair coin flip, and then this state is sent to you. Your task is to determine which of the two states it is, with as small an error probability as possible. Your distinguishing procedure can apply a unitary operation (of your choosing) to the qubit and then measure it (in the computational basis).

   (a) $|0\rangle$ and $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

   (b) $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$ (as usual, $i = \sqrt{-1}$)

   Now consider the same question except that you are given one of the following *three* states (each chosen with probability $\frac{1}{3}$) and your goal is to determine which one it is.

   (c) $|0\rangle$ and $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ and $-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

2. **Effects of rotating one qubit of an entangled state.** Suppose that Alice and Bob each possess one qubit of the Bell state

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle. \tag{1}$$

   (a) Suppose that Alice applies a rotation by some angle θ

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

   to her qubit, after which Bob measures his qubit in the computational basis. What are the probabilities of outcomes 0 and 1 for Bob?

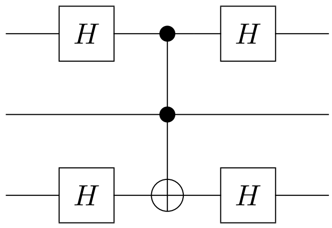
   (b) Suppose that Bob applies a rotation by θ to his qubit (instead of Alice doing so) and then Bob measures his qubit. What are the probabilities of outcomes 0 and 1 for Bob?

3. **A protocol that enables a state to be remotely created.** Suppose that Alice and Bob each possess one qubit of the Bell state in the previous question, and that Alice also possesses an arbitrary real number θ (and Bob does not possess θ). Let the goal be for Bob to end up with the state $\cos\theta|0\rangle + \sin\theta|1\rangle$. One way of doing this is for Alice to manufacture the state and then for her to teleport it to Bob. But recall that teleportation requires *two* bits of classical communication. Here, we consider how Alice and Bob can solve this problem (which is more restricted than teleportation) using only *one* classical bit of communication.

   The protocol begins by Alice applying a rotation by θ to her qubit. After this, Alice measures her qubit and sends the result (one classical bit) to Bob. Show how Bob can apply a unitary operation to his qubit—that depends on the bit he receives from Alice—so that his state is guaranteed to become $\cos\theta|0\rangle + \sin\theta|1\rangle$.

4. **A quantum circuit consisting of one Toffoli gate and four Hadamard gates.**

   Consider the following quantum circuit.



   (a) What is its effect on the eight computational basis states $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|101\rangle$, $|110\rangle$, $|111\rangle$?

   (b) In light of your results in part (a), give an equivalent circuit using fewer gates (among NOT, H, CNOT, and Toffoli gates). Try to use as few gates as possible.

5. **Various properties of Hadamard and swap gates.**

   (a) Give the eigenvalues and eigenvectors of the Hadamard matrix H.

   (b) Give the eigenvalues and eigenvectors of $H \otimes H$. (Try to simplify your calculations by building upon your results in part (a).)

   (c) Define the two-qubit gate U_{swap} as the linear transformation that maps $|a\rangle|b\rangle$ to $|b\rangle|a\rangle$, for all $a, b \in \{0, 1\}$. Give the 4×4 matrix corresponding to U_{swap}.

   (d) Show how to express U_{swap} as a sequence $CNOT$ gates (and no other gates).

   (e) Give a unitary matrix V that is a "square root of swap", in the sense that $V^2 = U_{\text{swap}}$.

6. **Entangled states and product states.** For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the states are *entangled*).

   (a) $\frac{1}{2}|00\rangle + \frac{1}{2}i|01\rangle + \frac{1}{2}i^2|10\rangle + \frac{1}{2}i^3|11\rangle$

   (b) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{8}}|01\rangle + \frac{1}{\sqrt{8}}|10\rangle + \frac{1}{\sqrt{4}}|11\rangle$

   (c) $\frac{1}{\sqrt{2}}|0\rangle(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle) - \frac{1}{\sqrt{2}}|1\rangle(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle)$