Quantum algorithms for formula evaluation

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joint work with
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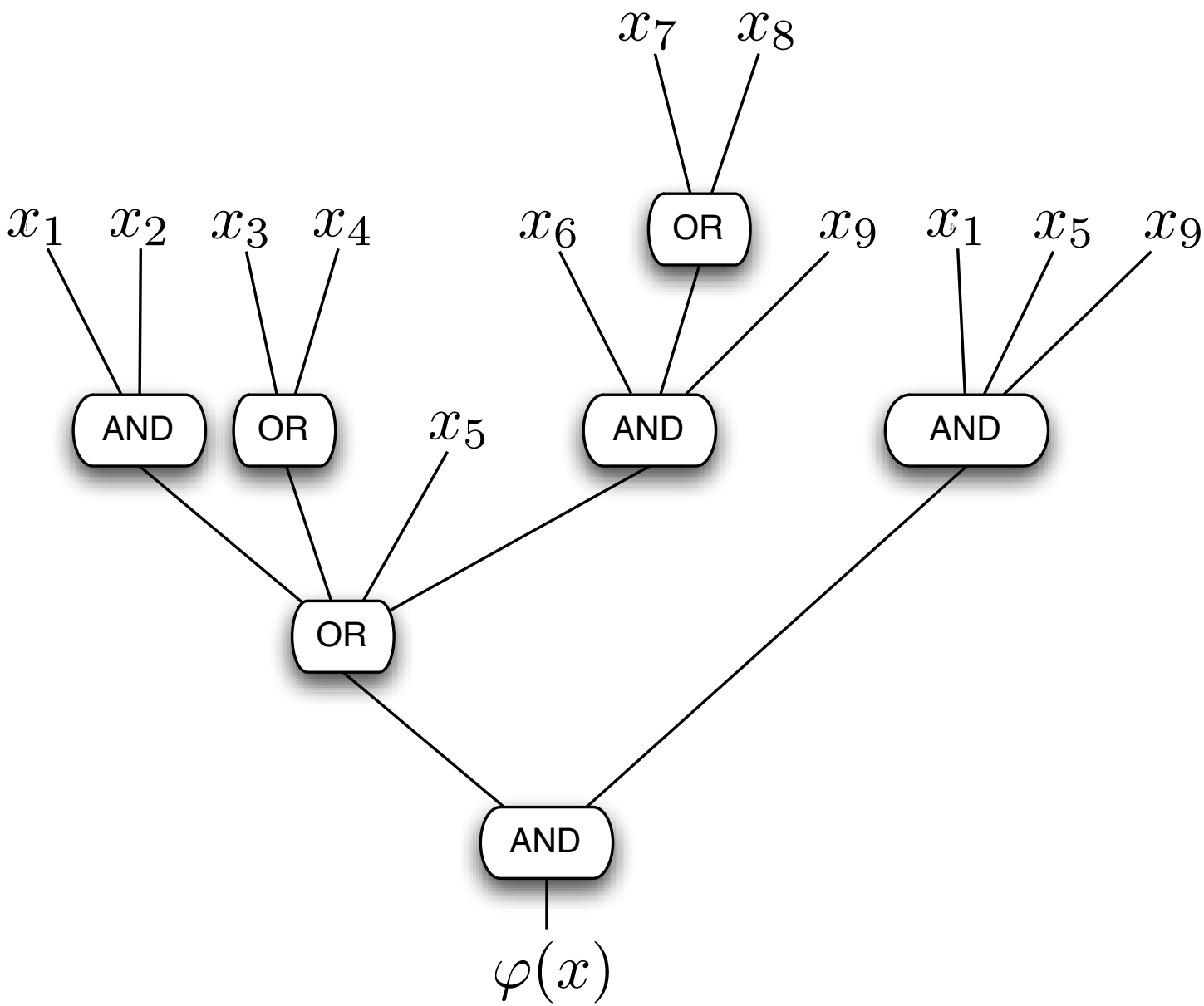
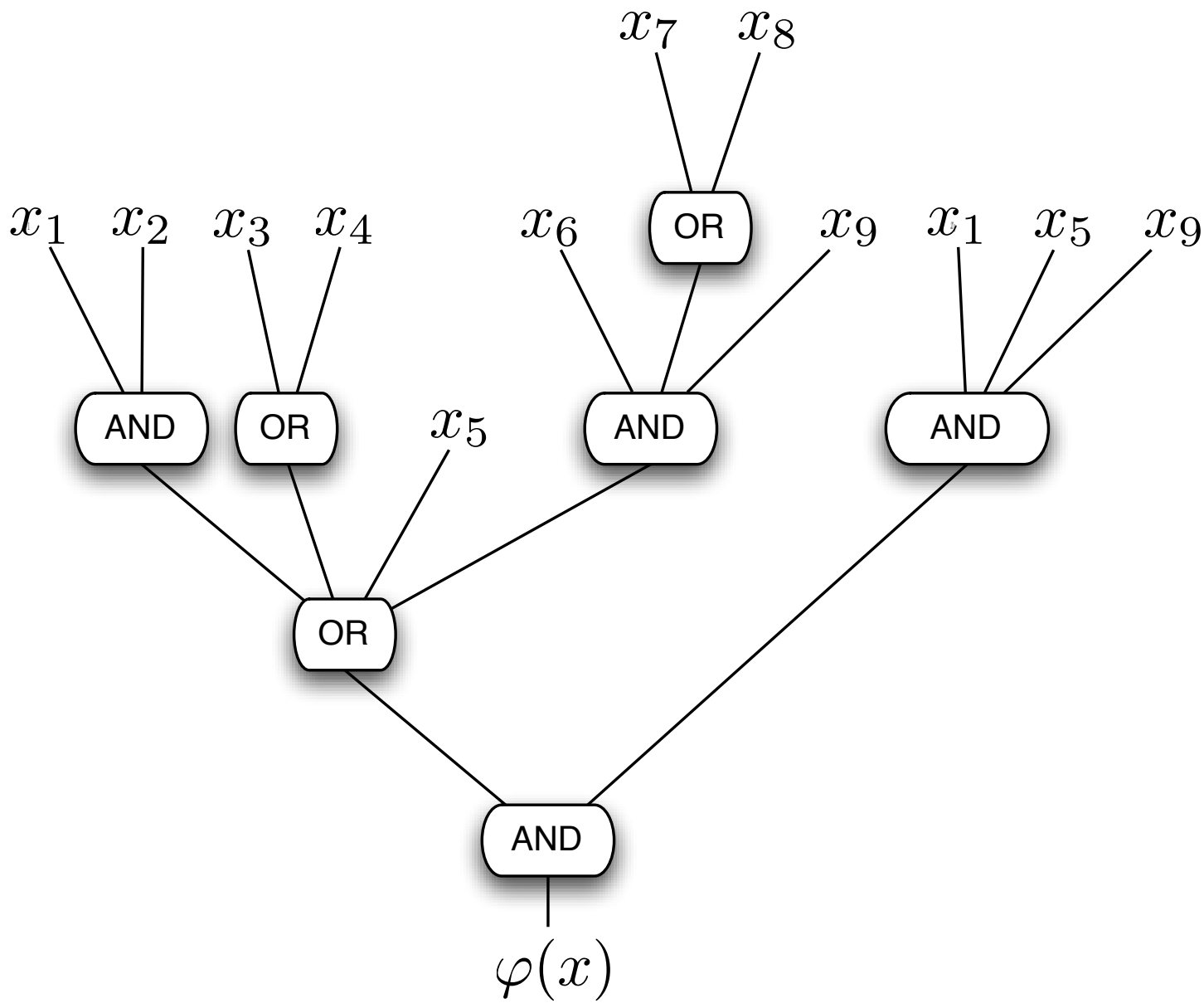
Problem: Evaluate the formula with minimal queries to the input bits x_i.



Def: {AND, OR, NOT} Formula = Tree of nested gates



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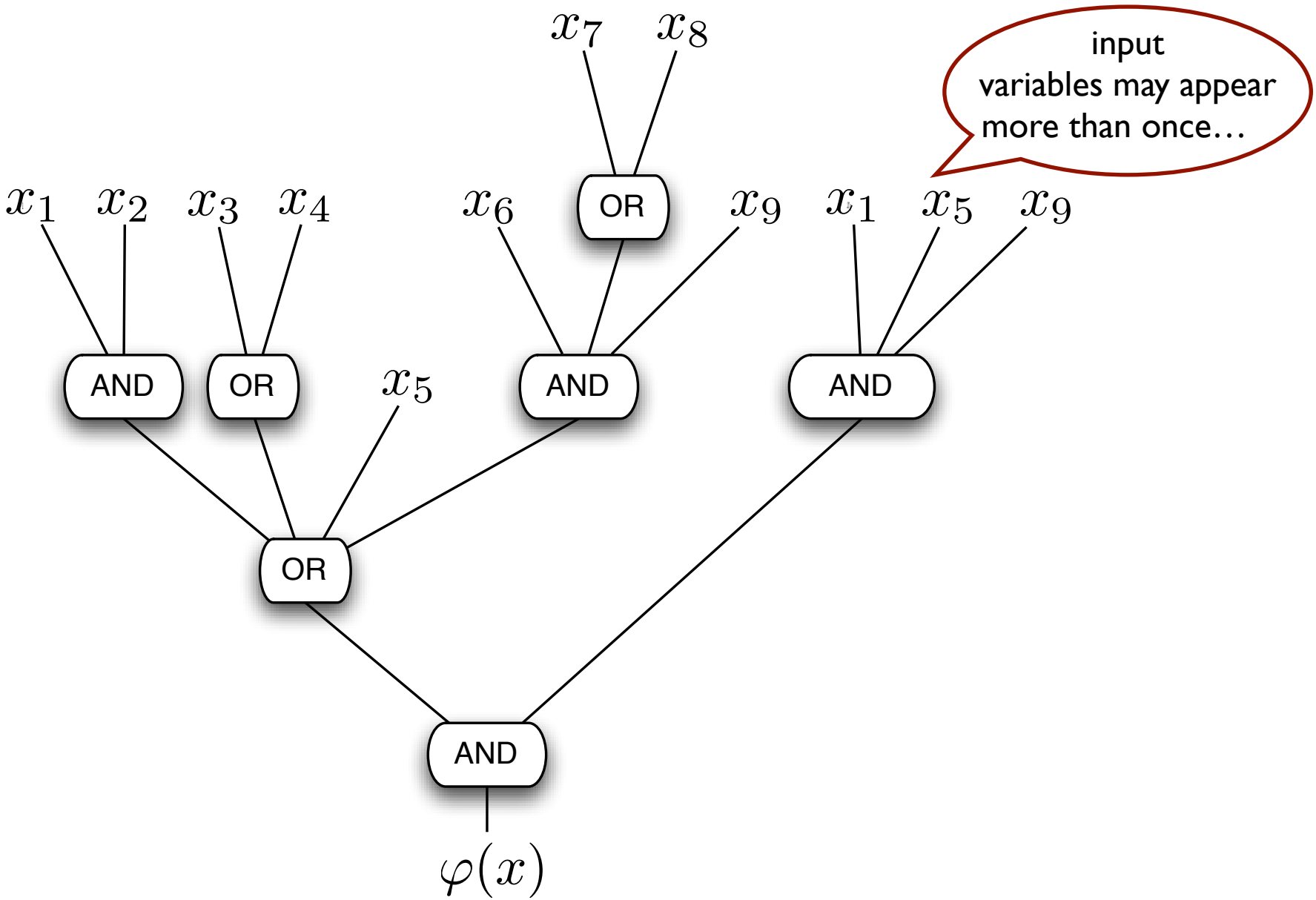

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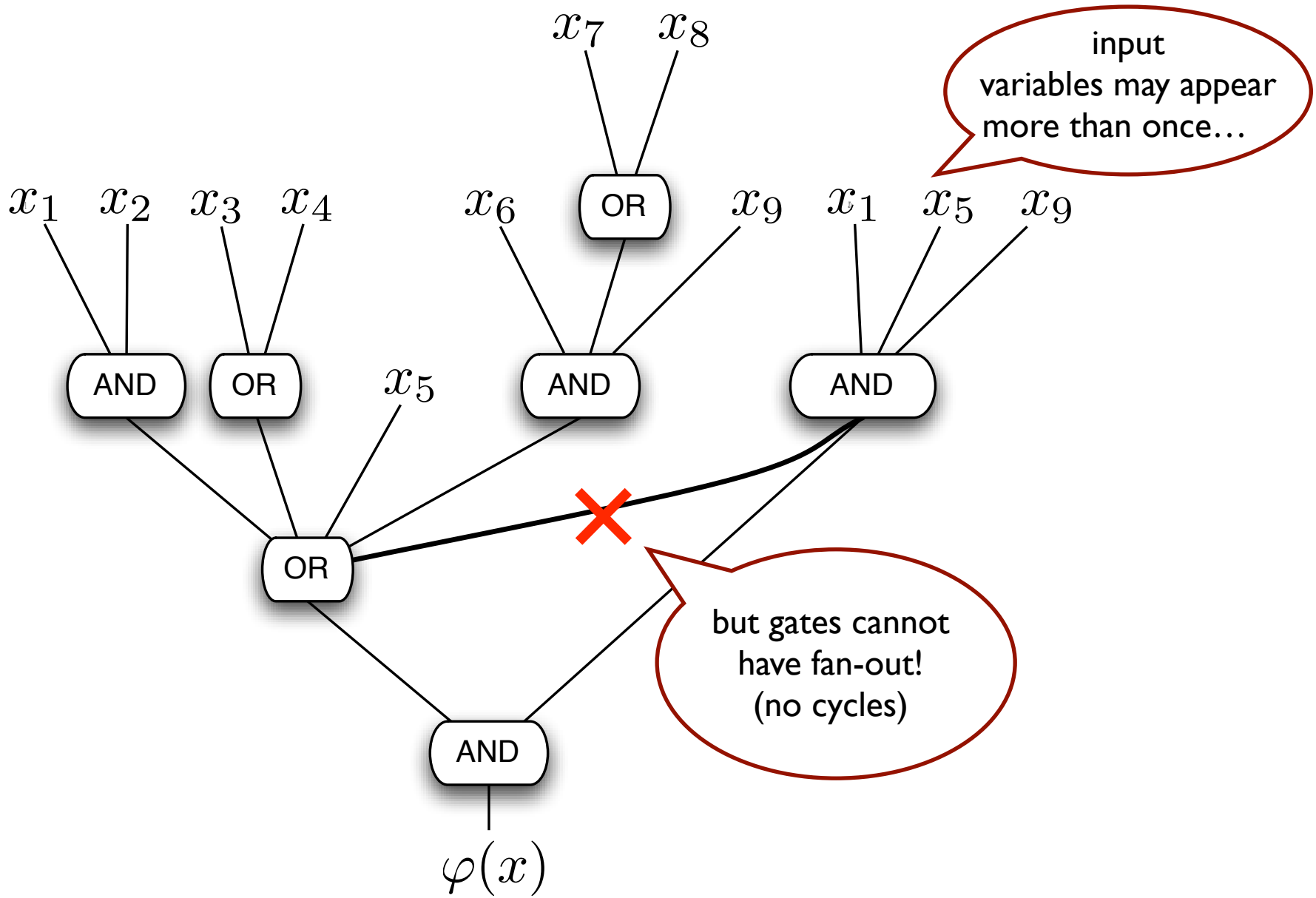
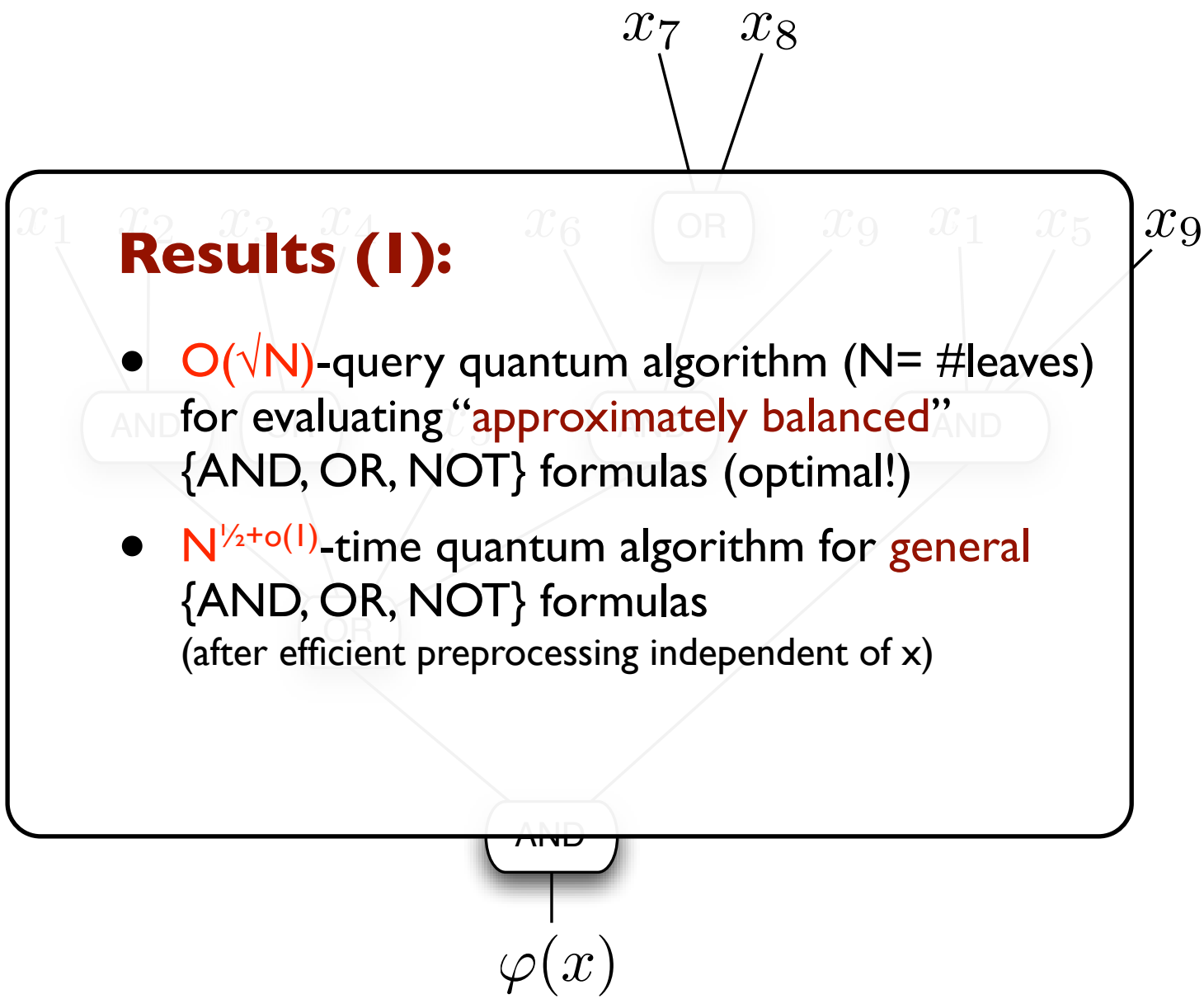


(in a *circuit*, cycles are allowed; but in a *formula*, subexpressions cannot be reused)

Problem: Evaluate the formula with minimal queries to the input bits x_i.



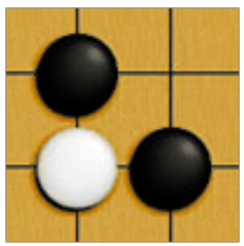
Results (1):

- $O(\sqrt{N})$-query quantum algorithm (N= #leaves) for evaluating "approximately balanced" {AND, OR, NOT} formulas (optimal!)
- $N^{1/2+o(1)}$-time quantum algorithm for general {AND, OR, NOT} formulas
  (after efficient preprocessing independent of x)

- Problem: Evaluate the formula, with minimal queries to the inputs bits x_i.
- Results:
  - $O(\sqrt{N})$-query quantum algorithm for "approximately balanced" AND-OR formulas
  - $N^{1/2+o(1)}$-time quantum algorithm for general AND-OR formulas (after preprocessing)

Problem Motivations:

- Playing "Go"



  - Nodes ↔ game histories
  - White wins if ∃ move s.t. ∀ black moves, ∃ move s.t. …

➡ Two-player game trees are formula trees with alternating levels of AND, OR gates

- Decision version of min-max tree evaluation
  - inputs are real numbers
  - want to decide if minimax is ≥10 or not
- Well-studied classical problem…

Problem history (1/2)

- Problem: Evaluate the formula, with minimal queries to the inputs bits x_i.
- Results:
  - $O(\sqrt{N})$-query quantum algorithm for "approximately balanced" AND-OR formulas
  - $N^{1/2+o(1)}$-time quantum algorithm for general AND-OR formulas (after preprocessing)
- Classical history
  - Deterministic algorithm requires time N
  - Randomized (Las Vegas) algorithm in E-time $O(N^{0.754})$ for balanced binary AND-OR formulas [Snir '85, Saks & Wigderson '86]
    - Flip coins to decide which subtree to evaluate next, short-circuit
    - Optimal [Santha '95]
  - For arbitrary AND-OR formulas $\Omega(N)$ time may be required

$$\log_d \lambda_{max}\left(\begin{pmatrix} 0 & d \\ 1 & \frac{d-1}{2} \end{pmatrix}\right)$$

Problem history (2/2)

- Classical history
  - Randomized algorithm in E-time $\Theta(N^{0.754})$ for balanced binary AND-OR formulas
  - Evaluating an arbitrary AND-OR formula may require $\Omega(N)$ time
- Quantum history
  - Adversary lower bound $\Omega(\sqrt{N})$ queries [Barnum, Saks '04]
  - Grover search: Evaluates $OR(x_1, x_2, \ldots, x_N) = \begin{cases} 1 & \text{if } \exists \text{ an } i : x_i = 1 \\ 0 & \text{otherwise} \end{cases}$

    using $O(\sqrt{N})$ queries ($O(\sqrt{N} \log \log N)$-time)
  - Can be applied recursively to evaluate shallow trees:
    - Evaluates regular depth-d AND-OR formula in $\sqrt{N}\, O(\log N)^{d-1}$ queries [BCW '98]
    - Search on faulty oracles [Høyer, Mosca, de Wolf '03] $\Rightarrow O(\sqrt{N}\, c^d)$ queries

Breakthrough!

- Classical history
  - Randomized algorithm in E-time $\Theta(N^{0.754})$ for balanced binary AND-OR formulas
  - Evaluating an arbitrary AND-OR formula may require $\Omega(N)$ time
- Quantum history
  - Adversary lower bound $\Omega(\sqrt{N})$ queries [Barnum, Saks '04]
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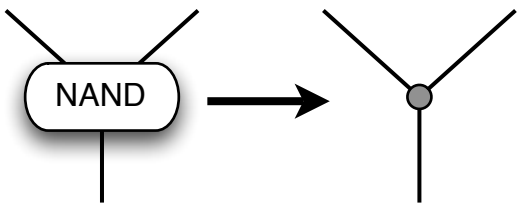
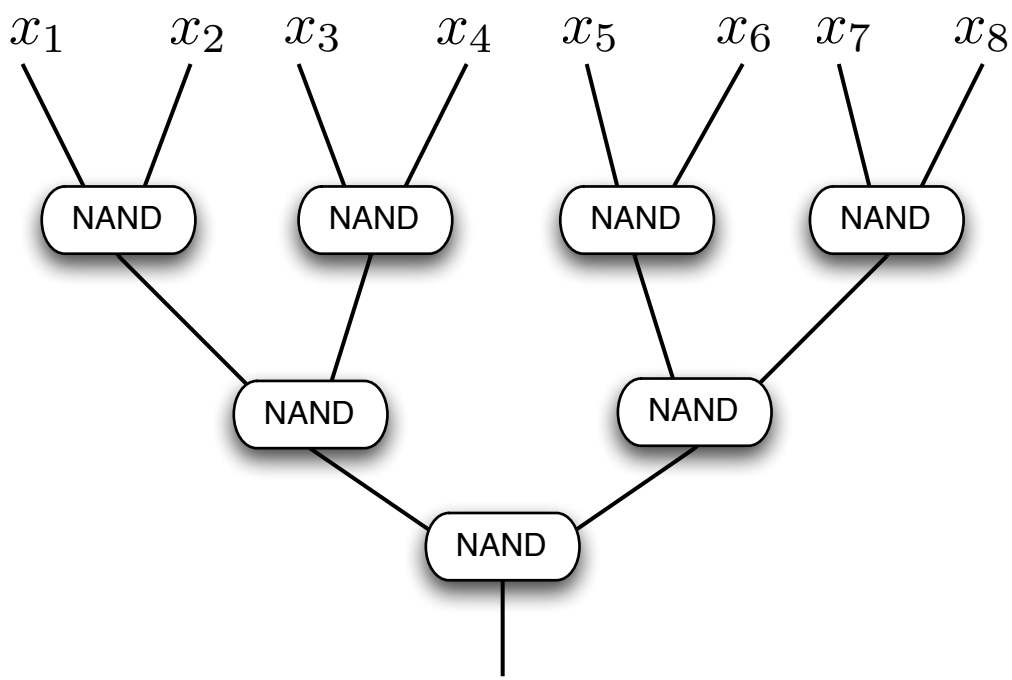
- Farhi, Goldstone, Gutmann 2007: Breakthrough continuous-time quantum algorithm for evaluating balanced binary NAND formula in $N^{1/2+o(1)}$ queries & time

Farhi, Goldstone, Gutmann '07 algorithm

- **Theorem** ([FGG '07, CCJY '07]): A balanced binary NAND formula can be evaluated in time $N^{1/2+o(1)}$.
- Convert formula to a tree:





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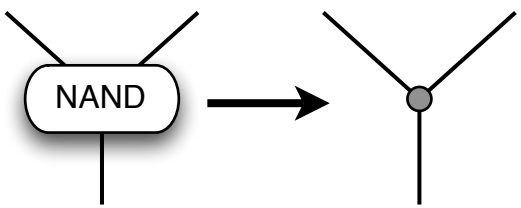

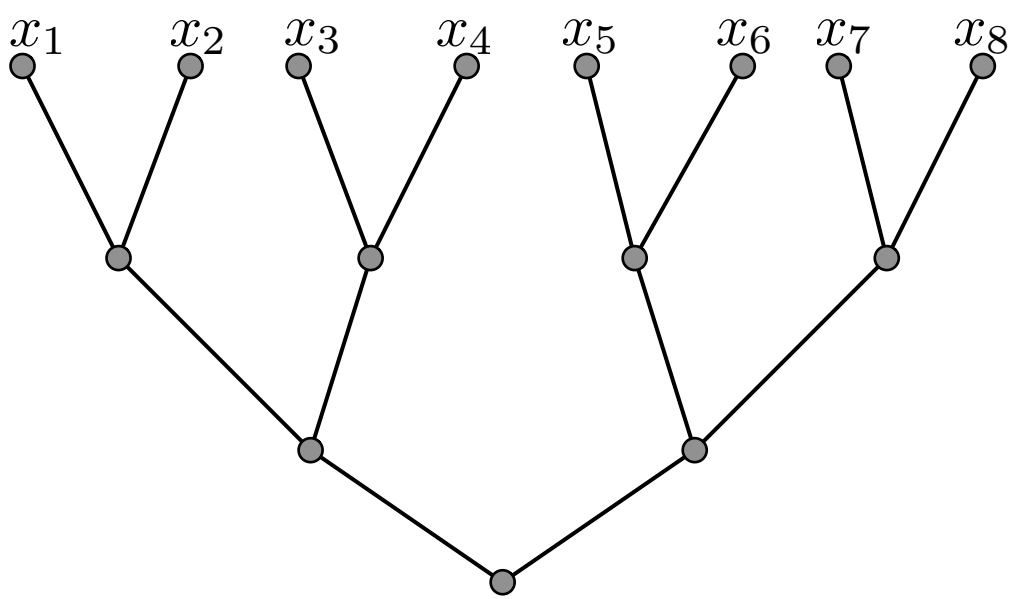

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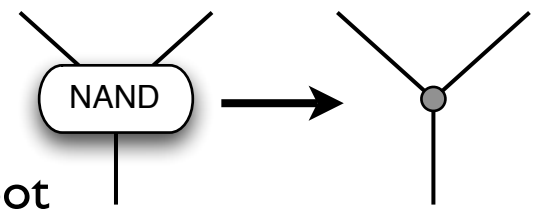
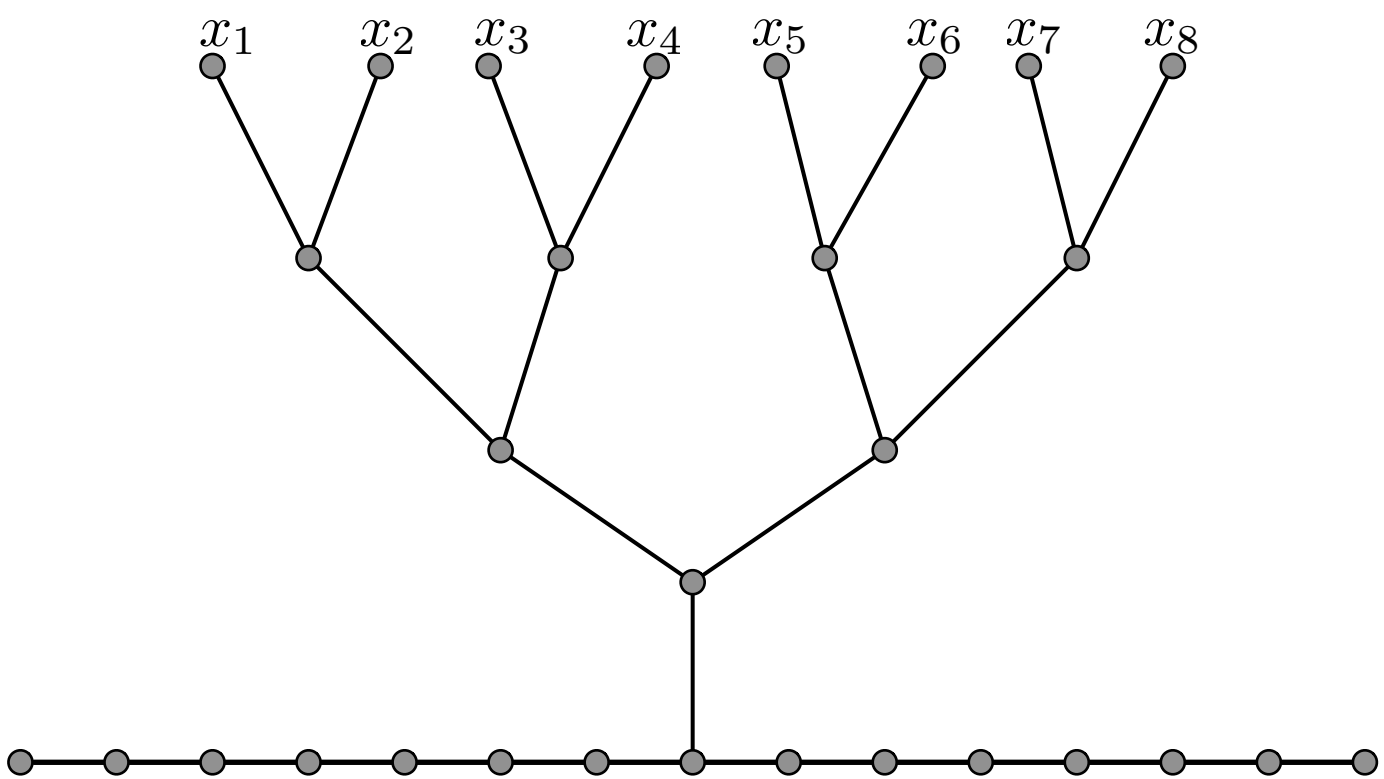
- Convert formula to a tree:
- Attach an infinite line to the root





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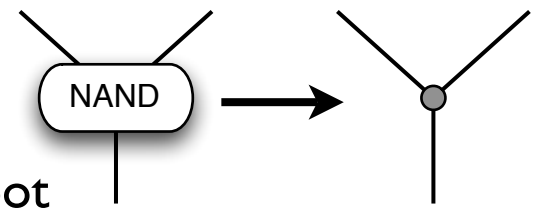
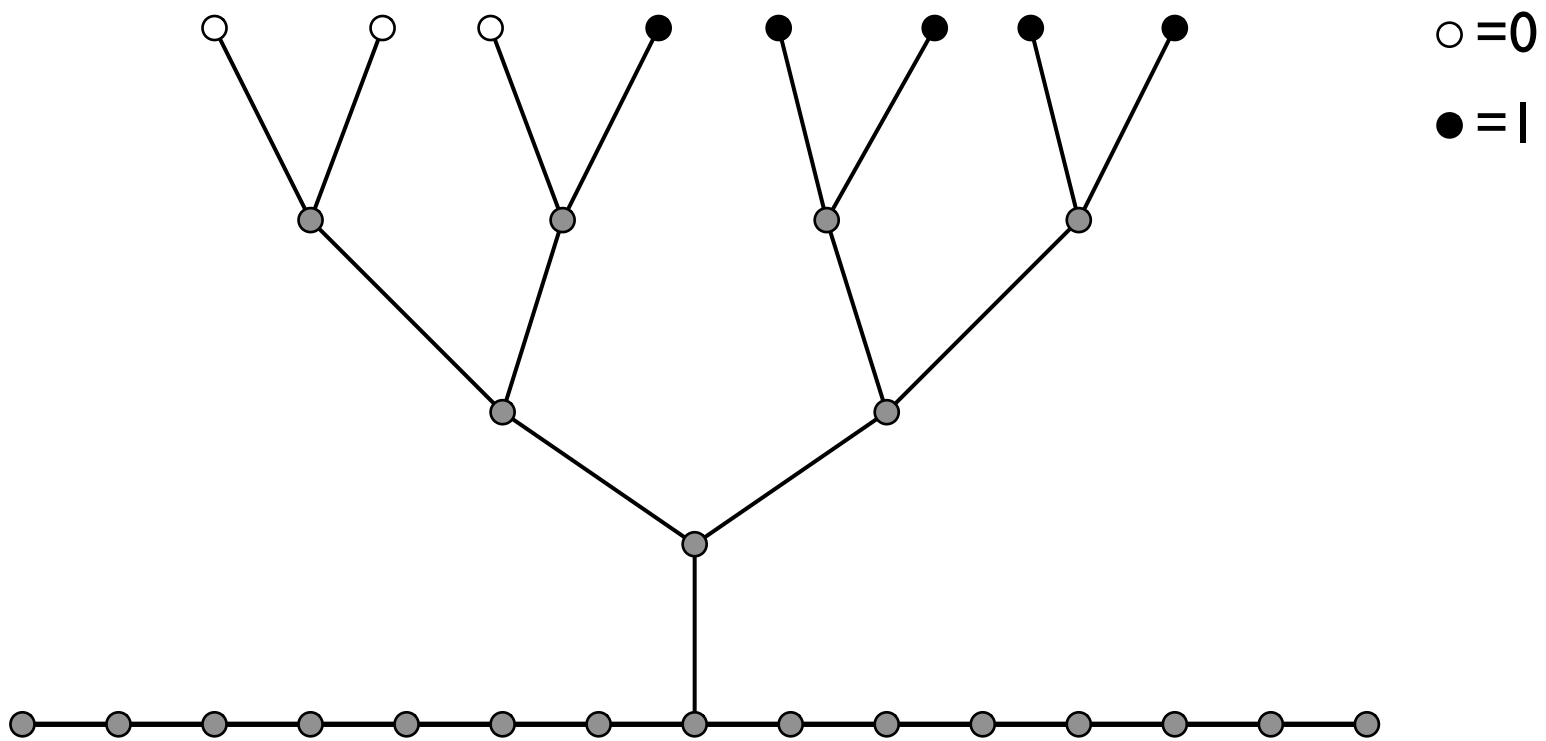
- **Theorem** ([FGG '07, CCJY '07]): A balanced binary NAND formula can be evaluated in time $N^{1/2+o(1)}$.
- Convert formula to a tree:
- Attach an infinite line to the root
- Add edges above leaf nodes evaluating to one…





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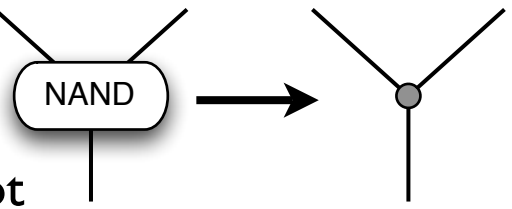

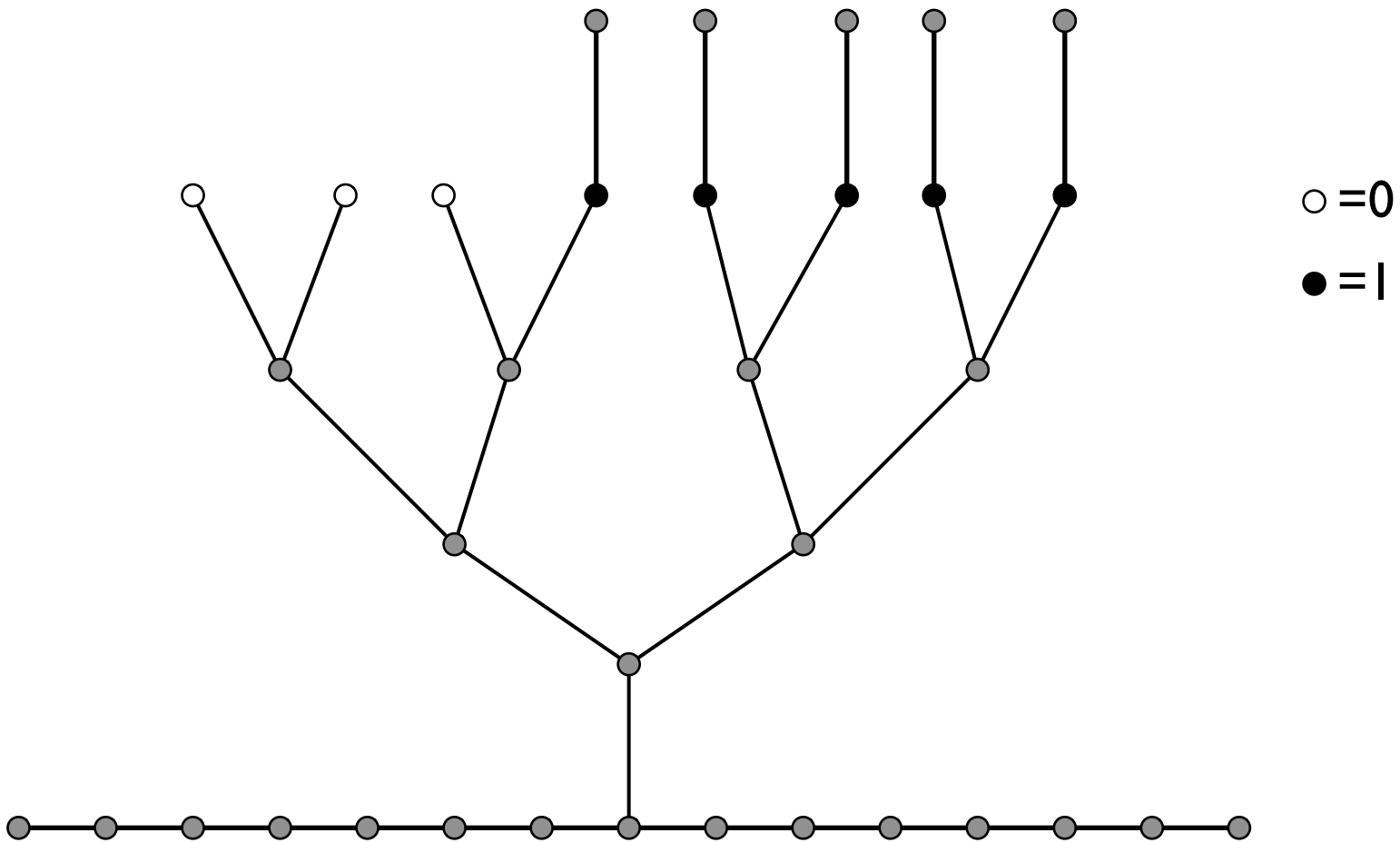

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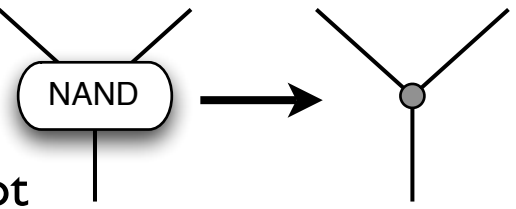
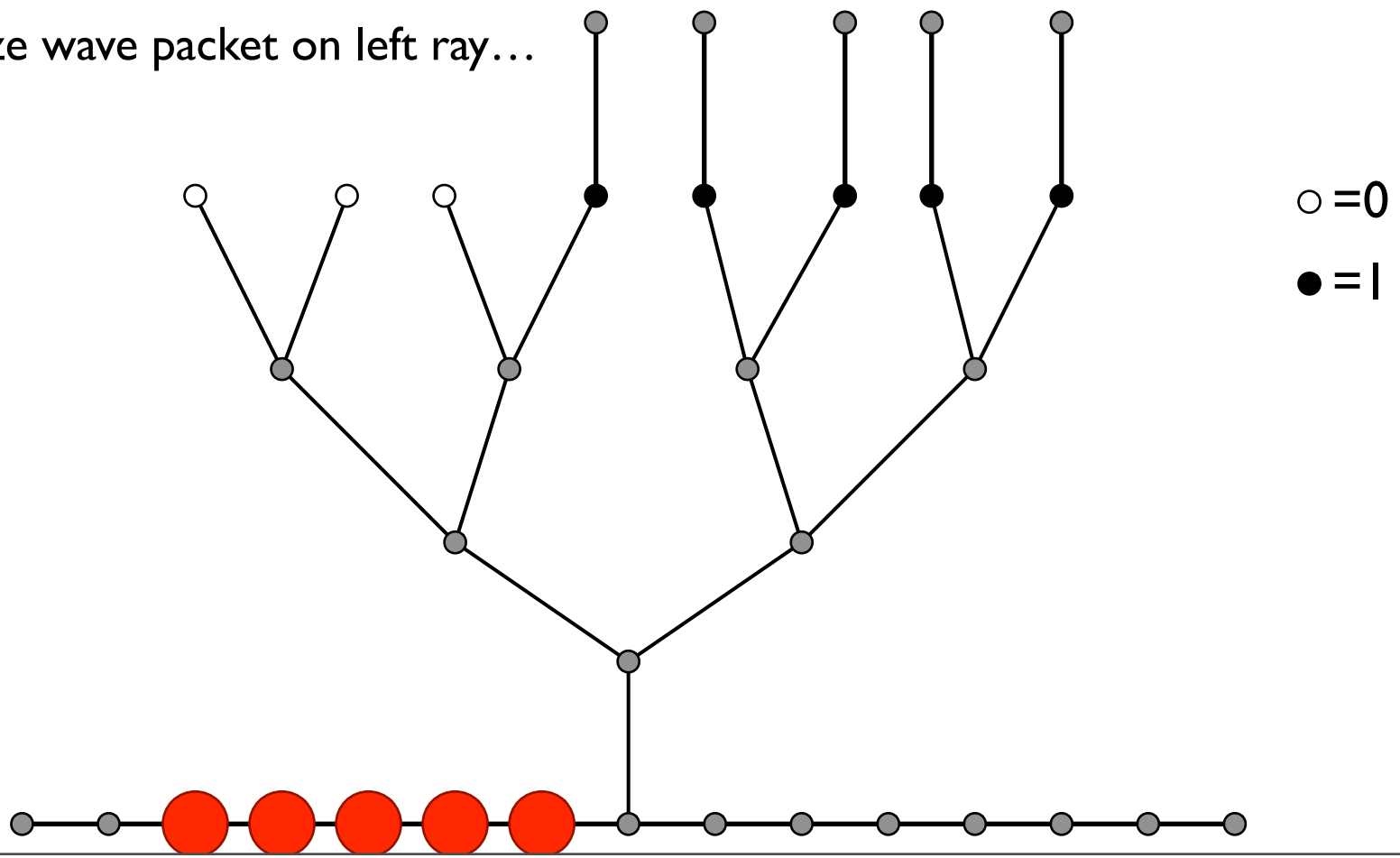
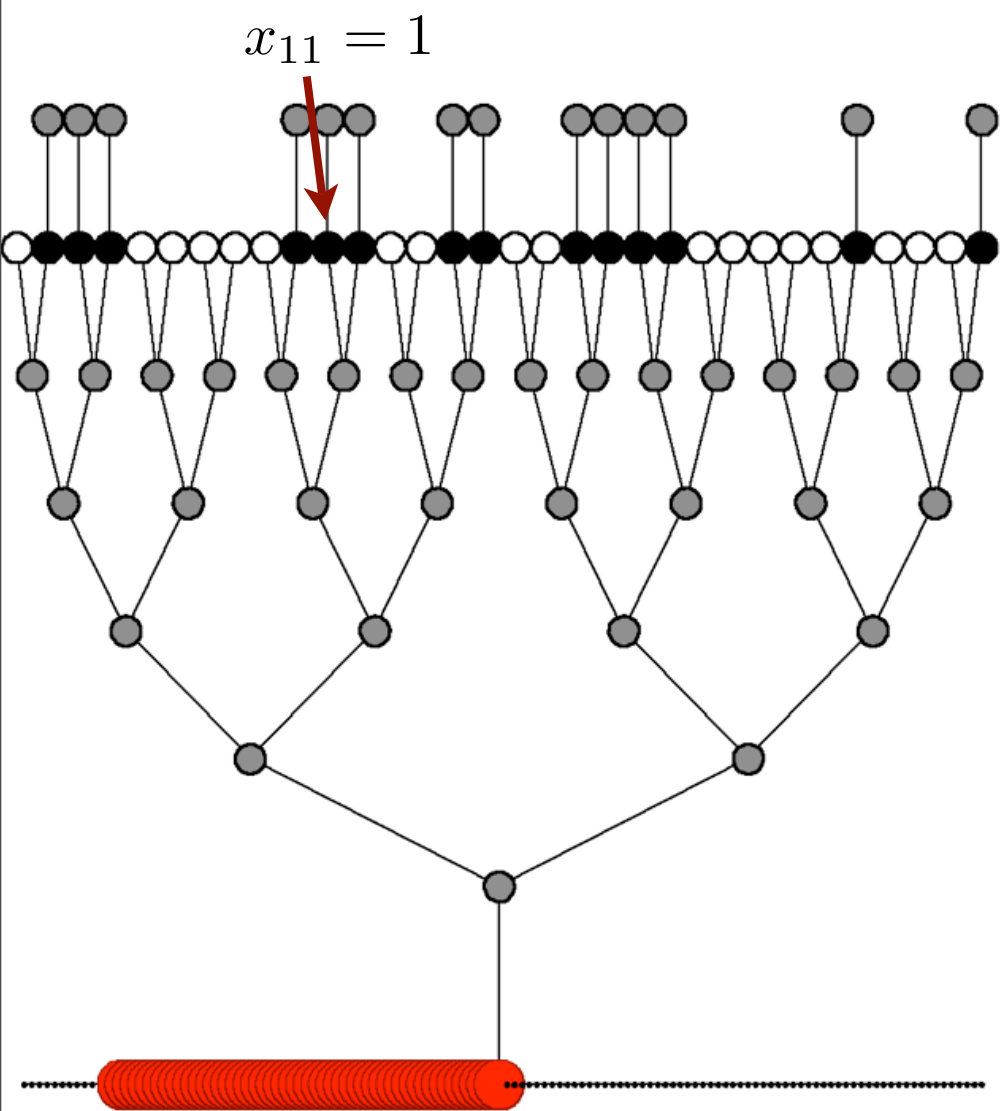
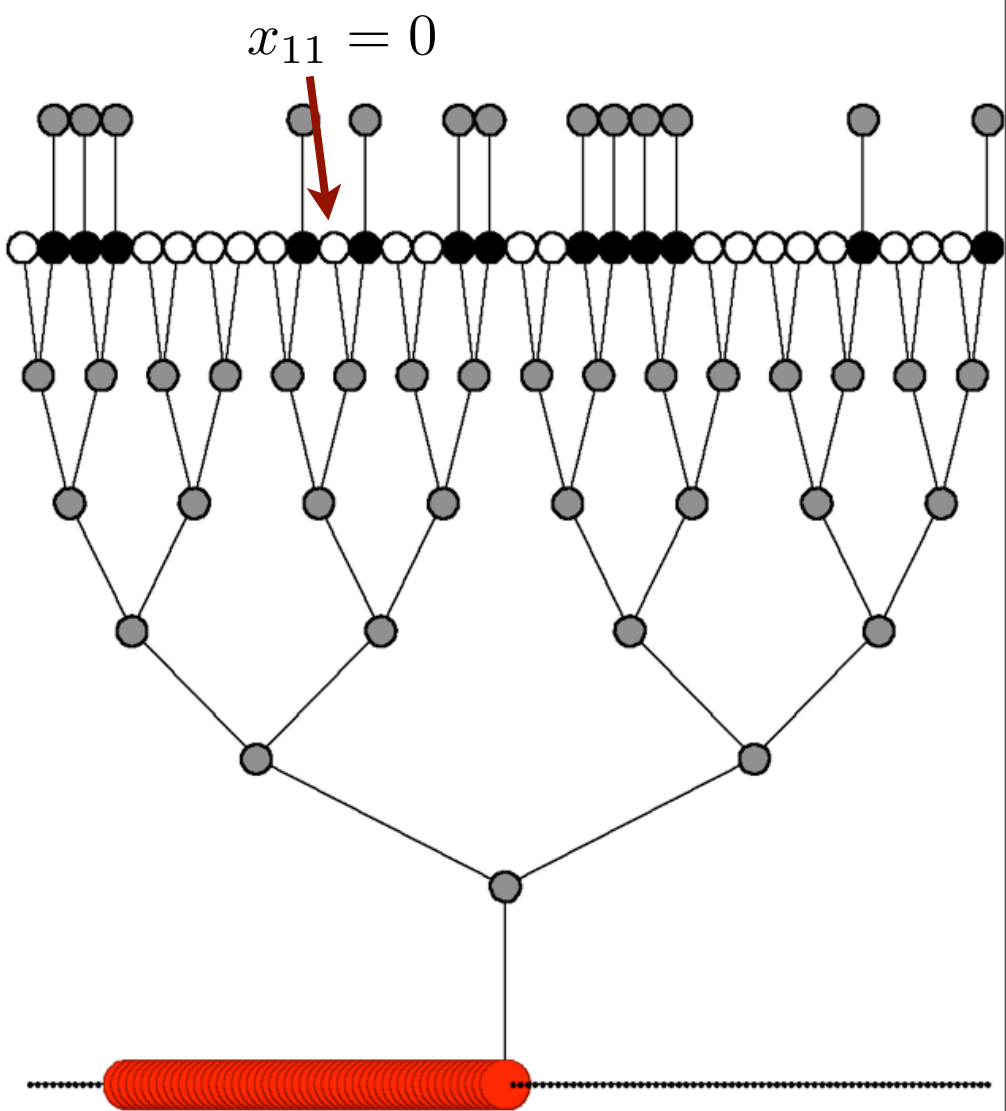
- Convert formula to a tree:
- Attach an infinite line to the root
- Add edges above leaf nodes evaluating to one
- Initialize wave packet on left ray…



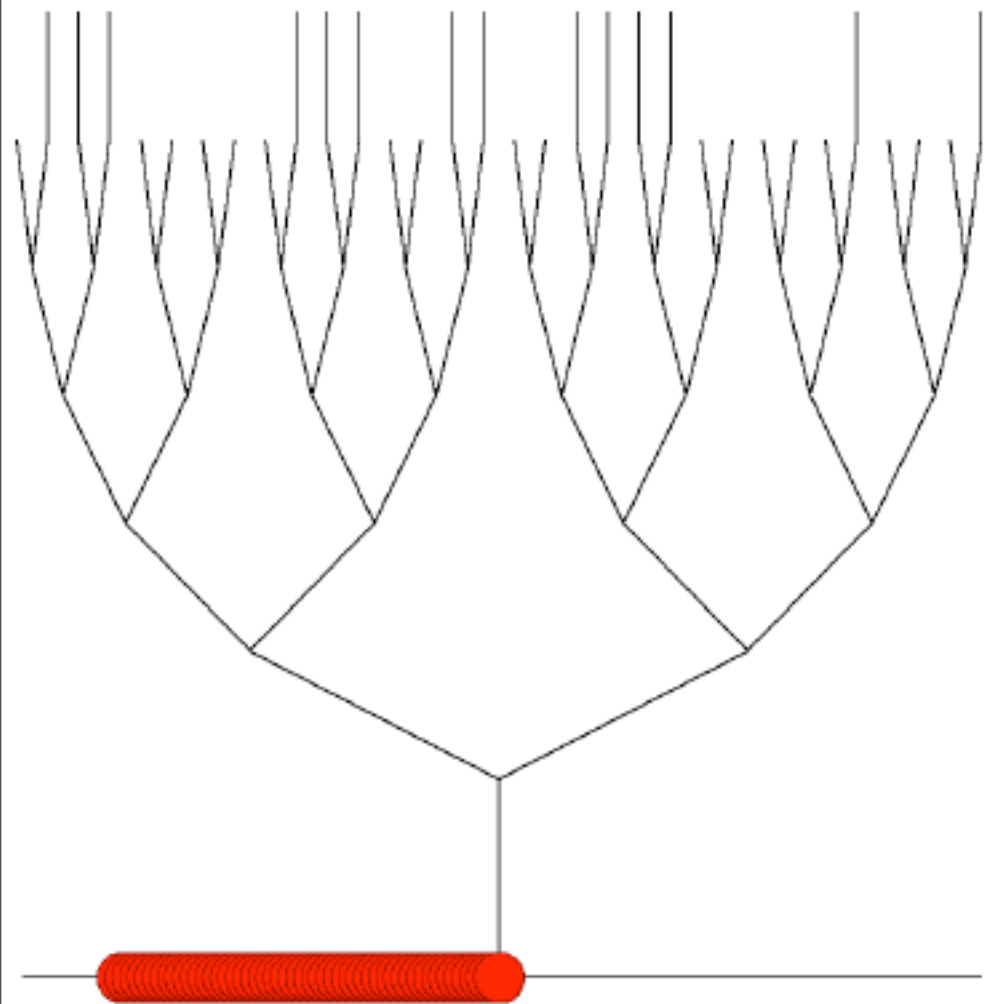


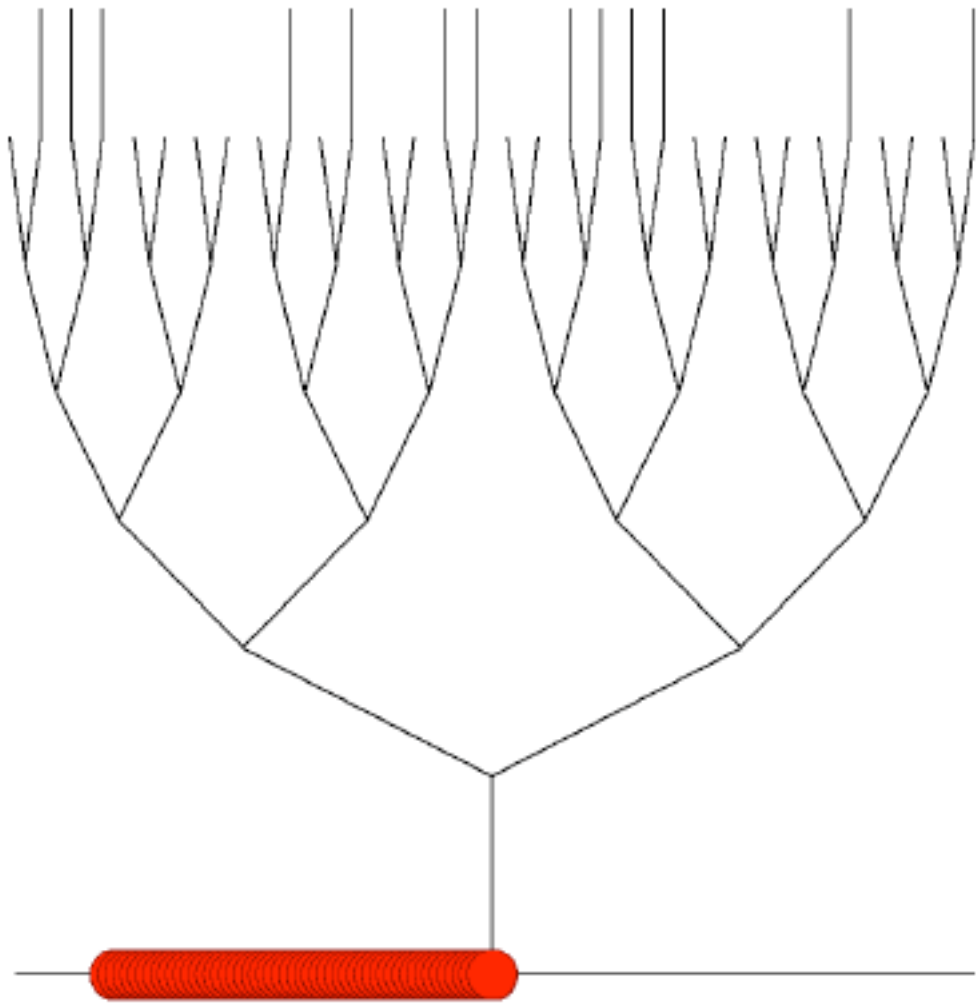
Continuous-time quantum walk [FGG '07]





FGG quantum walk $|\psi_t\rangle = e^{iA_G t}|\psi_0\rangle$





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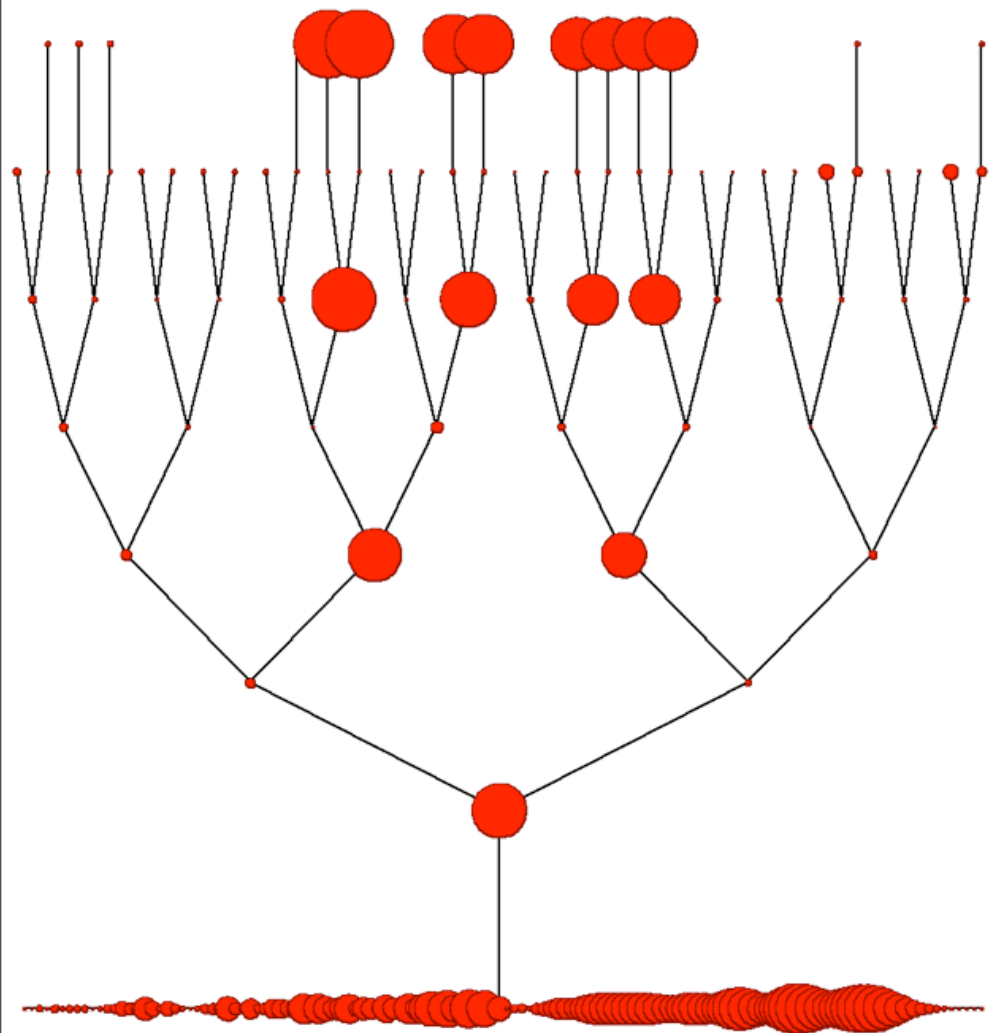

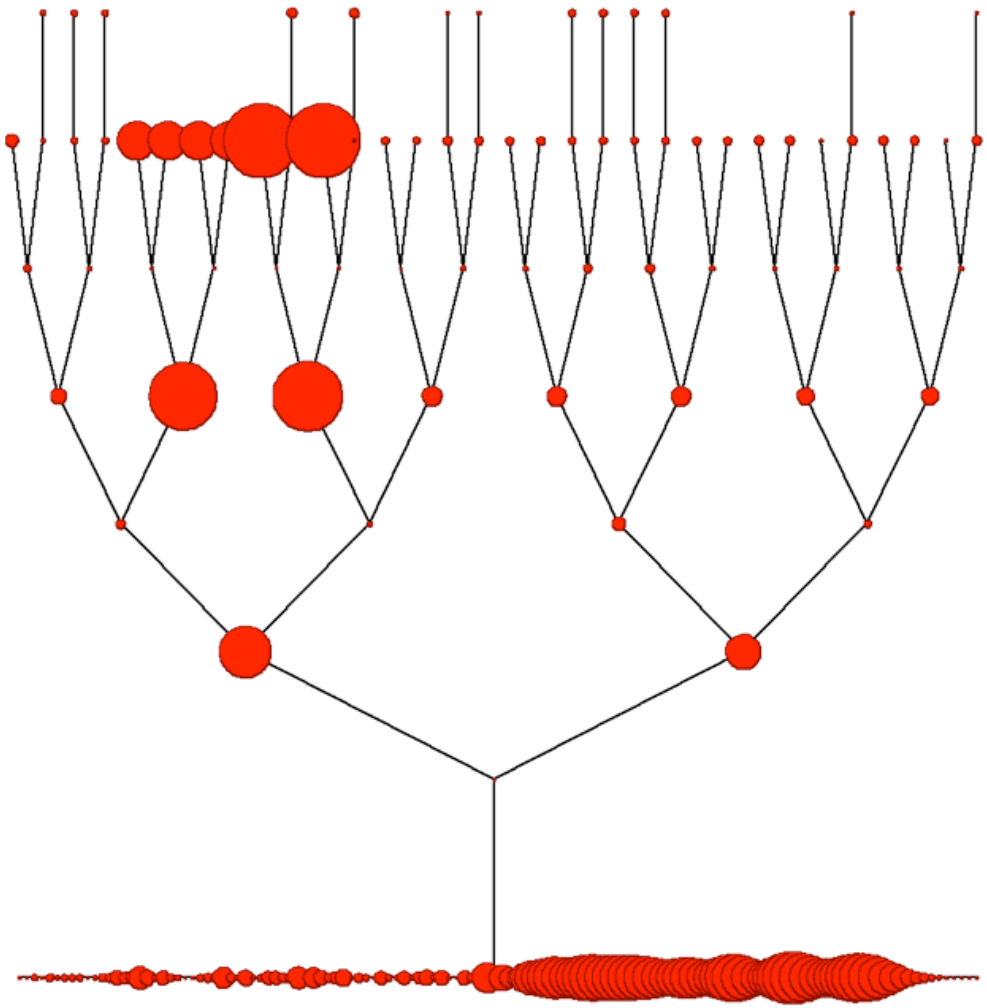

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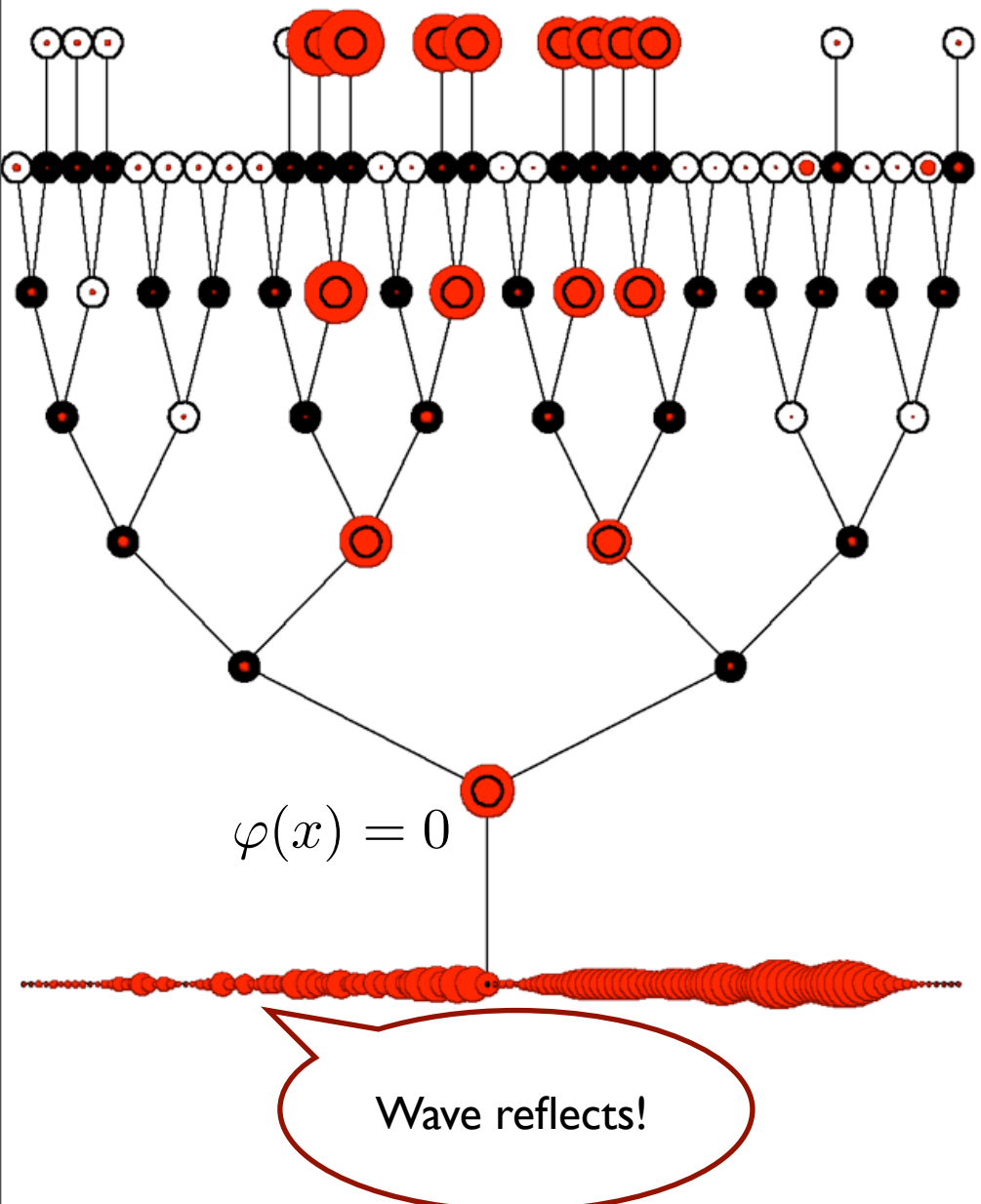

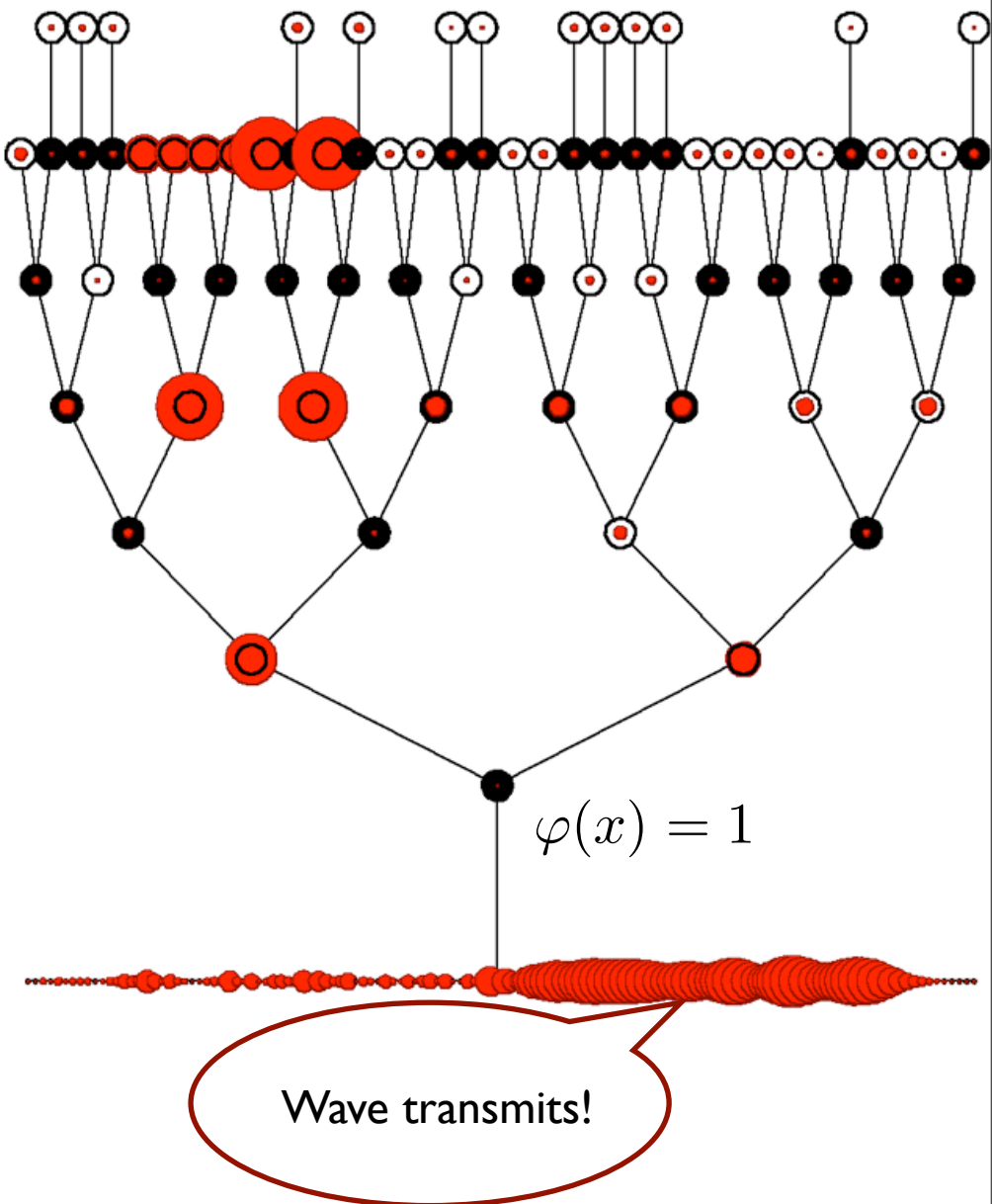

[FGG '07] algorithm

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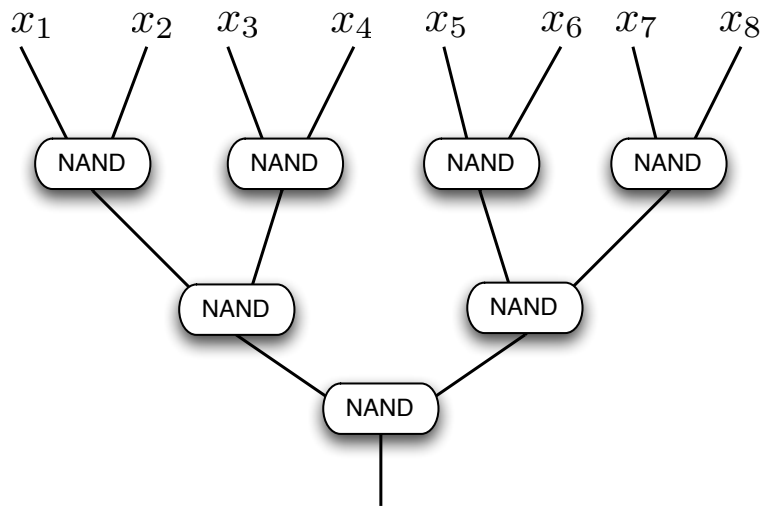
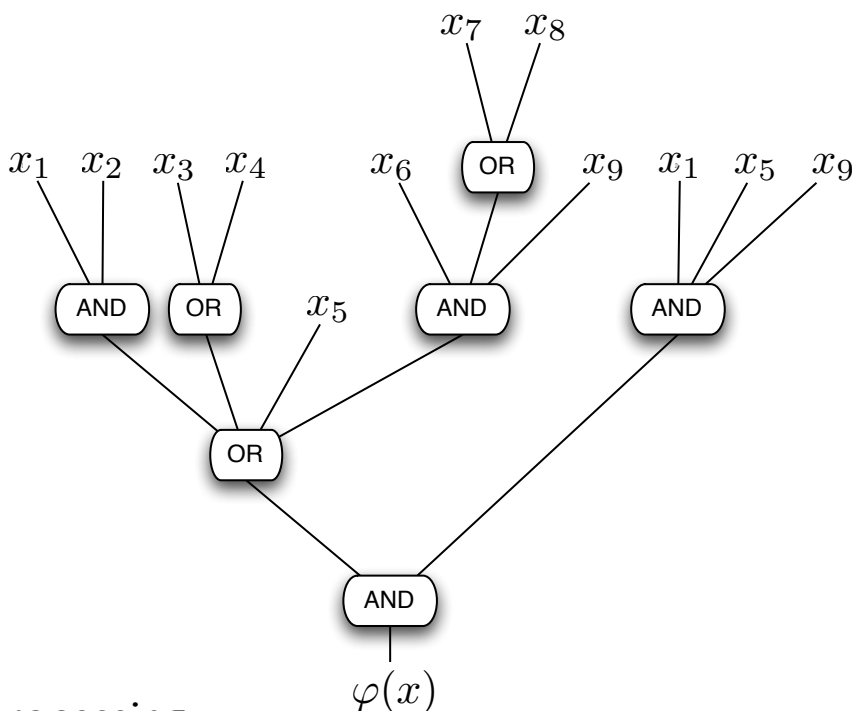


[CRŠZ '07] algorithm

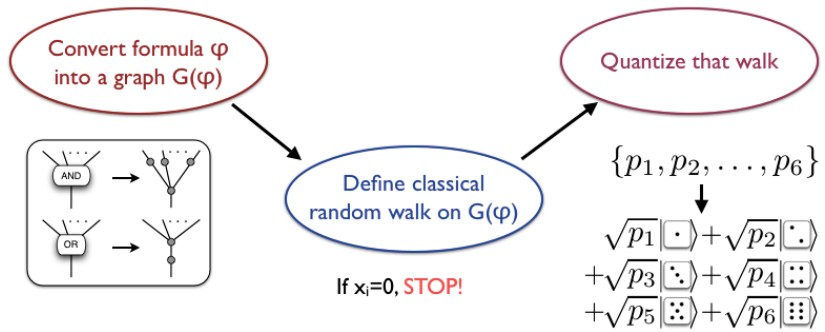
- **Theorem** ([CRŠZ '07]):
  - An "approximately balanced" {AND, OR, NOT} formula can be evaluated with $O(\sqrt{N})$ queries (optimal!).
  - A general {AND, OR, NOT} formula can be evaluated with $N^{1/2+o(1)}$ queries.



Running time is $N^{1/2+o(1)}$ in each case, after efficient preprocessing.

Talk outline

1. The Algorithm



2. Why It Works

Szegedy correspondence

Zero energy eigenstate analysis

3. Extensions

Talk outline

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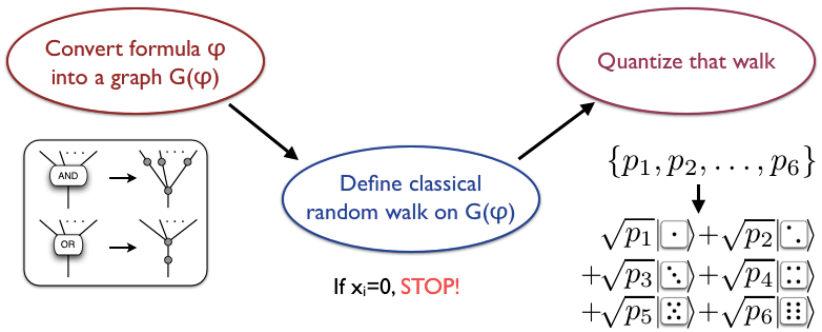

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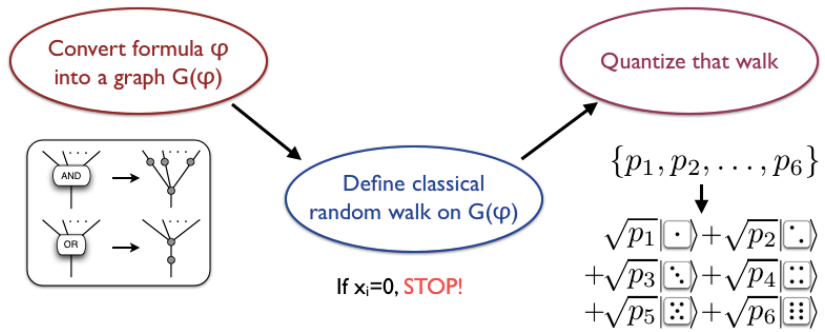

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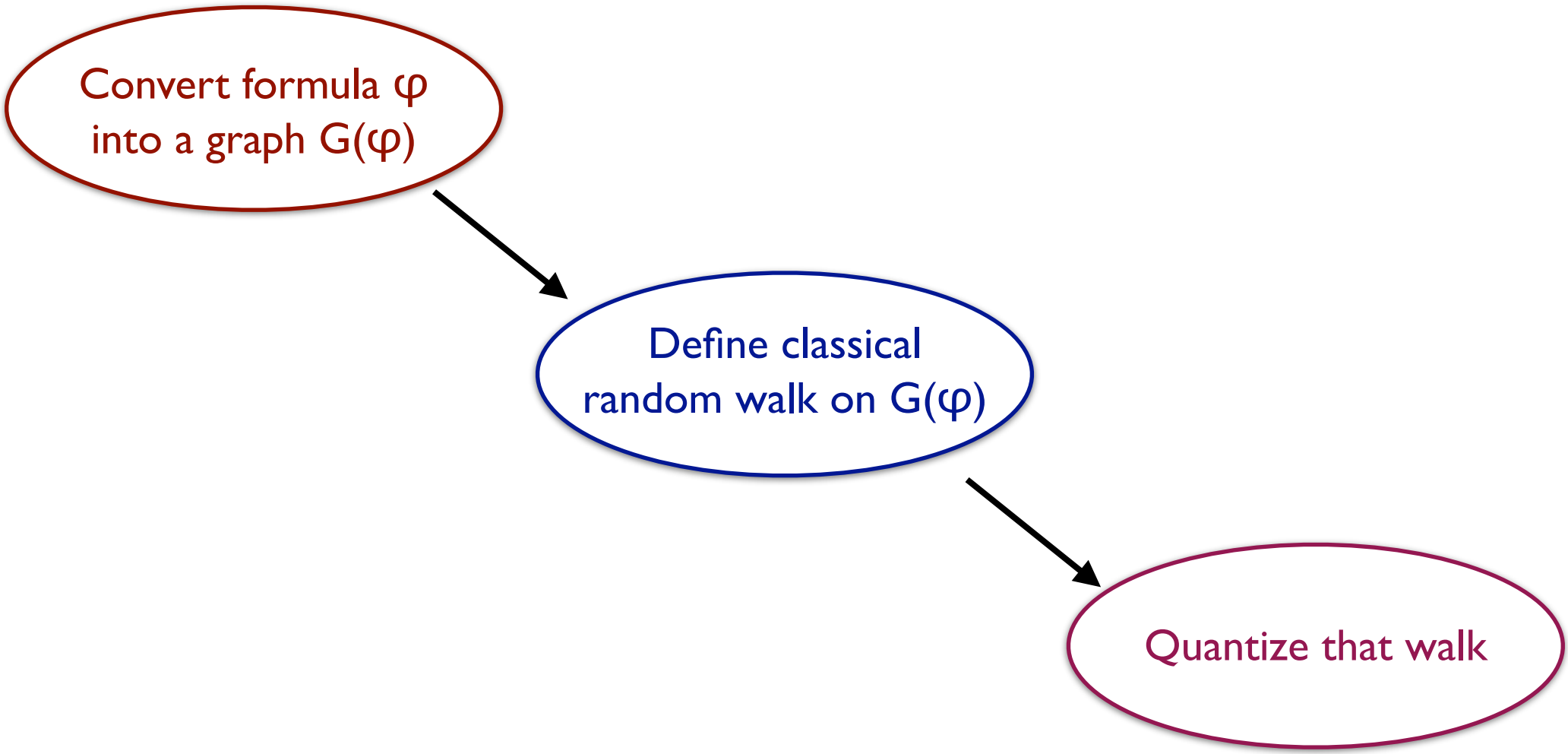
More Gates!

Fushimi-Inari

Chion-In

Katsura Imperial Villa

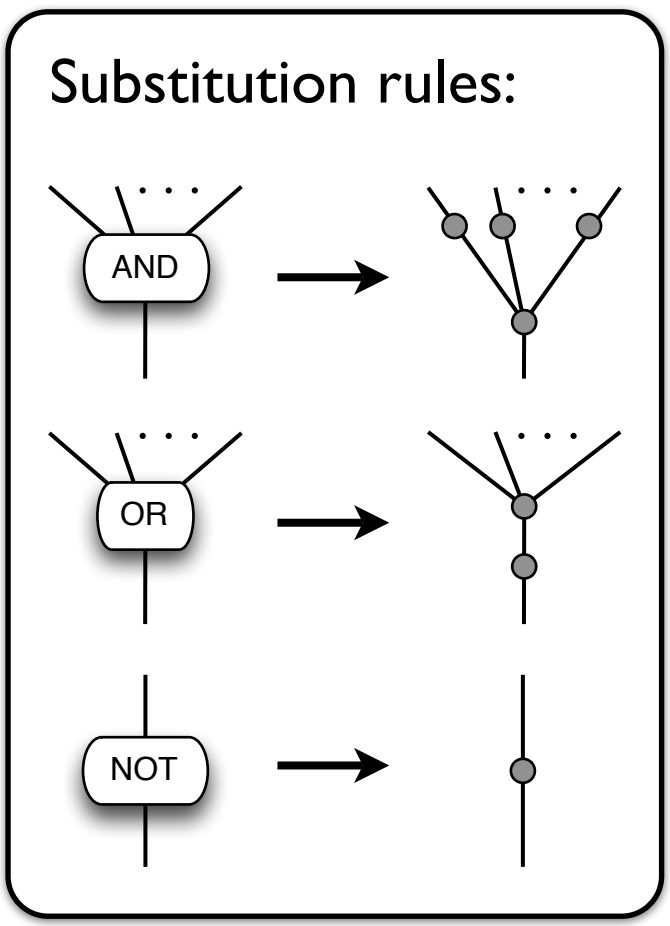
Formula evaluation algorithm

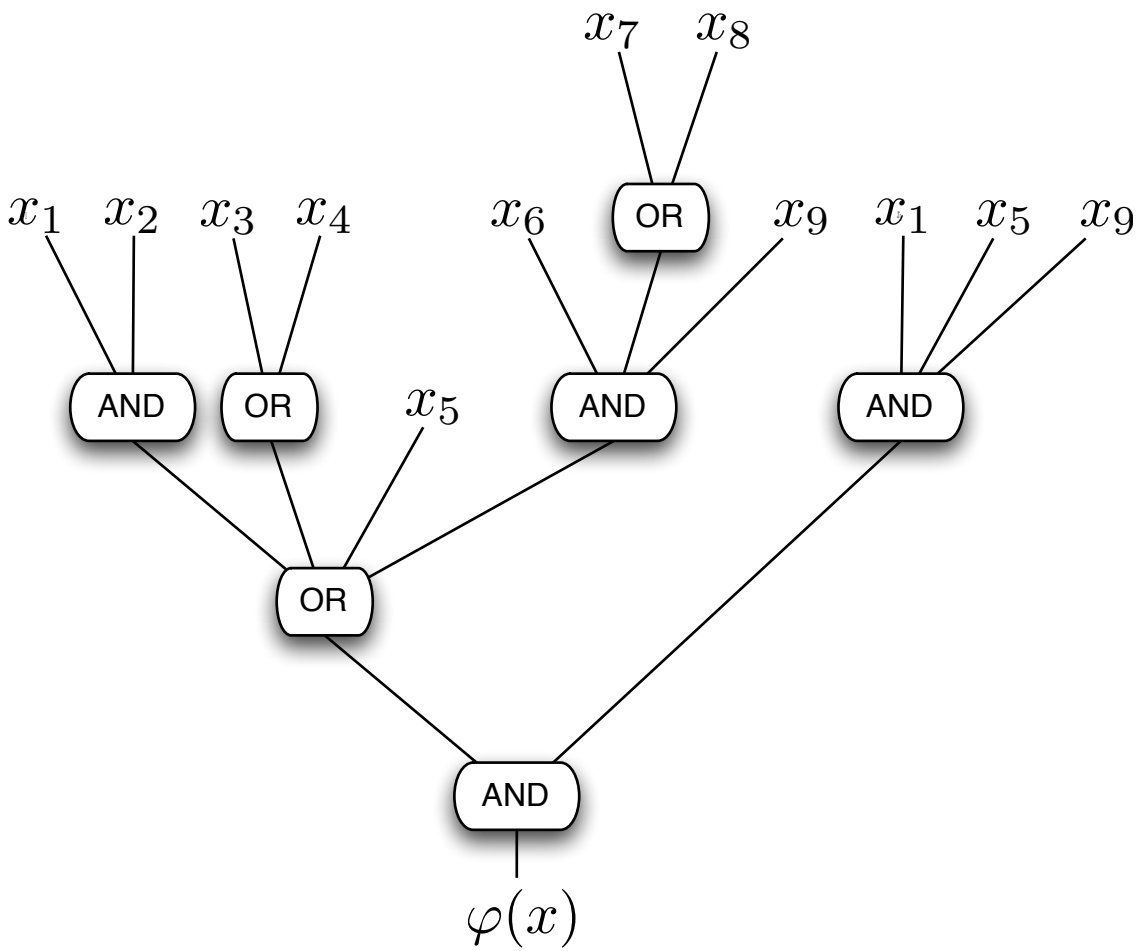


Convert formula φ into a graph G(φ)

Define classical random walk on G(φ)

Quantize that walk





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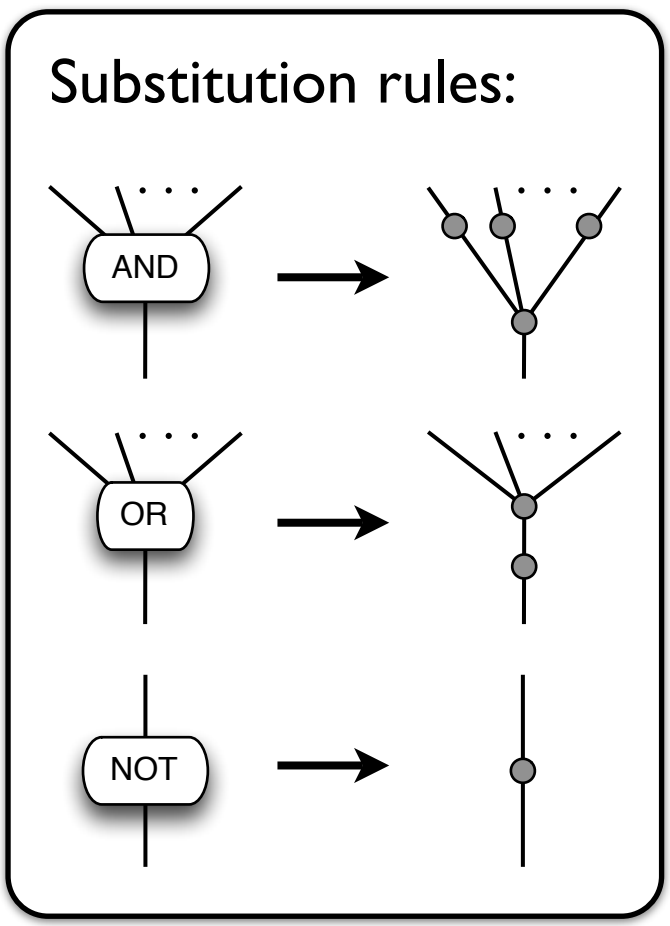

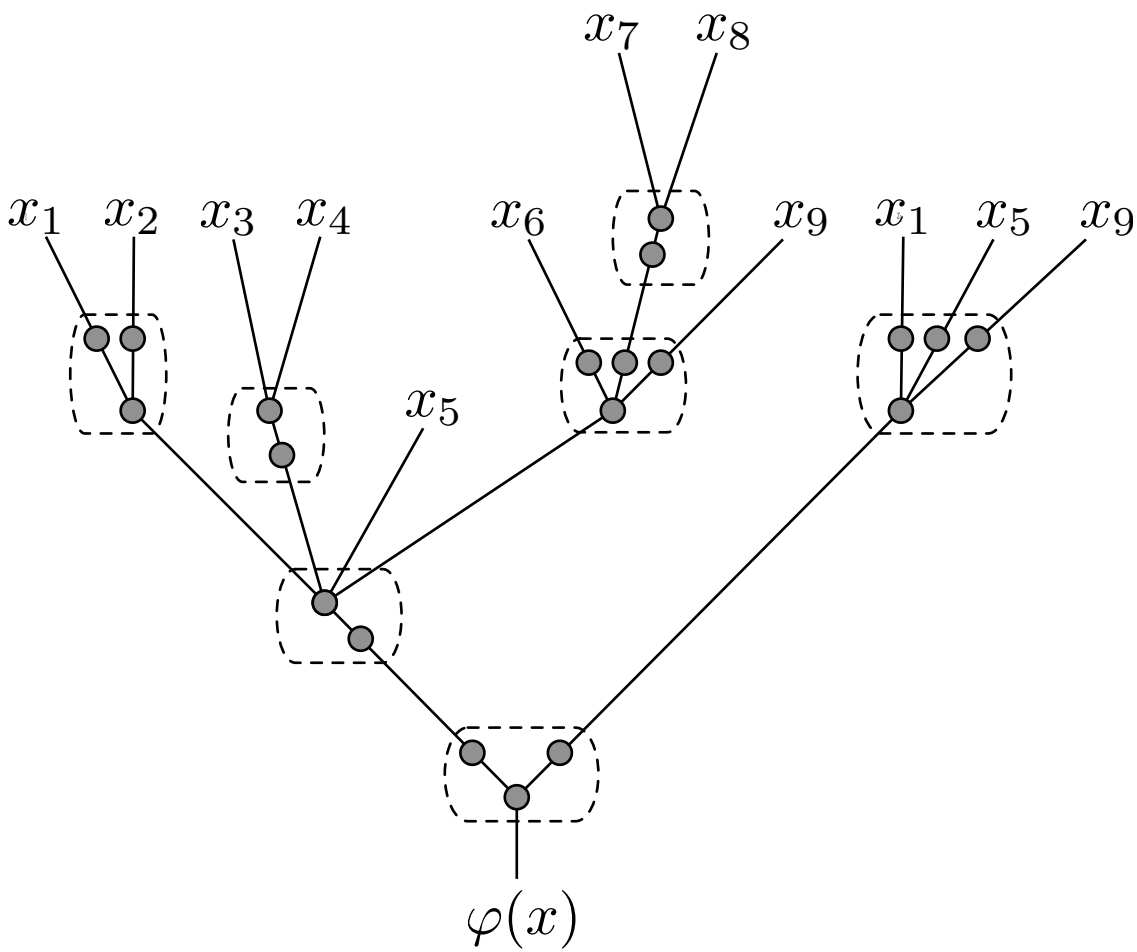

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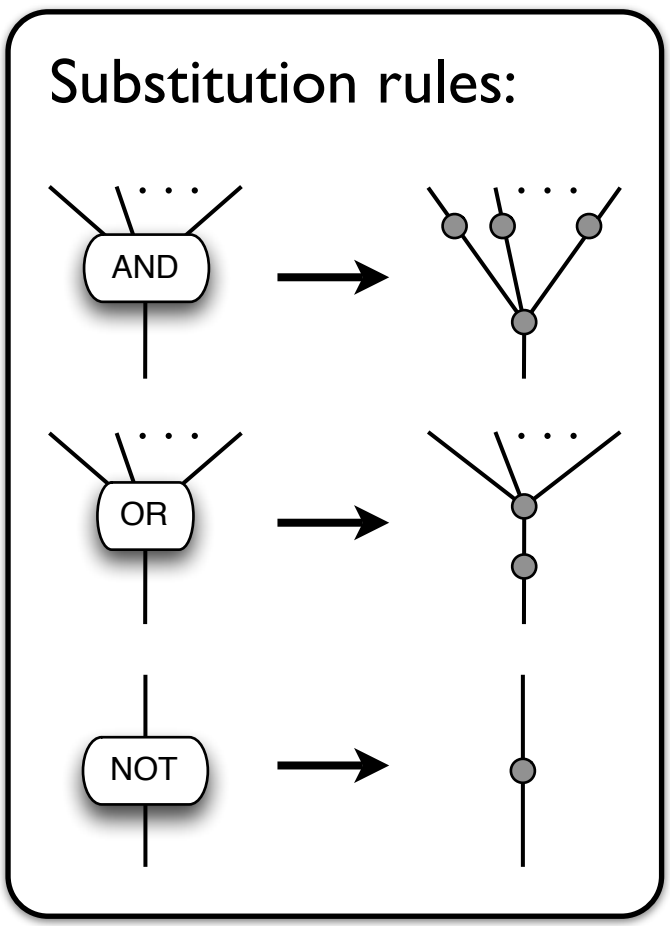

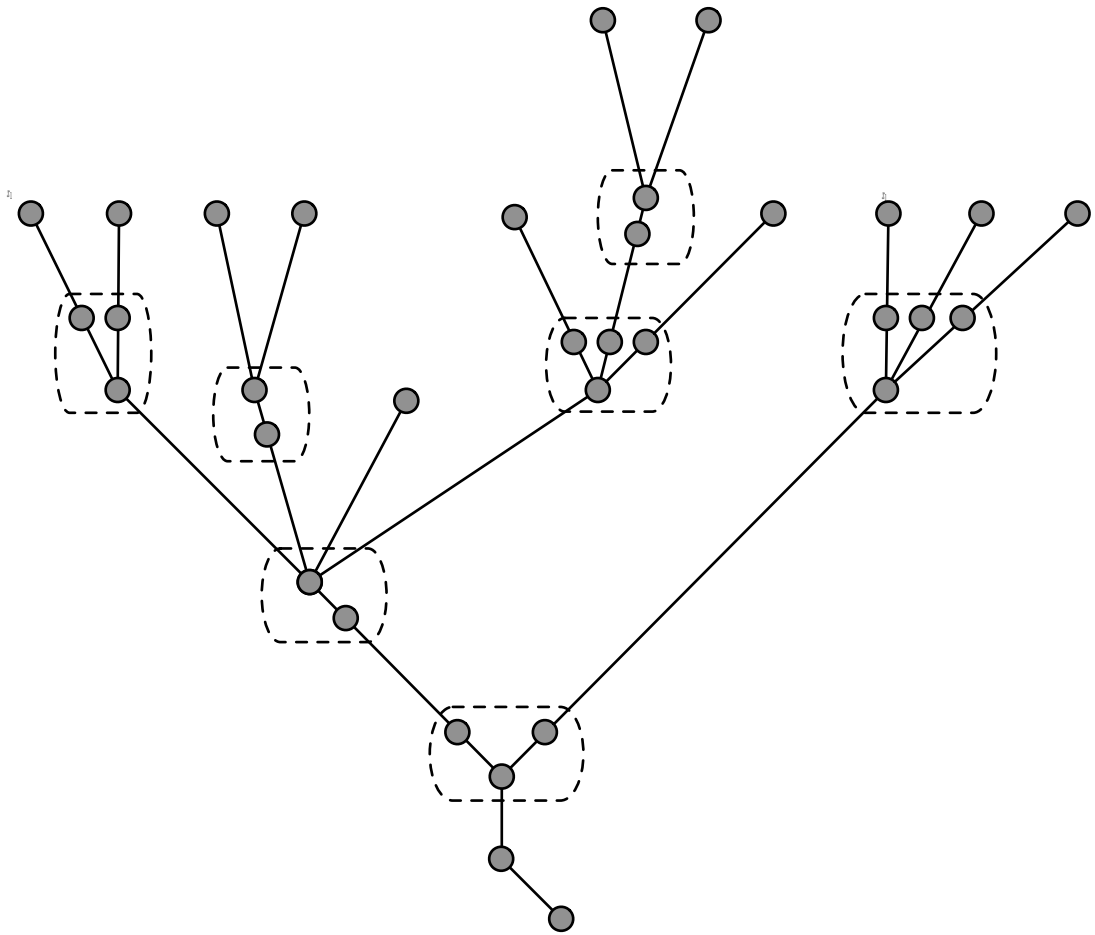

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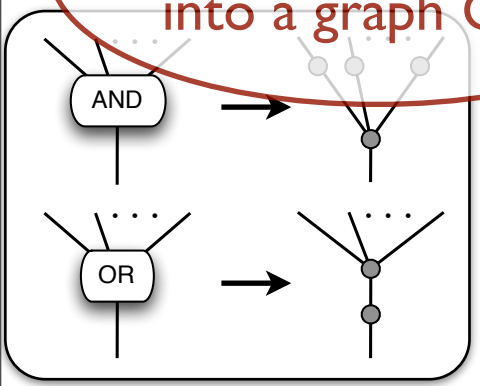
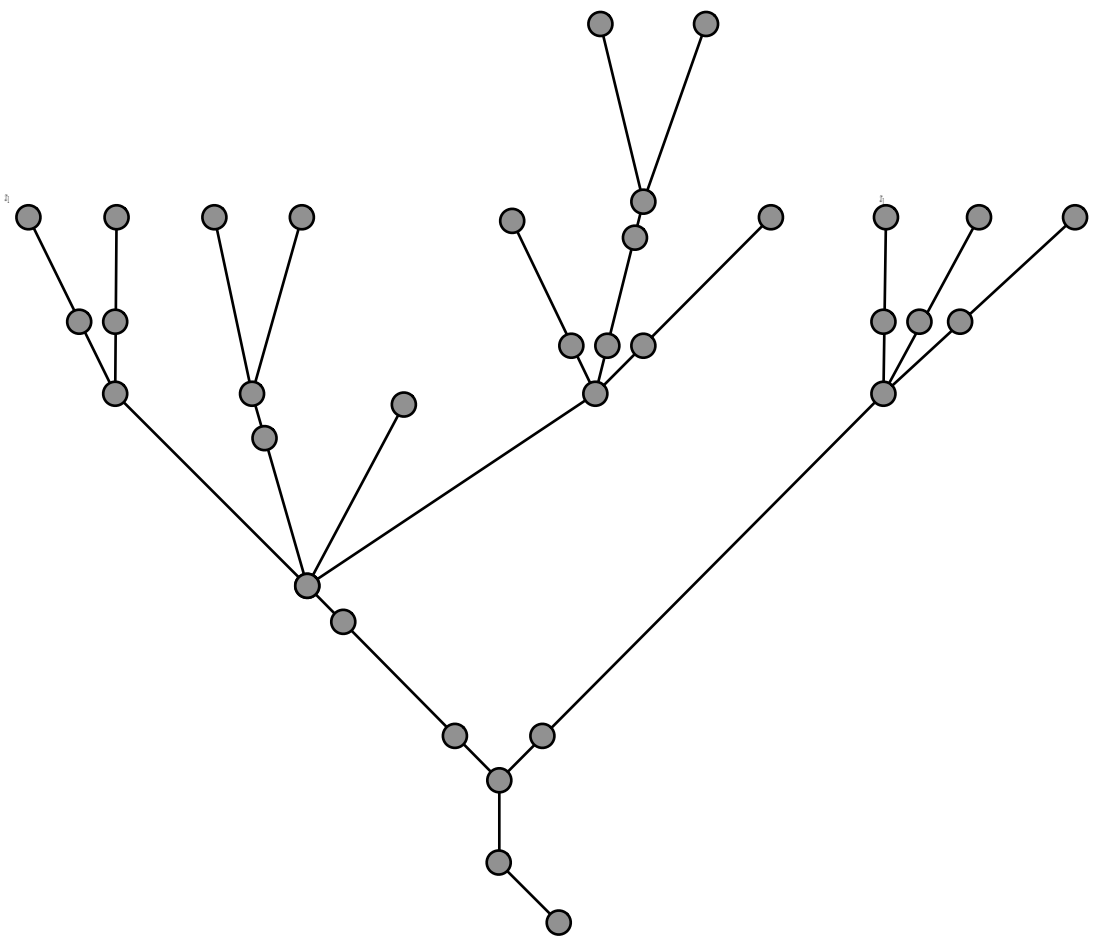


- P(stepping to subtree) ∝ √(size of that subtree)/√s_p
  - (For a balanced tree, walk is uniform)



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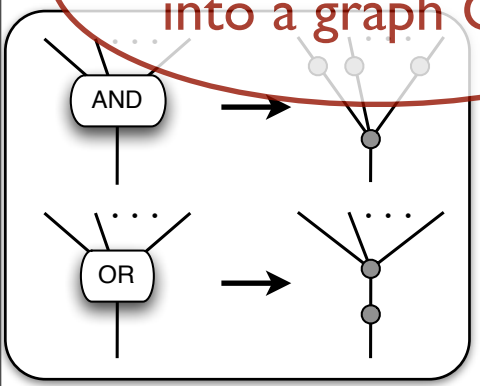

- P(stepping to subtree) ∝ √(size of that subtree)/√s_p
  - (For a balanced tree, walk is uniform)
- Make leaves (inputs) evaluating to 0 probability sinks

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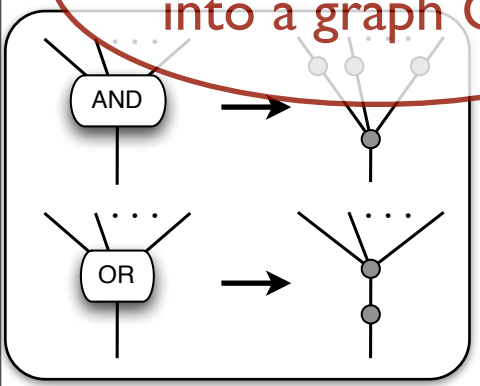

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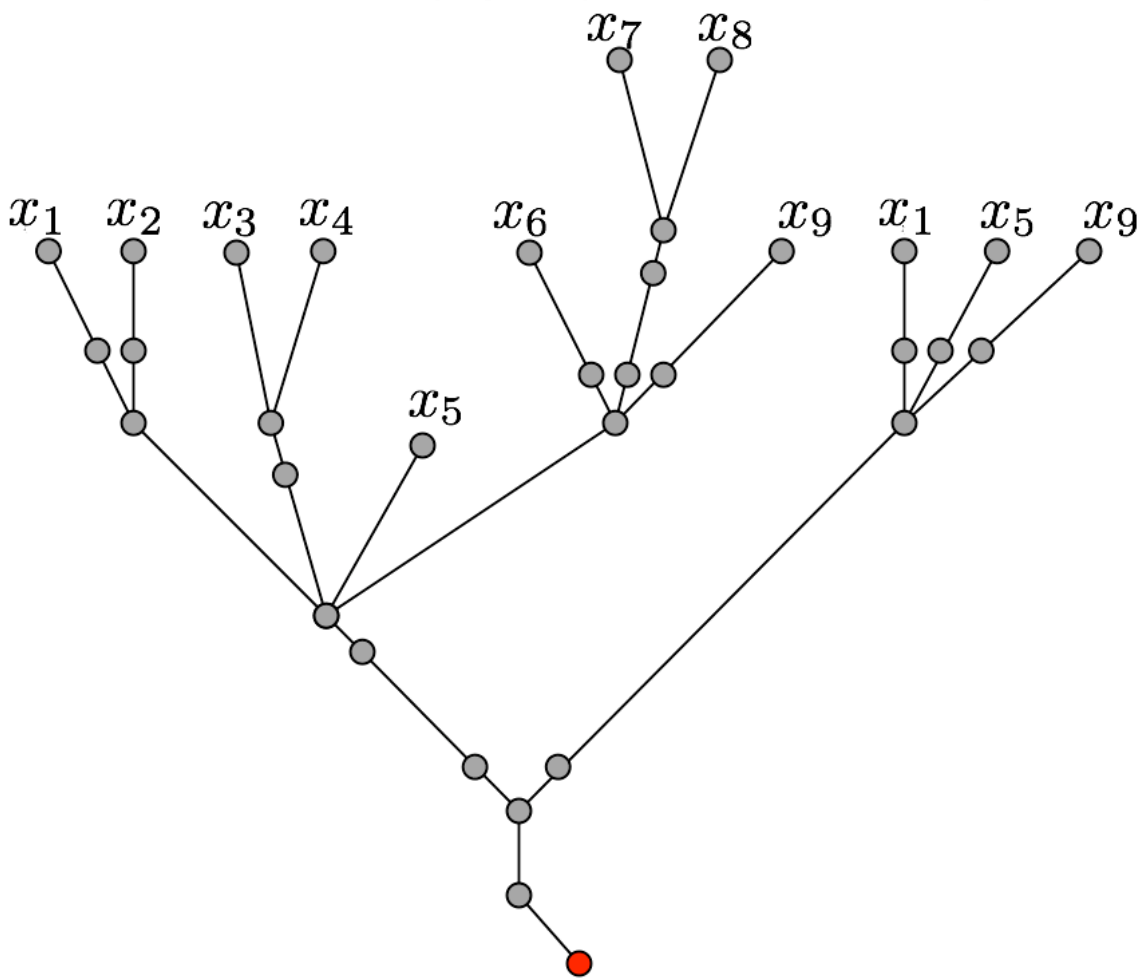

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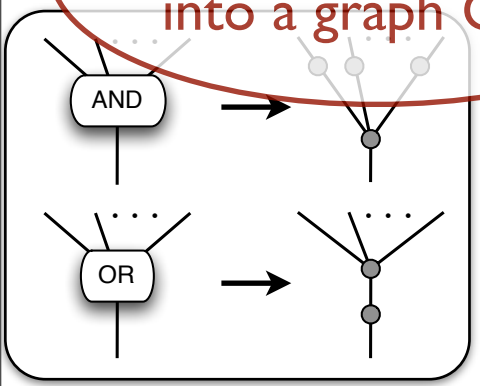

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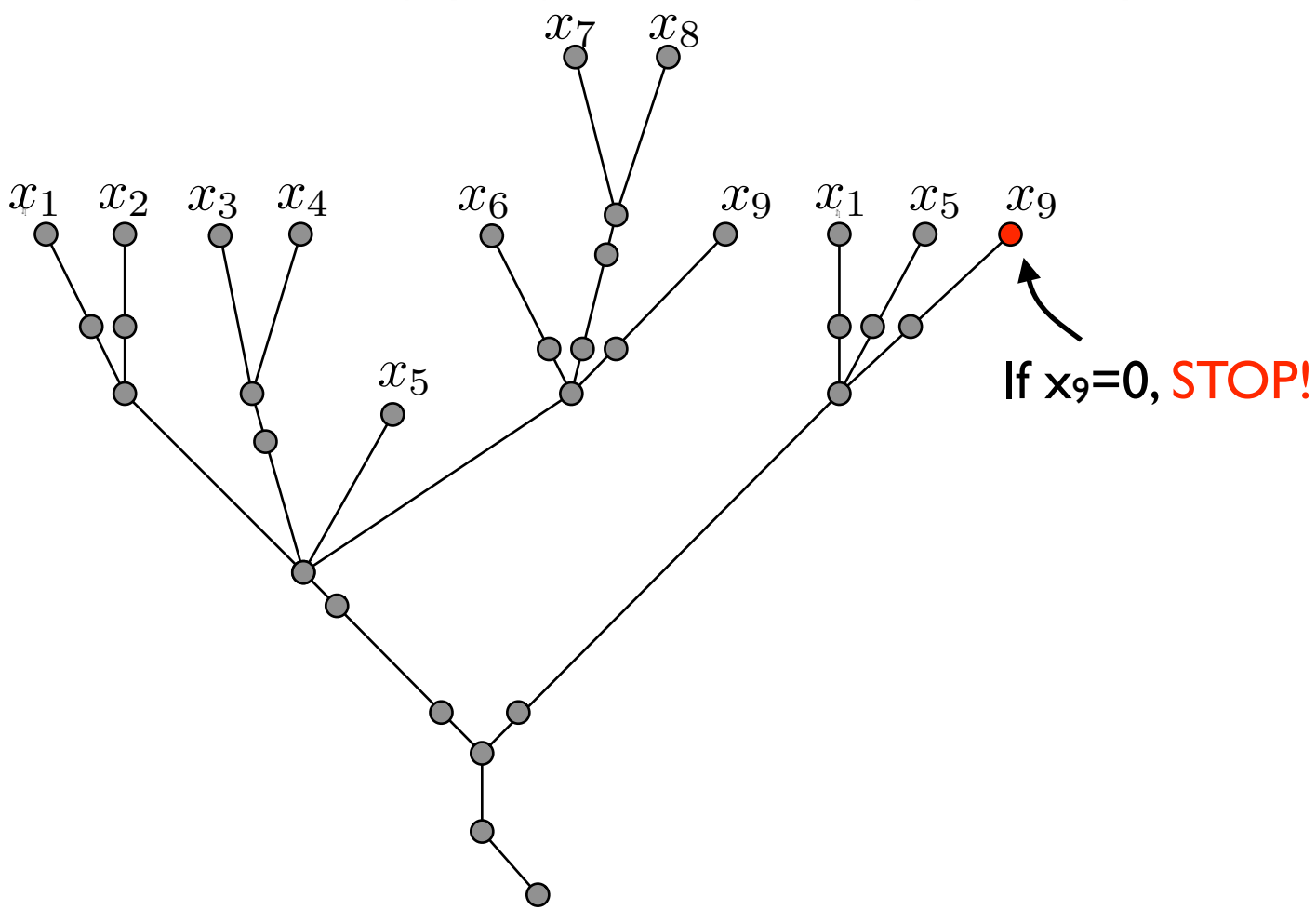
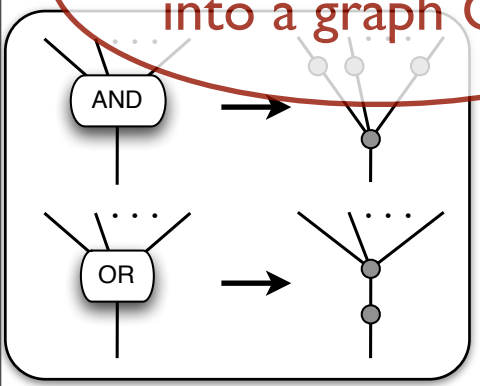
  - (For a balanced tree, walk is uniform)
- Make leaves (inputs) evaluating to 0 probability sinks



Convert formula φ into a graph G(φ)

Define classical random walk on G(φ)

Quantize that walk



If x_i=0, STOP!

- Classically, roll a dice to determine next step
- Quantumly, the dice is part of the quantum state. Instead of randomizing the dice between steps, apply a unitary operator to it.

Transition probabilities

$$\{p_1, p_2, \ldots, p_6\}$$

U = reflection about the state

$$\sqrt{p_1}\left|\boxdot\right\rangle + \sqrt{p_2}\left|\boxdot\right\rangle + \sqrt{p_3}\left|\boxdot\right\rangle + \sqrt{p_4}\left|\boxdot\right\rangle + \sqrt{p_5}\left|\boxdot\right\rangle + \sqrt{p_6}\left|\boxdot\right\rangle$$

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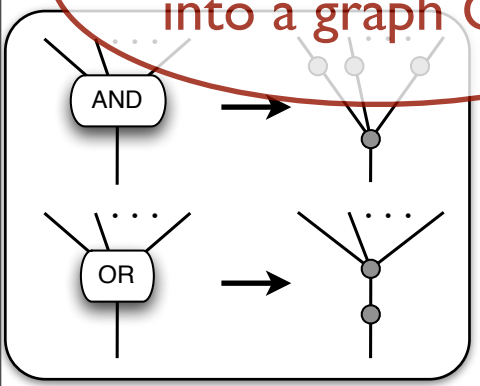

If x_i=0, STOP!

- Classically, roll a dice to determine next step
- Quantumly, the dice is part of the quantum state. Instead of randomizing the dice between steps, apply a unitary operator to it.
  - Probability sinks in the classical r.w. (inputs x_i=0) become phase flips in the qu. walk ⇒ standard phase flip oracle

Transition probabilities

$$\{p_1, p_2, \ldots, p_6\}$$

U = reflection about the state

$$\sqrt{p_1}\left|\boxdot\right\rangle+\sqrt{p_2}\left|\boxdot\right\rangle$$
$$+\sqrt{p_3}\left|\boxdot\right\rangle+\sqrt{p_4}\left|\boxdot\right\rangle$$
$$+\sqrt{p_5}\left|\boxdot\right\rangle+\sqrt{p_6}\left|\boxdot\right\rangle$$

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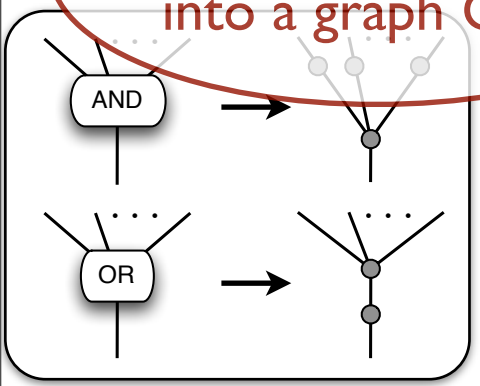
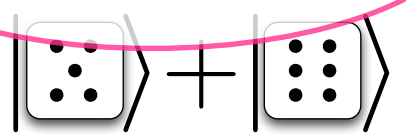
Quantize that walk



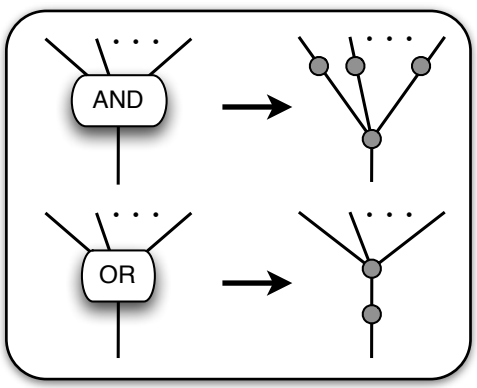


The Algorithm:

- Start at the root
- Apply phase estimation to the quantum walk with precision $1/\sqrt{N}$ (i.e., run the walk for time $\sqrt{N}$)
  - If phase is 0 or π, output "φ(x)=1"
  - Otherwise output "φ(x)=0"

Formula evaluation algorithm

Convert formula φ into a graph G(φ)



Define classical random walk on G(φ)

P(stepping to subtree)

∝ √(size of that subtree)/√s_p

If x_i=0, STOP!

Quantize that walk

$$\{p_1, p_2, \ldots, p_6\}$$

$$\sqrt{p_1}\left|\cdot\right\rangle + \sqrt{p_2}\left|\cdot\cdot\right\rangle + \sqrt{p_3}\left|\cdot\cdot\right\rangle + \sqrt{p_4}\left|\cdot\cdot\right\rangle + \sqrt{p_5}\left|\cdot\cdot\right\rangle + \sqrt{p_6}\left|\cdot\cdot\right\rangle$$

2. Why It Works

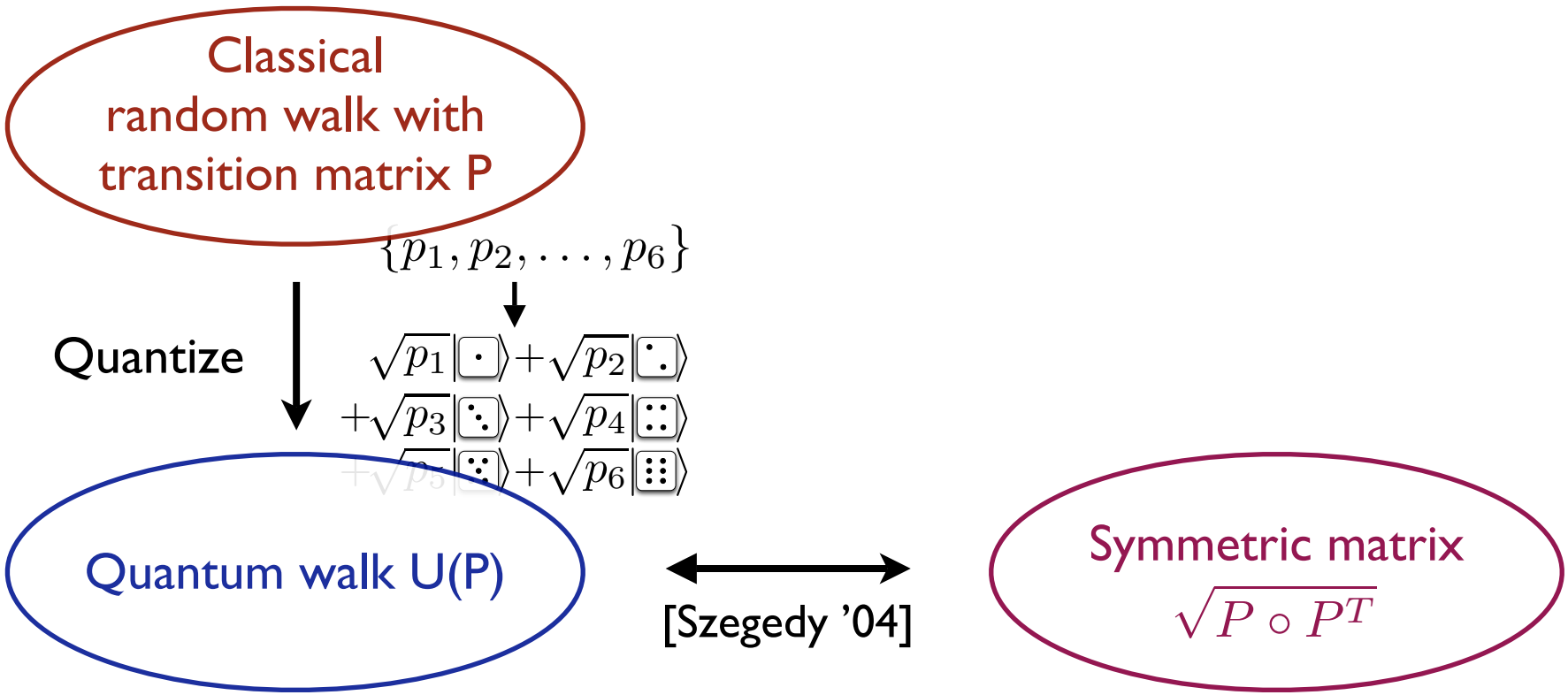
The Algorithm:

- Start at the root
- Apply phase estimation to the quantum walk with precision $1/\sqrt{N}$ (i.e., run the walk for time $\sqrt{N}$)
  - If phase is 0 or π, output "$\varphi(x)=1$"
  - Otherwise output "$\varphi(x)=0$"

- Outputs $\varphi(x)=1$ "iff" there is an eigenstate of the walk operator U that overlaps the root and has corresponding $|\text{eigenvalue}| < 1/\sqrt{N}$

$\therefore$ We need to carry out spectral analysis of the quantum walk U

Spectral analysis 1: Szegedy correspondence

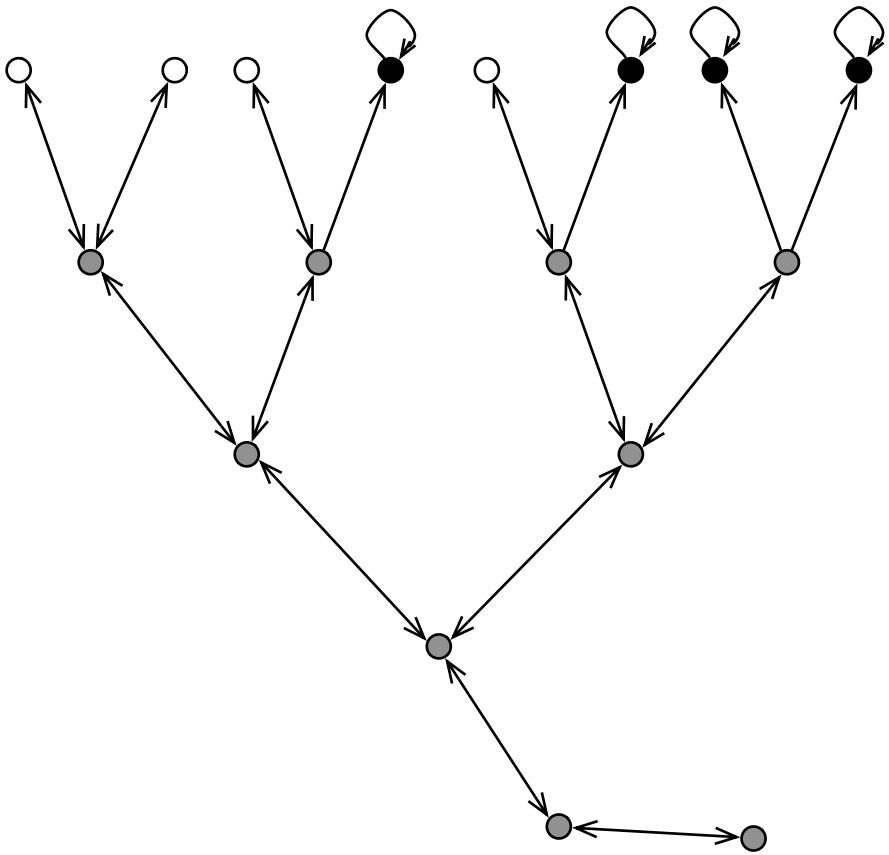


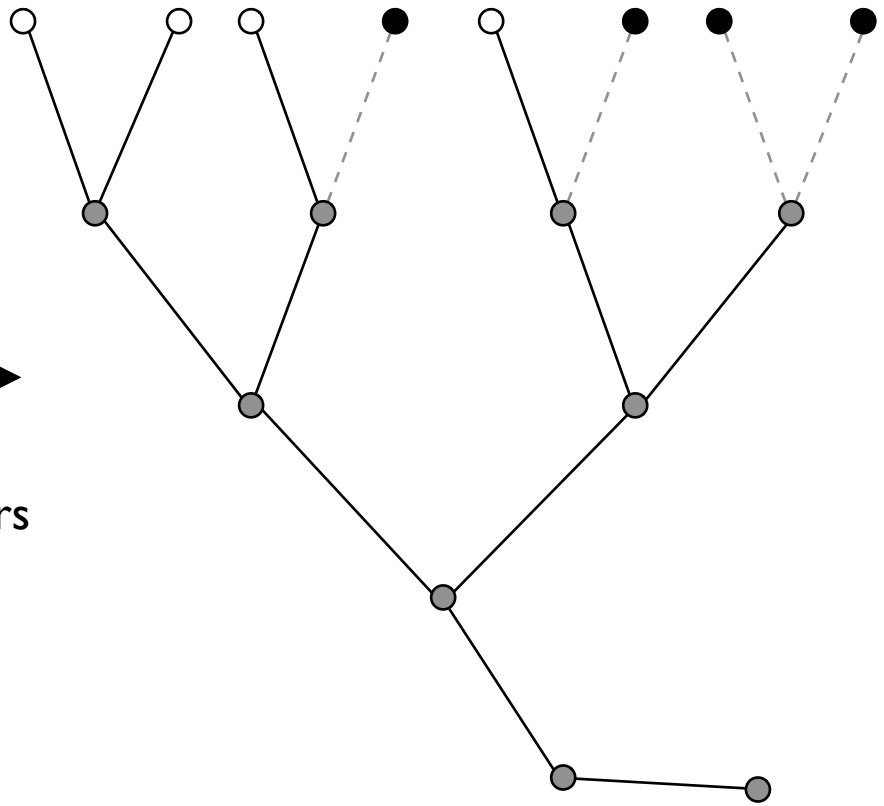
- Correspondence between spectrum and eigenvalues of quantum walk and those of a symmetric matrix
  (If P is symmetric, then P and U(P) have corresponding eigensystems)
  - Halves the dimensions
  - Real instead of complex operators

Spectral analysis 1: Szegedy correspondence

Quantum coined walk U on:

$\sqrt{P \circ P^T}$ = Weighted Adj. matrix of:





eigenvalues
& eigenvectors

○ = 1

● = 0

2|E| dimensions

|V| dimensions

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- Apply phase estimation to the quantum walk with precision $1/\sqrt{N}$ (i.e., run the walk for time $\sqrt{N}$)
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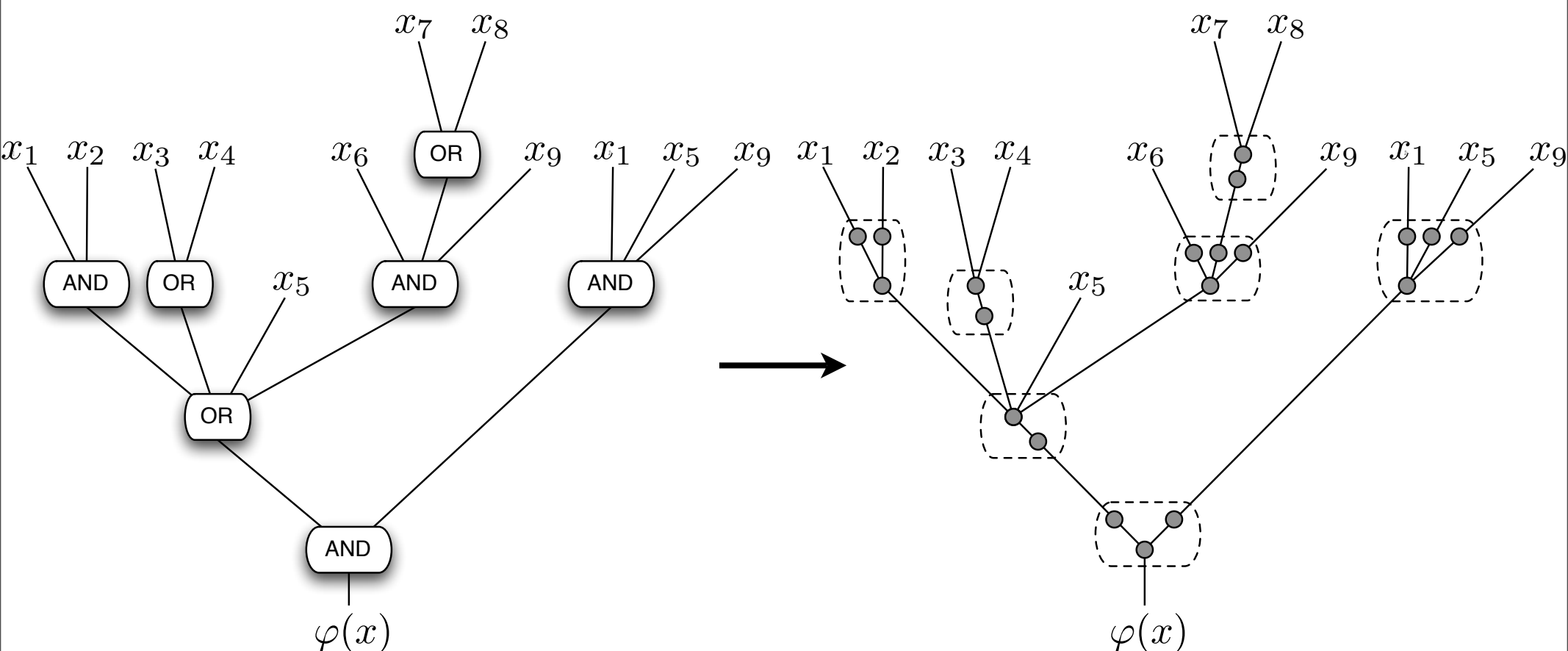
- Outputs φ(x)=1 "iff" there is an eigenstate of $A_G = \sqrt{P \circ P^T}$ that overlaps the root and has corresponding |eigenvalue| < $1/\sqrt{N}$

∴ We need to carry out spectral analysis of A_G

- **Main Theorem:**
  - Adjacency matrix A_G has eigenvalue E=0 eigenvector with Ω(1) support on r" when φ(x)=1.
  - A_G has no eigenvalues $E \in (-1/\sqrt{N}, 1/\sqrt{N})$ with support on r" when φ(x)=0.

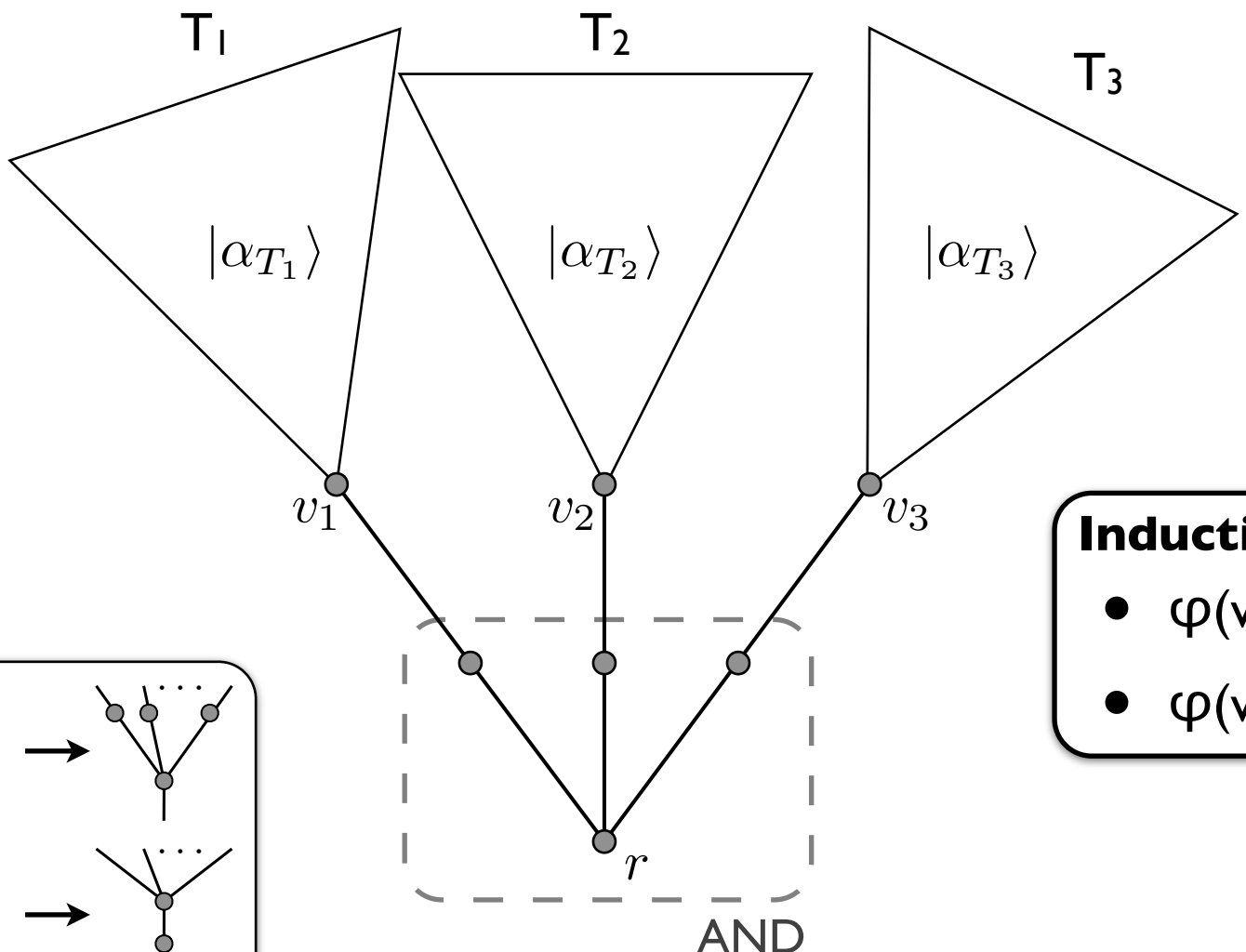
- **Theorem:** φ(x)=1 $\Leftrightarrow$ ∃ an E=0 eigenstate of A_G supported on root r.

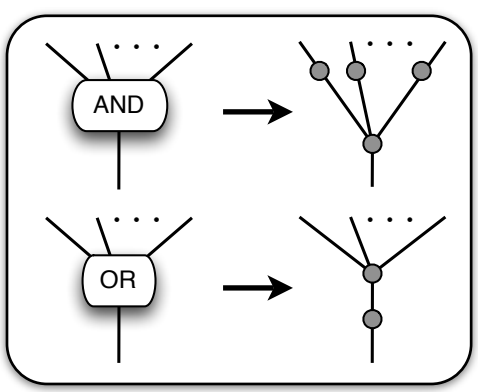
Proof



- **Theorem:** $\varphi(x)=1 \iff \exists$ an E=0 eigenstate of A_G supported on root r.

Construct eigenstate $|\alpha\rangle = \sum_v \alpha_v |v\rangle$ by induction





Inductive Hypothesis:

- $\varphi(v)=0 \Rightarrow \alpha_v=0$
- $\varphi(v)=1 \Rightarrow \alpha_v$ can be $\neq 0$

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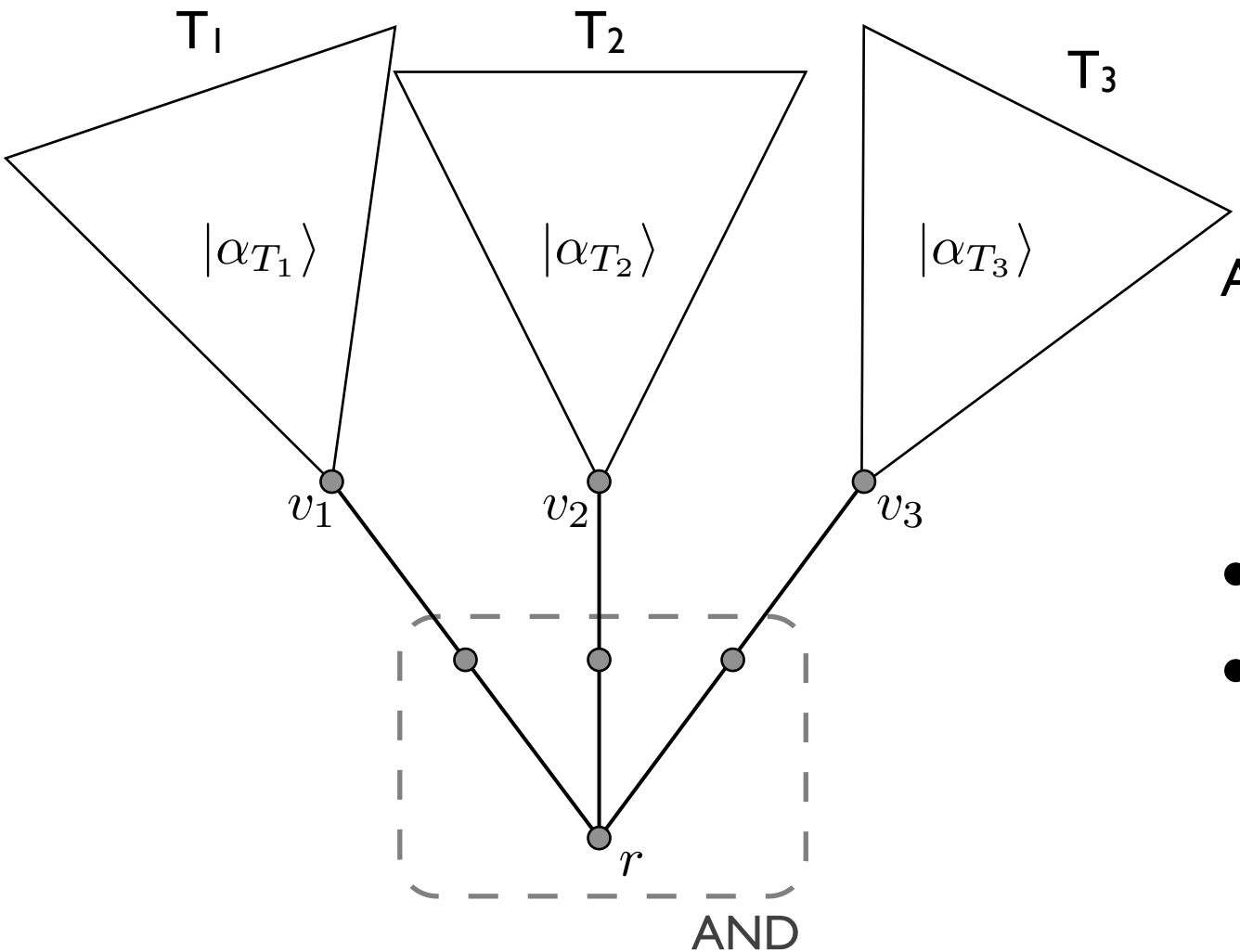

AND gate gadget constraints:

$$\alpha_{v_1} + \alpha_r = 0$$
$$\alpha_{v_2} + \alpha_r = 0$$
$$\alpha_{v_3} + \alpha_r = 0$$

- If any $\varphi(v_i)=0$, $\alpha_{v_i}=0 \Rightarrow \alpha_r=0$
- If all $\varphi(v_i)=1$, can scale each $|\alpha_{T_i}\rangle$ so $\alpha_{v_1}=\alpha_{v_2}=\alpha_{v_3}\neq 0$, then set $\alpha_r=-\alpha_{v_i}\neq 0$

✓ AND

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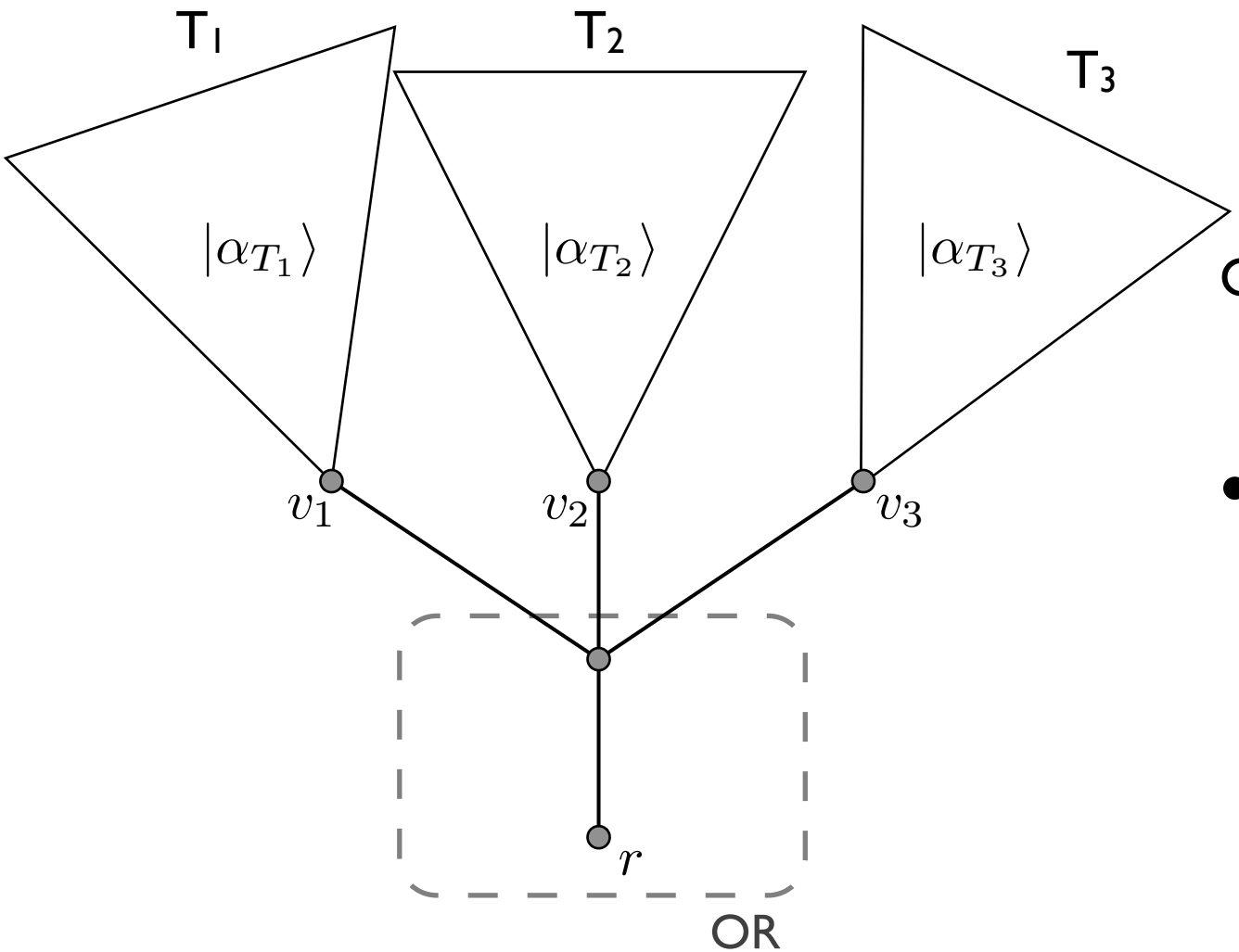


OR gate gadget constraint:

$$\alpha_{v_1} + \alpha_{v_2} + \alpha_{v_3} + \alpha_r = 0$$

- α_r can be $\neq 0 \Leftrightarrow$ at least one $\alpha_{v_i} \neq 0 \Leftrightarrow$ at least one $\varphi(v_i)=1$

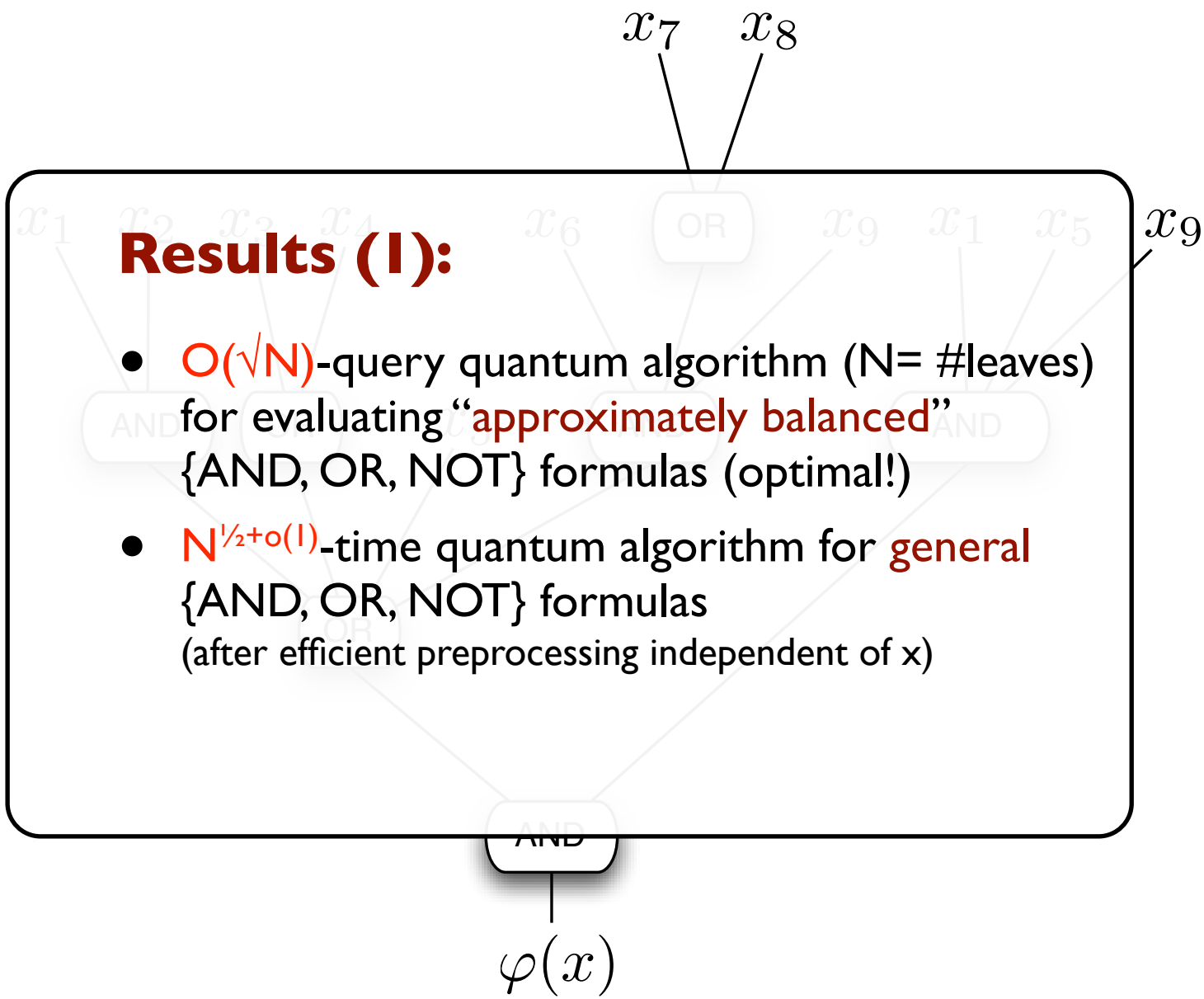
✓ OR

- **Theorem:** $\varphi(x)=1 \iff \exists$ an $E=0$ eigenstate of A_G supported on root r. ✓

- **Main Theorem:**
  - Adjacency matrix A_G has eigenvalue $E=0$ eigenvector with $\Omega(1)$ support on r" when $\varphi(x)=1$.
  - A_G has no eigenvalues $E \in (-1/\sqrt{N}, 1/\sqrt{N})$ with support on r" when $\varphi(x)=0$.

- Remains to show support α_r is large ($\Omega(1)$) when $\varphi(r)=0$, and that there is a large spectral gap ($1/\sqrt{N}$) away from $E=0$ when $\varphi(r)=1$.
- Proofs by same induction but quantitative.

Problem: Evaluate the formula with minimal queries to the input bits x_i.



Results (1):

- $O(\sqrt{N})$-query quantum algorithm (N= #leaves) for evaluating "approximately balanced" {AND, OR, NOT} formulas (optimal!)
- $N^{1/2+o(1)}$-time quantum algorithm for general {AND, OR, NOT} formulas
  (after efficient preprocessing independent of x)

3. Extensions

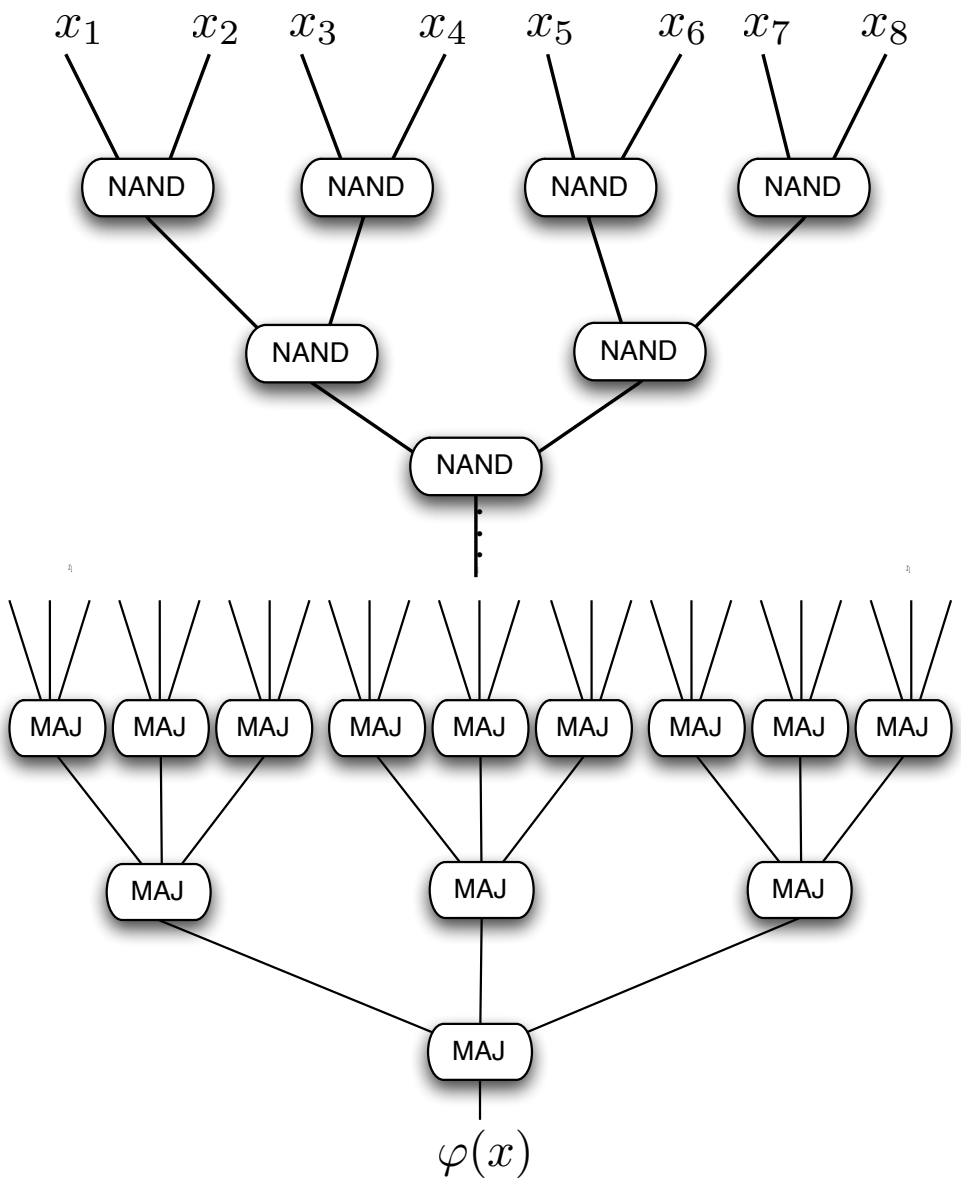

3. Extensions


Your Good Times are
Just Begining With
Lucky TORII

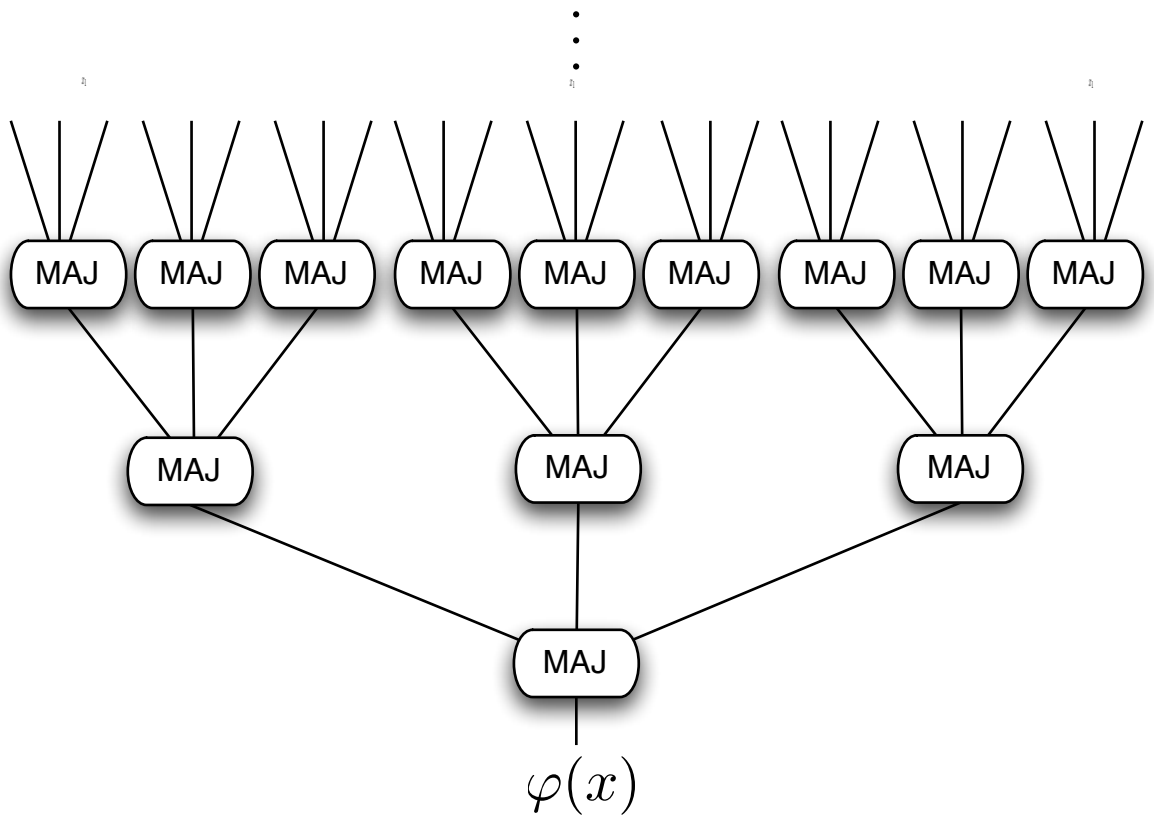
joint work with Robert Špalek

Extension: Formulas on different gate sets

- Cost to evaluate a formula that uses other gates besides {AND, OR, NOT}?
- First step: Balanced iterative functions (the same function composed on itself)
  - [Farhi, Goldstone, Gutmann '07]: Balanced recursive NAND gate formula
  - Other balanced iterative functions?



Recursive 3-bit majority tree



- What is the classical complexity of evaluating recursive MAJ_3 tree?

  [Boppana '86]

- Answer: Unknown!
  - Between $\Omega\left((7/3)^d\right)$ and $o\left((8/3)^d\right)$ for depth d

    [Jayram, Kumar & Sivakumar '03]

Recursive 3-bit majority tree

- Classical complexity to evaluate recursive MAJ3-gate tree is unknown:
  - Between $\Omega\left((7/3)^d\right)$ and $o\left((8/3)^d\right)$ [Jayram, Kumar & Sivakumar '03]
- Best quantum lower bound is $\Omega\left(\sqrt{C_0(f)C_1(f)}\right) = \Omega(2^d)$
- Quantum algorithm:
  - Expand majority into {AND, OR} gates:

$$\mathrm{MAJ3}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_3 \wedge (x_1 \vee x_2))$$

∴ {AND, OR} formula size increases is 5^d

∴ $O(\sqrt{5^d}) = O(2.24^d)$-query algorithm!

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$< 7/3$ better than classical lower bound

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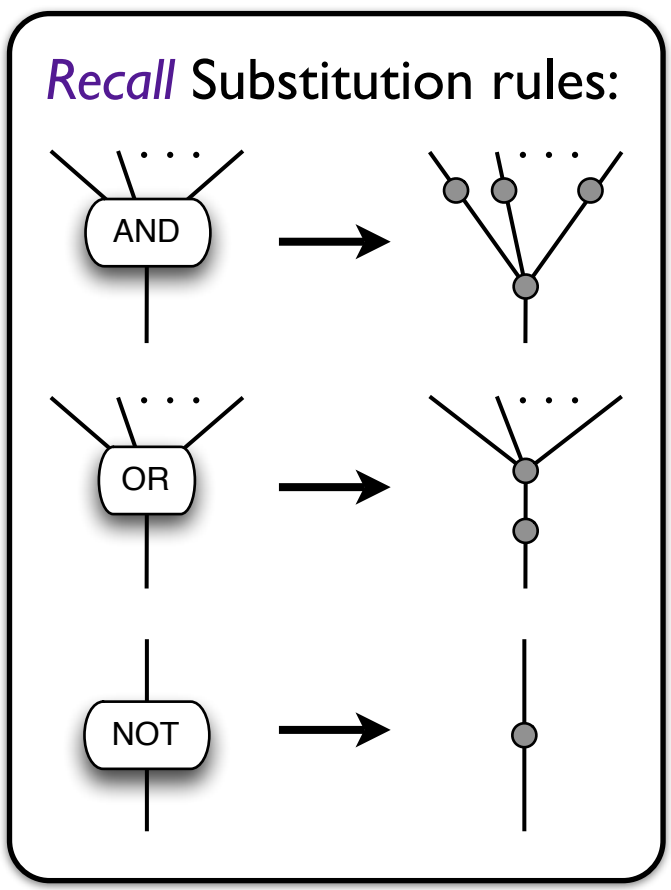
  - ∴ {AND, OR} formula size increases is 5^d
  - ∴ $O(\sqrt{5^d}) = O(2.24^d)$-query algorithm
  - In fact, inputs are not arbitrary; worst-case inputs are promised not to occur
  - ∴ Improved analysis gives $O\left(\left(\sqrt{3+\sqrt{2}}\right)^d\right) = O\left(2.101\ldots^d\right)$

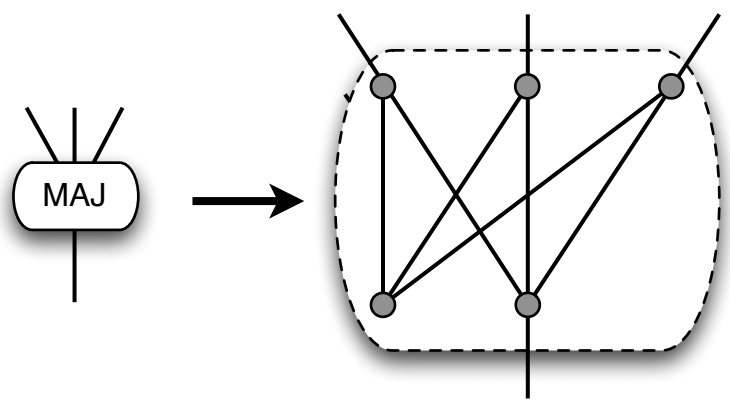
Different gate sets: Gate gadgets

- Classical complexity between $\Omega\left((7/3)^d\right)$ and $O\left((2.655\ldots)^d\right)$
- Quantum lower bound: $\Omega(2^d)$
- Quantum upper bound: $\mathrm{MAJ3} = (x_1 \wedge x_2) \vee (x_3 \wedge (x_1 \vee x_2))$ ⇒ O($\sqrt{5}^d$) = O(2.236…^{d})
  - improved analysis ⇒ O($\sqrt{(3+\sqrt{2})}^d$) = O(2.101…^{d})

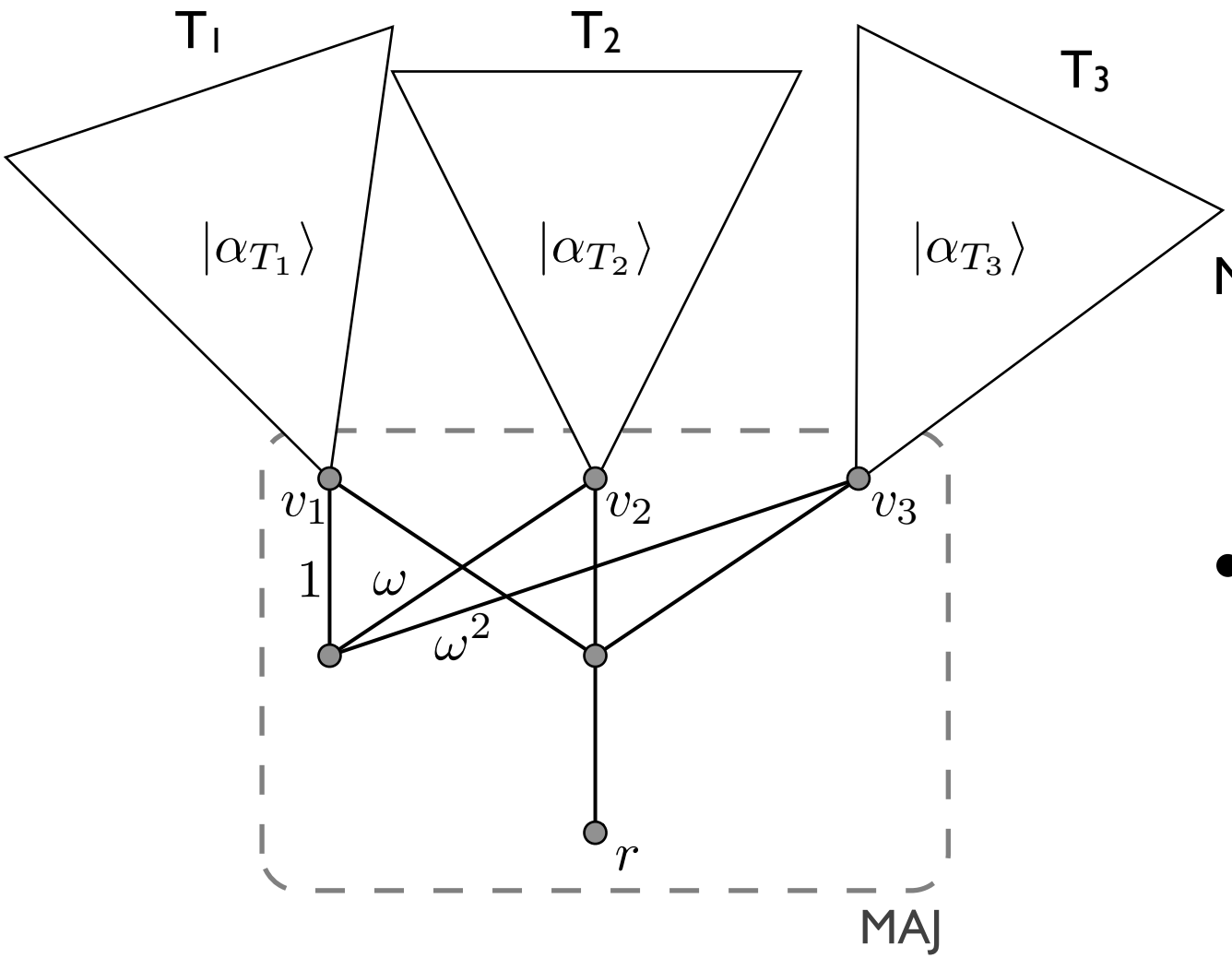
- Gate gadgets:



New MAJ3 substitution rule:



- **Inductive Hypothesis:**
  - $\varphi(v)=0 \Rightarrow \alpha_v=0$
  - $\varphi(v)=1 \Rightarrow \alpha_v$ can be $\neq 0$



MAJ3 gate gadget constraints:

$$-\alpha_r = \alpha_{v_1} + \alpha_{v_2} + \alpha_{v_3}$$

$$\alpha_{v_1} + \omega\alpha_{v_2} + \omega^2\alpha_{v_3} = 0$$

- At least two $\varphi(v_i)$ must be 1 to satisfy second constraint nontrivially.

✓ MAJ

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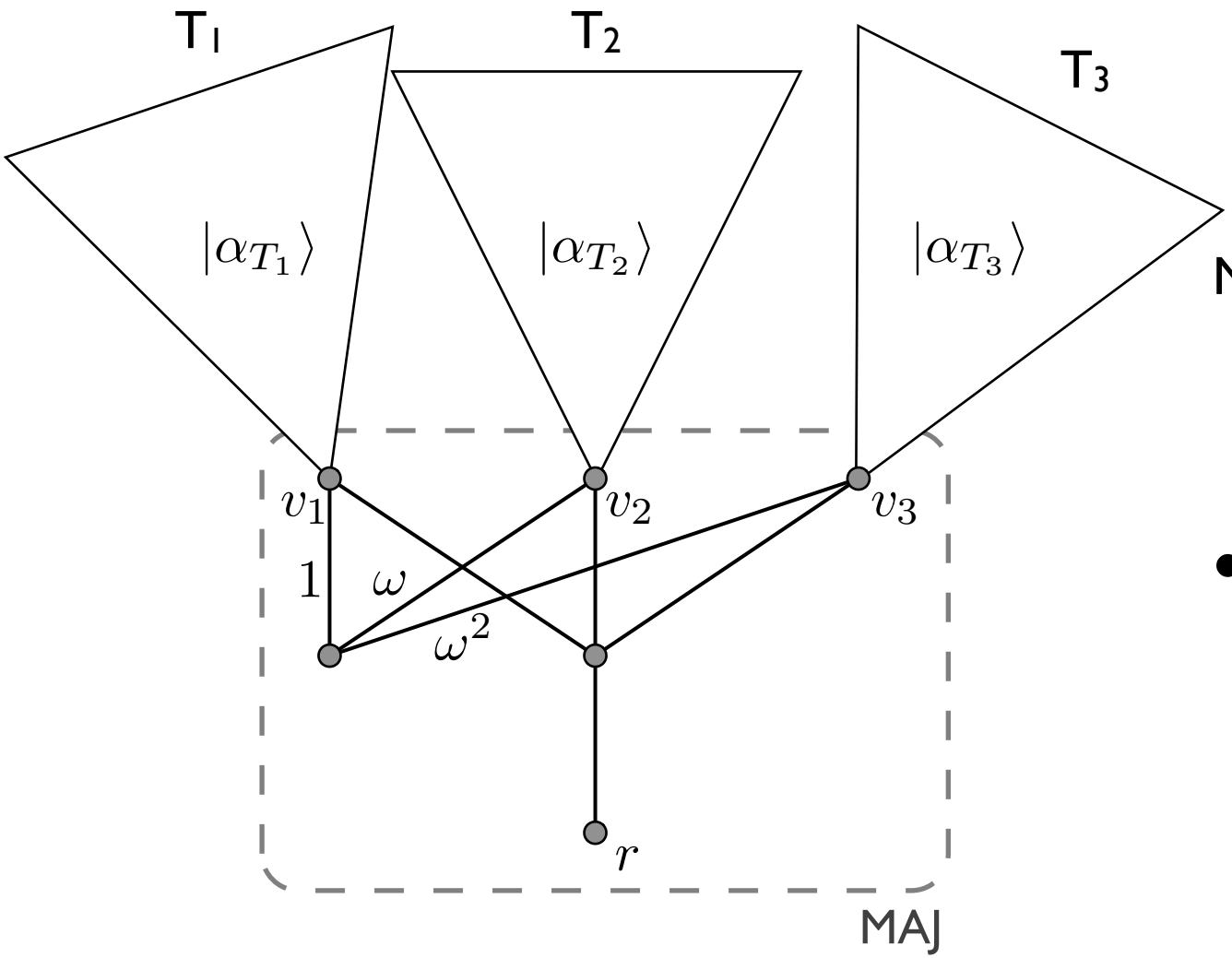

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✓ MAJ

$\Rightarrow$ (Near) Optimal $O(2^{d(1+o(1))}) = O(N^{\log_3 2+o(1)})$-query balanced recursive MAJ_3 formula evaluation algorithm

More results…

⇒ (Near) Optimal $O(2^{d(1+o(1))}) = O(N^{\log_3 2+o(1)})$-query balanced recursive MAJ_3 formula evaluation algorithm, based on new substitution gadget

- Furthermore, either by
  - Analysis of AND-OR formula expansion on promised inputs,
  - Or by constructing new "gadget" substitution rules

AKA "**Span programs**," well-studied in classical complexity theory

⇒ Nearly optimal algorithms for iterative versions of **all 3-bit functions**, some 4-bit functions (38 of 208 inequivalent functions)

Remarks on formula evaluation algorithms:

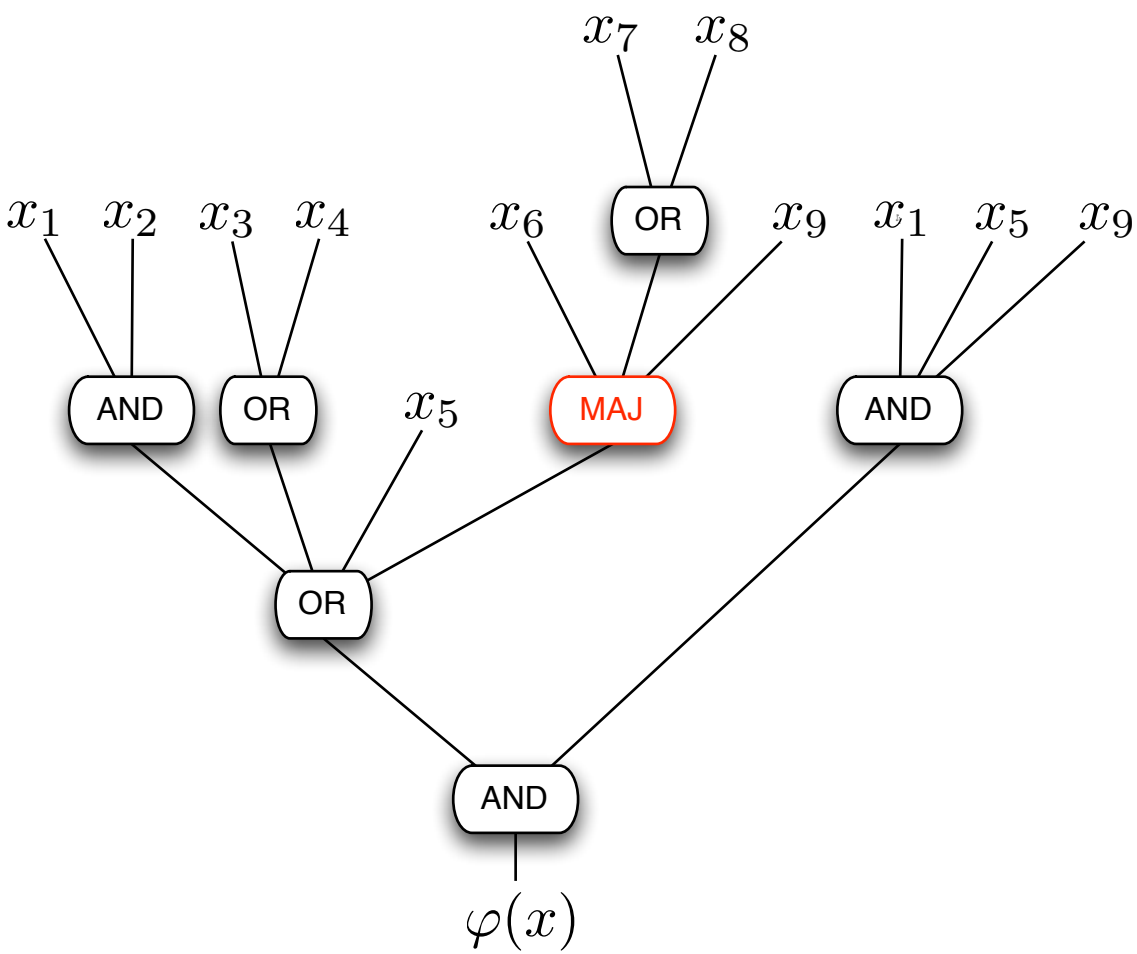
Classical vs. Quantum

- Classical complexity of evaluating balanced k-ary alternating AND-OR tree is $(k/2)^{depth} = N^{\sim(1-1/\log_2 k)}$ — approaches N as k increases
  - Classical complexity of evaluating general AND-OR formulas is not known?
- Classical complexity of evaluating iterative MAJ_3 formula is unknown: between $\Omega\left((7/3)^d\right)$ and $o\left((8/3)^d\right)$
  - (the generalization of the optimal AND-OR algorithm is *not* optimal when applied to MAJ_3 trees)

- Quantumly, complexity is $N^{1/2}$ queries always, all the way up to k=N (i.e., evaluating $OR(x_1,\ldots,x_N)$, Grover search)
  - General AND-OR formulas can be evaluated with $N^{1/2+o(1)}$ queries
- Quantumly, the AND-OR algorithm generalizes to give optimal algorithm for evaluating iterated f, where f is any 3-bit function

Further extensions: Mixing different gates

- Algorithm works for unbalanced formulas with mixtures of different gates — but is it optimal?



Further extensions: Mixing different gates

- Algorithm works for unbalanced formulas with mixtures of different gates — but is it optimal?
- Answer: Work in progress. Unclear, but in general, probably *not*.

Promising:

- "Layered" balanced formulas (at each depth, one gate type used) are okay
- {AND, OR, +} compose well on top; e.g., $\mathrm{AND}(c_1,\ldots,c_k) = \sqrt{(c_1^2+\ldots+c_k^2)}$
- "Adversary-balanced" formulas on gate set S:
  - S' = {arbitrary two- or three-bit gates, $\mathrm{EQUAL}_{O(1)}$ gates}
  - S = { bounded-size {AND, OR, NOT, PARITY} formulas on inputs that are themselves possibly gates from S' }

Discouraging:

- With the exception of {AND, OR, PARITY}, we do not know how to evaluate optimally formulas with inputs that do not have balanced $\mathrm{ADV}^+ = \mathrm{ADV}^\pm$ bounds
  - Simple examples (~~e.g., $\mathrm{MAJ}_3(x_1,x_2,x_3 \oplus x_4)$, $\mathrm{MAJ}_3(x_1,x_2,x_3 \wedge x_4)$~~) suggest that gadget weights cannot be optimized to match lower bounds

(x1 & (x2 v x3)) v (!x1 & (x4 v (!x2 & !x3)))

- Results:
  - $N^{1/2+o(1)}$-time quantum algorithm for general {AND, OR, NOT} formulas (after efficient preprocessing); $O(\sqrt{N})$-query quantum algorithm for "approximately balanced" {AND, OR, NOT} trees (optimal!)
  - Nearly optimal query qu. algs. for iterative versions of all 3-bit functions (e.g., MAJ_3), and for 38 of 208 inequivalent 4-bit functions—extended to "adversary balanced" S-formulas

Open problems

- More optimal formula types:
  - Extension to allow other gates
  - Mix different gates in the same formula. Inductive hypotheses are compatible, so algorithm works — but it may or may not be optimal.
  - Better lower bounds: Want to understand qu. lower bounds ADV^+ versus $ADV^\pm$ [Høyer, Lee, Špalek '07], esp. gate composition
- Can a witness set be extracted from the eigenstate?
- Classical connections:
  - More significant connections to classical "span programs"
  - Significance of classical random walk?
  - Open Classical ?: Is [BCE'91] formula rebalancing optimal?
    - Does there exist formula φ, k such that every equivalent φ' of depth at most k log N has size(φ') $\geq N^{1+1/\log k}$?