Shortest Descending Paths through Given Faces

Mustaq Ahmed* Anna Lubiw*

Abstract

A path from s to t on a polyhedral terrain is *descending* if the height of a point p never increases while we move p along the path from s to t. No efficient algorithm is known to find a shortest descending path from s to t in a polyhedral terrain. We give some properties of such paths. In the case where the face sequence is specified, we show that the shortest descending path is unique, and give an ϵ-approximation algorithm that computes the path in $O(n^{3.5}\log(\frac{1}{\epsilon}))$ time.

1 Introduction

The problem of determining a shortest path in a polyhedral terrain has many applications in robotics, industrial automation, Geographic Information Systems and wire routing. In certain applications, the feasibility of a path is determined by the height of the points. For example, for laying a canal of minimum length from the source of water at the top of a mountain to fields for irrigation purpose [7], and for skiing down a mountain along a shortest route, we need to compute a shortest path whose height never increases as we move from source to destination. The problem of finding *descending paths in a polyhedral terrain* was first studied by de Berg and van Kreveld [2], who gave an $O(n \log n)$ time algorithm to decide if there is a descending path between two points. They stated as open the problem of finding a shortest descending path (SDP). In a subsequent paper, Roy, Das and Nandy [7] consider some special cases; in particular, they give an $O(n \log n)$ time algorithm to compute an SDP through a sequence of parallel edges.

In this paper, we first establish a set of characteristics of a locally shortest descending path, and show that locally shortest descending paths are much more complicated than geodesic paths (and in particular that the previous work [7] has failed to recognize some of the subtlety). Then we turn to the case where the path must go through a given sequence of faces. By formulating the problem as a convex optimization problem, we prove that the SDP is unique, and give an ϵ-approximation algorithm to compute the path in $O(n^{3.5}\log(\frac{1}{\epsilon}))$ time.

*David R. Cheriton School of Computer Science, University of Waterloo, ON N2L 3G1, Canada, {m6ahmed,alubiw}@uwaterloo.ca

In a forthcoming paper, we give our main result on this problem: an approximation algorithm to find an SDP from s to t in a polyhedral terrain. We use the uniqueness result from the current paper, but we need significantly more. To be precise, our algorithm requires the ability to extend a locally shortest path. This is easy for geodesic paths because they unfold to straight lines, but for descending paths we need a significantly more detailed analysis of the bend angles. Although the convex optimization technique offered in the current paper is not part of our general solution, we think it is interesting in its own right.

2 Preliminaries

A terrain is a 2-dimensional surface in 3-dimensional space with the property that every line parallel to the z-axis intersects it in a point [3]. We assume that the terrain is triangulated. For any point p in the terrain, $h(p)$ denotes the height of p, i.e., the z-coordinate of p.

A path P from s to t on the terrain is *descending* if the z-coordinate of a point p never increases while we move p along the path from s to t. A line segment of a descending path in face f is called a *free segment* if moving either of its endpoints by an arbitrarily small amount to a new position in f keeps the segment descending. Otherwise, the segment is called a *constrained segment*. All the points in a constrained segment are at the same height, though not all constant height segments are constrained. For example, a segment in a horizontal face is free, although all its points are at the same height. A path consisting solely of constrained segments is called a *constrained path*.

We will now define a *locally shortest descending path (LSDP)*, which is analogous to a *geodesic path* (i.e., a *locally shortest path*) [5]. An LSDP between two nodes is a descending path that cannot be shortened by slight perturbation of the intermediate nodes. Note that perturbing a single node in a descending path may make the path infeasible (i.e., not descending), and hence, we allow more than one node to be perturbed simultaneously. For example, if we increase the height of a node p to H, all the points before p on the path must be moved to height at least H to keep the path descending. Also note that a constrained path is an LSDP.

For ease of discussion, we will use the term "edge" to denote a line segment of the terrain, and the term "segment" to denote a line segment of a path. Similarly, an endpoint of an edge is called a "vertex", while an endpoint of a segment is called a "node". We assume that all paths in our discussion are directed.

3 Characteristics of an LSDP

An LSDP and a geodesic path over a terrain are similar in many respects. The following lemmas establish two properties of an LSDP that make an LSDP analogous to a geodesic path [5].

Lemma 1 *Any subpath of an LSDP is an LSDP.*

Lemma 2 *An LSDP consists of straight line segments, and bends only at the edges of the terrain.*

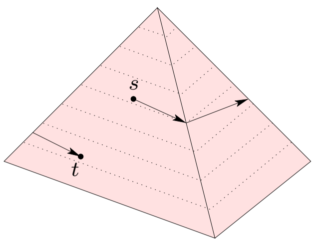


Figure 1: An LSDP visiting a face twice

As in the case of a geodesic path [5], an LSDP may visit a single face more than once. For example, a string tightly wrapped around a pyramid as shown in Figure 1 is an LSDP from s to t, and it visits a face twice. However, like a shortest path, an SDP visits a face at most once:

Lemma 3 *The intersection of an SDP P with a face of the terrain is either empty or a line segment.*

One important difference between an LSDP and a geodesic path is that unlike a geodesic path [5], two consecutive segments of an LSDP through an edge ab do not always become a straight line segment when the two faces of the terrain adjacent to ab are unfolded onto a plane. Before proving this claim, we define two angles at every edge intersected by an LSDP to quantify the amount of deflection at that edge. Let $P = (p, q, r)$ be a descending path from an interior point p in face f_1 to an interior point r in face f_2 adjacent to f_1 such that P crosses edge $ab = f_1 \cap f_2$ at q where $h(a) \geq h(b)$ (Figure 2). Let $p'r'$ be a line segment perpendicular to ab at q such that $p' \in f_1$ and $r' \in f_2$. The angle $\angle pqp'$ is called the *entering angle of P at ab*, and is considered positive if and only if p and b are on the same side of $p'r'$. The angle $\angle rqr'$ is called the *exiting angle of P at ab*, and is considered positive if and only if r and a are on the same side of $p'r'$. In Figure 2, α and β are respectively the entering angle and the exiting angle of P at ab. When $h(a) > h(b)$, we say that P deflects downward at q if $\alpha > \beta$, and that P deflects upward at q if $\alpha < \beta$. Note that if $h(a) = h(b)$, entering and leaving angles can be defined in two ways. Our discussion is valid for any of these definitions.

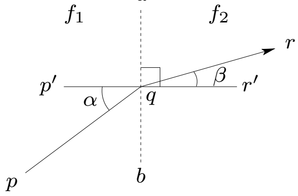


Figure 2: Entering and exiting angles

Lemma 4 *The path $P = (p, q, r)$ is an LSDP if and only if one of the following holds: (i) $\alpha = \beta$; (ii) $\alpha > \beta$, and qr is constrained; or (iii) $\alpha < \beta$, and pq is constrained.*

Proof. $\Leftarrow$: We prove only Case (ii) here because Case (i) is trivial, and Case (iii) is similar to Case (ii).

If $\alpha > \beta$, and qr is constrained, then for any point $q' \in ab$ such that $h(q') > h(q)$, the path (p, q', r) is longer than P. On the other hand, for any point $q' \in ab$ such that $h(q') < h(q)$, (p, q', r) is not a descending path because $h(r) = h(q) > h(q')$. Therefore, P is an LSDP.

$\Rightarrow$: If $\alpha < \beta$ and pq is free, let q' be a point on ab slightly above q. The path $P' = (p, q', r)$ is shorter than P. Because qr is descending, the segment $q'r$ is also descending. The segment pq' is descending since pq is free and q' is arbitrarily close to q. Therefore, P' is a descending path. So, P is not an LSDP. We can similarly show that P is not an LSDP if $\alpha > \beta$ and qr is free. $\square$

Note that in their attempt to compute SDP's in a convex terrain, Roy, Das and Nandy [7, Lemma 1] claim that an LSDP can never bend downward at an intermediate node. However, Lemma 4 shows that an LSDP can bend downward even in a convex terrain. For instance, the terrain in Figure 3, in which the dotted lines are horizontal lines, is convex, and it can be shown easily that the path from s to t is an SDP when the first and the last segments are parallel to each other. Consequently, their first algorithm, i.e., the algorithm for computing an SDP in a convex terrain, is wrong.

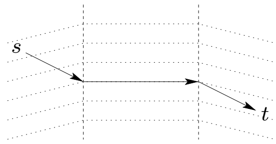
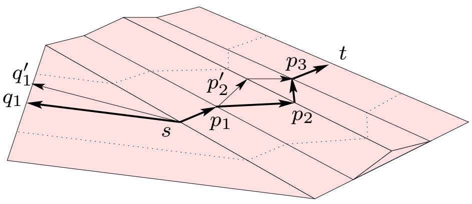


Figure 3: Downward deflection in a convex terrain

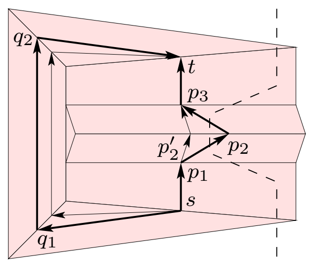
In spite of all the similarities between an LSDP and a geodesic path, an SDP and a shortest path can be completely different from each other in every respect. The following lemma proves this claim.

Lemma 5 *Let P_T and P'_T denote respectively an SDP and a shortest path from s to t in terrain T. There exists a terrain T for which one (or more) of the following holds: (i) the ratio of the lengths of P_T and P'_T is arbitrarily large; (ii) P_T and P'_T pass through two different face sequences; and (iii) there is no descending path through the face sequence crossed by P'_T.*

Proof. Consider a polyhedron that has a perspective view and a top view as in Figure 4. The dotted lines in the perspective view are horizontal lines. Let s and t be two points of equal heights as shown.



(a)



(b)

Figure 4: Proof of Lemma 5

Let T_1 be the terrain consisting of the faces crossed by the constrained path (s, p_1, p_2, p_3, t) as shown. Clearly, (s, p_1, p_2, p_3, t) is an SDP in T_1. Also (s, p_1, p'_2, p_3, t) is a shortest path, where $p_1p'_2 \perp p_2p'_2$ (and $p_3p'_2 \perp p_2p'_2$ by symmetry). Now imagine rotating T_1 around the axis defined by the line through s and t. This rotation keeps the length of (s, p_1, p'_2, p_3, t) unchanged, but changes the length of (s, p_1, p_2, p_3, t). If we rotate T_1 until the face adjacent to s becomes almost horizontal, the length of (s, p_1, p_2, p_3, t) becomes arbitrarily large. This proves the first part of the lemma.

Let T_2 be the terrain consisting of the faces visible in the top view in Figure 4. It is not hard to see that from s to t, there are exactly two LSDP's (s, p_1, p_2, p_3, t) and (s, q_1q_2, t), and exactly two geodesic paths (s, p_1, p'_2, p_3, t) and $(s, q'_1q'_2, t)$ in T_2. In the figure, the path (s, p_1, p'_2, p_3, t) is shorter than the path $(s, q'_1q'_2, t)$. So, $(s, p_1p'_2, p_3, t)$ is the shortest path from s to t. We can make the length of (s, p_1, p_2, p_3, t) greater than that of (s, q_1q_2, t) by rotating the faces crossed by (s, p_1p_2, p_3, t) as in the first part of the proof, while keeping the slopes of other faces unchanged. This makes (s, q_1q_2, t) an SDP in T_2. Clearly, the SDP and the shortest path in T_2 pass through disjoint sets of faces, which proves the second part.

If we modify T_2 by removing the part of it to the right of the dashed lines in Figure 4(b), it is no longer possible to construct any descending path through the face sequence crossed by the shortest path (s, p_1, p'_2, p_3, t). $\square$

Lemma 5 implies that it is unlikely that one can determine an SDP from s to t by using a shortest path between those two points. Note that the third algorithm of Roy, Das and Nandy [7, Sec. 5] tries to trace an approximate SDP P from s to t by following a shortest path P' until reaching a point where P' is not descending, and following constrained segments from that point until the traced path P either reaches t, or reunites with P' in which case P start following P' again. The last part of Lemma 5 implies that the traced path fails to reach destination t in certain cases. Even when the algorithm successfully traces a descending path from s to t, Parts (i) and (ii) of Lemma 5 imply that there is no guarantee that this path approximates an SDP.

4 Uniqueness of an LSDP

In this section, we show that LSDP's are unique by formulating the problem of computing an LSDP as a convex optimization problem. The uniqueness of a geodesic path is evident from the fact that an unfolded geodesic path is a straight line segment. Since an unfolded LSDP is not a straight line segment, the uniqueness of an LSDP is not obvious. In our proof below, we use $\pi_k = (f_0, f_1, \ldots, f_k)$ to denote the given face sequence. We assume without loss of generality that source s is an interior point of f_0, destination t is an interior point of f_k, and for all $i \in [1, k]$, the edge between f_{i-1} and f_i is a_ib_i with $h(a_i) \geq h(b_i)$.

Let $F(x_1, x_2, \ldots, x_k)$ denote the general path consisting of the line segments sp_1, p_1p_2, p_2p_3, $\ldots$, $p_{k-1}p_k$ and p_kt in this order, where for all $i \in [1, k]$, p_i is any point on *line* a_ib_i, and x_i is a parameter to denote the position of p_i on line a_ib_i. For all $i \in [1, k]$ such that a_ib_i is non-horizontal, the height of p_i uniquely determines its position. So, in these cases, we use the height of p_i as parameter x_i. For each horizontal edge a_ib_i, we use as parameter x_i the signed distance of p_i from b_i. More precisely, $x_i = \overrightarrow{b_ip_i} \cdot \overrightarrow{b_ia_i}/|a_ib_i|$.

Let $L(x_1, x_2, \ldots, x_k)$ be the length of the path $F(x_1, x_2, \ldots, x_k)$, i.e., $\sum_{i=0}^{k} |p_i p_{i+1}|$, where $p_0 = s$ and $p_{k+1} = t$. Using elementary vector arithmetic, one can prove the following lemma:

Lemma 6 *$L(x_1, x_2, \ldots, x_k)$ is a strictly convex function.*

We now determine the constraints on the variables x_i, $1 \le i \le k$, that ensure that $F(x_1, x_2, \ldots, x_k)$ is a descending path through π_k. For all $i \in [1, k]$, the following constraints ensure that the intermediate nodes of $F(x_1, x_2, \ldots, x_k)$ are not outside the corresponding edges:

$$h(b_i) \le x_i \le h(a_i), \qquad \text{when } h(a_i) \ne h(b_i), \quad (1)$$

$$\text{and } \ 0 \le x_i \le |a_i b_i|, \qquad \text{when } h(a_i) = h(b_i). \quad (2)$$

The constraints that ensure that $F(x_1, x_2, \ldots, x_k)$ is a descending path are: $h(p_i) \ge h(p_{i+1})$ for all $i \in [0, k]$. For each i such that $a_i b_i$ is horizontal, $h(p_i)$ is a constant of value $H_i = h(a_i)$. Moreover, $h(p_0)$ and $h(p_k)$ are also constants of values $H_0 = h(s)$ and $H_{k+1} = h(t)$ respectively. For all other $i \in [1, k]$, $h(p_i) = x_i$. Therefore, the height constraints in terms of variables x_i's has the form:

$$v_i \ge v_{i+1}, \quad (3)$$

where for all i, v_i denotes either variable x_i or constant H_i. Note that when both v_i and v_{i+1} are constants, the corresponding constraint is either always satisfied, or never satisfied. Clearly, the constraint is redundant in the former case, and there is no descending path through π_k from s to t in the latter case.

Since L is strictly convex (Lemma 6), and the constraints in Equations (1) to (3) are convex (more precisely, linear), there is at most one local minimum of L in the domain defined by the constraints in Equations (1) to (3) [1, Sec.4.2.1]. Now, any LSDP through π_k from s to t is an instance of $F(x_1, x_2, \ldots, x_k)$ because an LSDP bends only on the edges of the terrain (Lemma 2). Moreover, the length of an LSDP through π_k from s to t corresponds to a local minimum of the length of $F(x_1, x_2, \ldots, x_k)$, i.e., a local minimum of L. These facts establish the following lemma:

Lemma 7 *There is at most one LSDP through π_k from s to t.*

5 Algorithm

It follows from Lemma 7 that we can determine an SDP through π_k by solving the following convex optimization problem:

$$\begin{array}{ll} \text{minimize} & L(x_1, x_2, \ldots, x_k) = \displaystyle\sum_{i=0}^{k} |p_i p_{i+1}| \\ \text{subject to} & \text{the constraints in Equations (1) to (3).} \end{array}$$

We now convert the above problem into the following equivalent problem on variables $x_1, x_2, \ldots, x_k, t_0, t_1, t_2, \ldots, t_k$:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=0}^{k} t_i & \\ \text{subject to} & |p_i p_{i+1}| \le t_i, \quad \text{for } i \in [0, k], & (4) \\ & \text{and the constraints in Equations (1) to (3).} & \end{array}$$

It is easy to show that the coordinates of p_i vary linearly with x_i for all $i \in [1, k]$. As a result, the constraint in Equation (4) can be written in the form $|A_i x_i + B_i x_{i+1} + C_i| \le t_i$ for some scalar constants A_i, B_i and C_i for all $i \in [0, k]$. This makes the above optimization problem a Second-order Cone Program [4], for which finding an ϵ-approximate solution takes $O(k^{3.5} \log(\frac{1}{\epsilon}))$ time [6].

Theorem 8 *Determining an ϵ-approximate SDP through a sequence of k faces from s to t takes $O(k^{3.5} \log(\frac{1}{\epsilon}))$ time.*

References

[1] Stephen Boyd and Lieven Vandenberghe. *Convex optimization.* Cambridge University Press, Cambridge, 2004.

[2] Mark de Berg and Marc J. van Kreveld. Trekking in the Alps without freezing or getting tired. *Algorithmica*, 18(3):306–323, 1997.

[3] Mark de Berg, Marc J. van Kreveld, Mark Overmars, and Otfried Schwarzkopf. *Computational Geometry: Algorithms and Applications.* Springer-Verlag, Berlin, Germany, 2nd edition, 2000.

[4] Miguel Sousa Lobo, Lieven Vandenberghe, Stephen Boyd, and Hervé Lebret. Applications of second-order cone programming. *Linear Algebra and its Applications*, 284:193–228, 1998.

[5] Joseph S. B. Mitchell, David M. Mount, and Christos H. Papadimitriou. The discrete geodesic problem. *SIAM J. Comput.*, 16(4):647–668, 1987.

[6] Valentin Polishchuk and Joseph S. B. Mitchell. Touring convex bodies – a conic programming solution. In *Proceedings of the 17th Canadian Conference on Computational Geometry*, pages 290–293, 2005.

[7] Sasanka Roy, Sandip Das, and Subhas C. Nandy. Shortest monotone descent path problem in polyhedral terrain. In *STACS 2005*, volume 3404 of *Lecture Notes in Comput. Sci.*, pages 281–292. Springer, Berlin, 2005.