Geometric Spanners

Given points in the plane, defining a complete graph G with Euclidean lengths, find a sparse subgraph H that approximates distances in G, i.e., we want

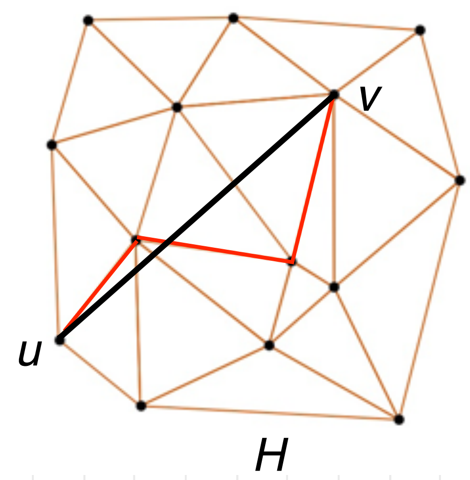
$$d_H(u,v) \leq t\, d_G(u,v) = t\,|uv| \quad \forall u,v \in V$$

the min t is called the ***spanning ratio*** or "stretch factor" of H. H is a ***spanner***.

(More generally spanners can be defined for any edge-weighted graph G.)

Criteria for H :

- minimize the number of edges
- minimize the sum of edge weights
- make H "nice" — bounded degree, planar, etc.
- fault-tolerance (H should be well-connected)



survey Joseph SB Mitchell and Wolfgang Mulzer. "PROXIMITY ALGORITHMS." Chapter 32 in Handbook of Discrete and Computational Geometry, 2016. (see Lecture 1 slides for link to the Handbook in the library)

Geometric Spanners

Given points in the plane, defining a complete graph G with Euclidean lengths, find a sparse subgraph *H* that approximates distances in *G*, i.e., we want

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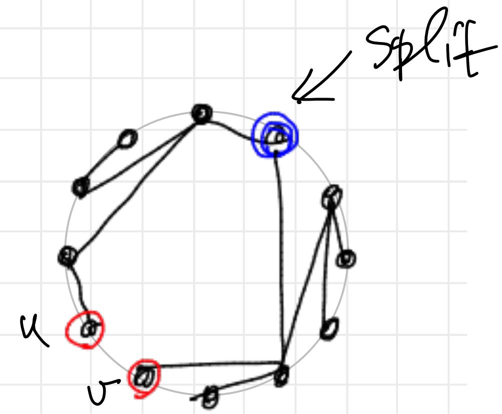
(More generally spanners can be defined for any edge-weighted graph.)

Can H be a tree?

The Minimum Spanning Tree (MST) has n-1 edges, and min sum of weights but its spanning ratio is Theta(n) in the worst case.



What about a tree of minimum spanning ratio (called the "minimum dilation spanning tree")?
NP-hard even for points in the plane

Otfried Cheong, Herman Haverkort, and Mira Lee. "Computing a minimum-dilation spanning tree is NP-hard." *Computational Geometry* 41, no. 3 (2008): 188-205.

https://doi.org/10.1016/j.comgeo.2007.12.001

Any tree can be split at some vertex into 1/3 – 2/3
This gives ~~pts~~ u, v close in |uv| but far in the tree.

The greedy spanner

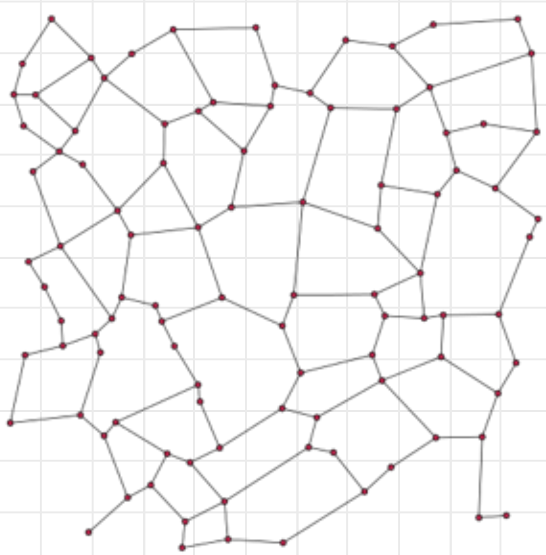
input: point set and desired spanning ratio t
H := ∅
For each edge e = (u,v) of G in order from min to max weight
  if $d_H(u,v) > t\, d_G(u,v)$ then add e to H

for point in the plane

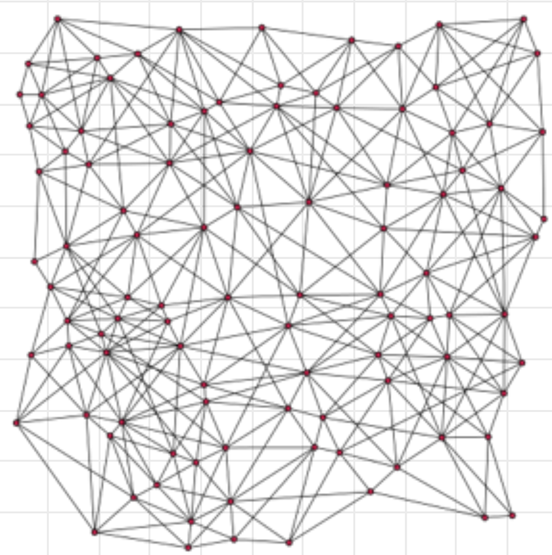
Can be implemented in $O(n^2 \log n)$. For constant t, H has bounded degree, hence O(n) edges. Total weight is O((log n)weight(MST)).

Ingo Althöfer, Gautam Das, David Dobkin, Deborah Joseph, and José Soares. "On sparse spanners of weighted graphs." *Discrete & Computational Geometry* 9, no. 1 (1993): 81-100. https://doi.org/10.1007/BF02189308

t = 2



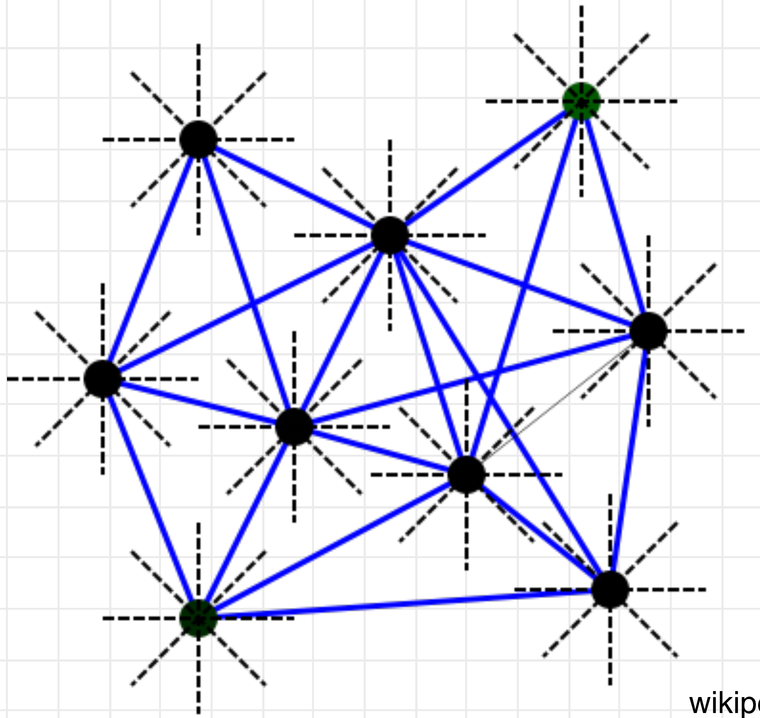
t = 1.1



https://en.wikipedia.org/wiki/Greedy_geometric_spanner

The Yao graph — a bounded degree spanner

To construct Y_k: Make k equal-size cones around each point, and connect the point to the nearest neighbour in each cone.



wikipedia

k = 8

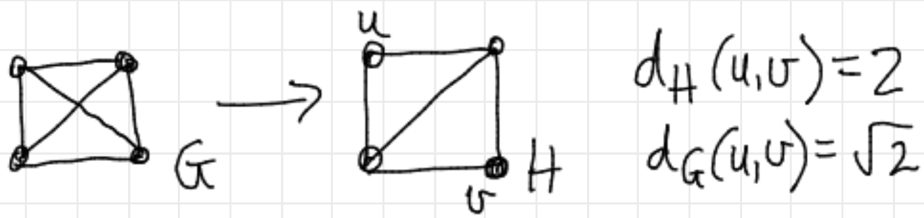
Theorem. Y_k is a t-spanner for $t = 1 + O(1/k)$

This is easy to prove.

Planar Spanners

For points in the plane, what is the min spanning ratio achievable by a planar spanner (in the worst case)?

We cannot do better than $\sqrt{2}$ spanner:

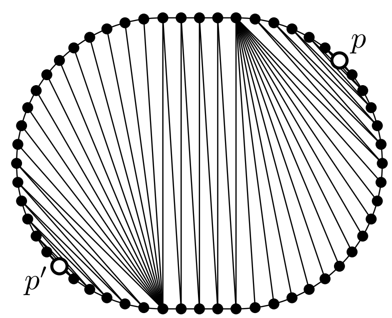


This lower bound was improved to 1.4308 in 2016.

Best known upper bound is 1.998 — achieved by the Delaunay triangulation.

Theorem. The Delaunay graph is a t-spanner for $1.5932 \le t \le 1.998$

[Xia 2011]



a lower bound

Theorem. The Delaunay graph is a t-spanner for $1.5932 \leq t \leq 1.998$

The idea of proving that the Delaunay graph is a t-spanner (for a larger t).

Consider sequence of Voronoi regions cut by xy

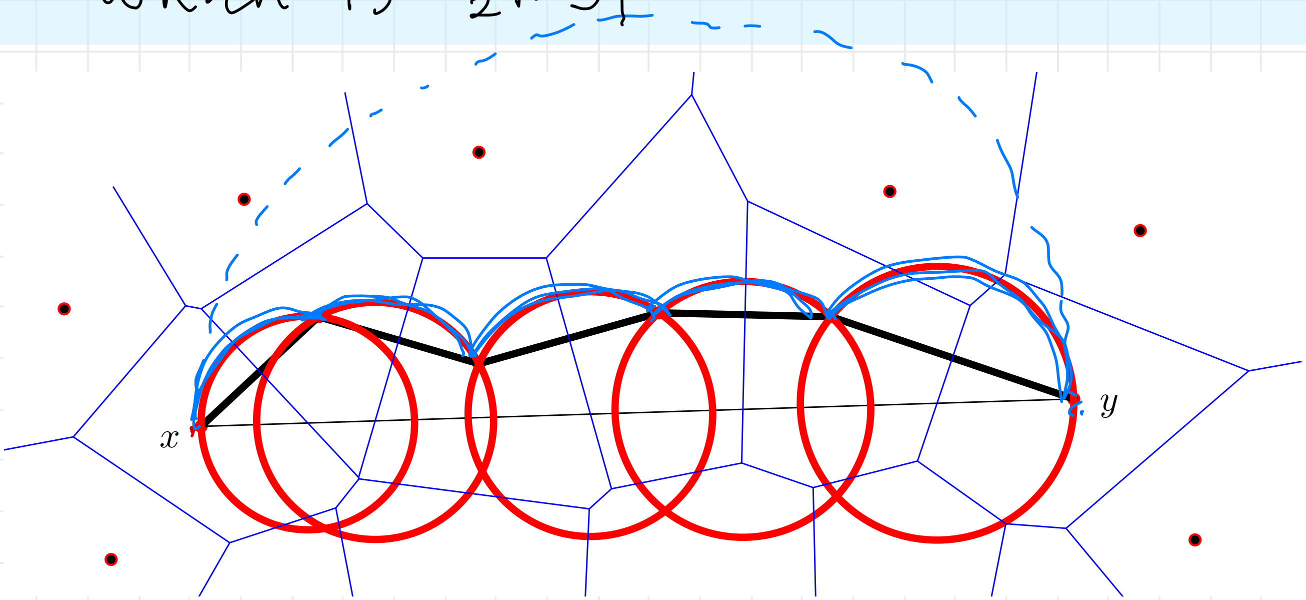
This gives path in Delaunay triangulation (in black)

Replace each black edge by arc on circle (see blue arcs)

length (of black path) grows.

and is $\leq$ length of half circle with diameter xy

which is $\frac{\pi}{2}|xy|$



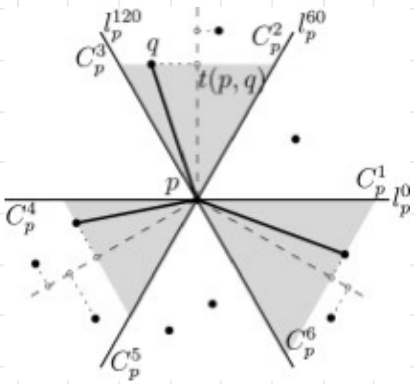
Bose and Smid

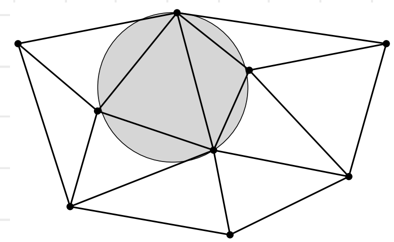
Another interesting planar spanner

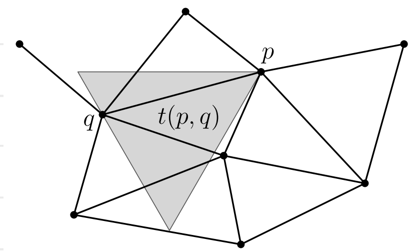
Half-θ_6 graph = TD-Delaunay graph TD= “triangular distance”

two equivalent definitions

defining like Yao graph = theorem defining like Delaunay triangulation

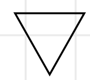






Biniaz, Maheshwari, Smid

From each point p, put three edges to the “nearest” points in three wedges.

Instead of an edge for an empty circle, put an edge for an empty equilateral triangle  (no rotations).

Theorem. The Half-θ_6 graph has spanning ratio 2.

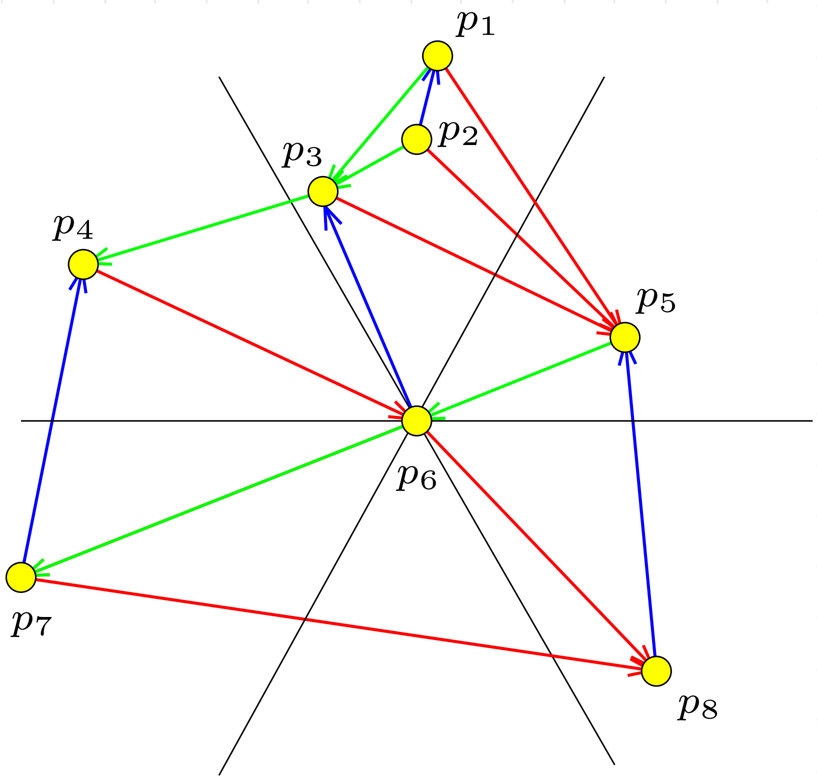
Bonichon, Nicolas, Cyril Gavoille, Nicolas Hanusse, and David Ilcinkas. "Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces." In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pp. 266-278. Springer, Berlin, Heidelberg, 2010. https://doi.org/10.1007/978-3-642-16926-7_25

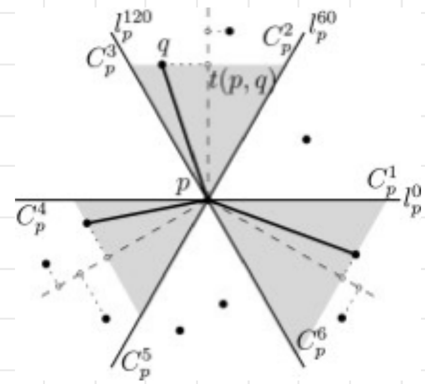
Another interesting planar spanner

Half-θ_6 graph = TD-Delaunay graph TD= "triangular distance"

two equivalent definitions

Example





the result is a Schnyder drawing http://page.math.tu-berlin.de/~felsner/Slides/dagstuhl.pdf

Bonichon, Nicolas, Cyril Gavoille, Nicolas Hanusse, and David Ilcinkas. "Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces." In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pp. 266-278. Springer, Berlin, Heidelberg, 2010. https://doi.org/10.1007/978-3-642-16926-7_25

Planar Spanners of bounded degree

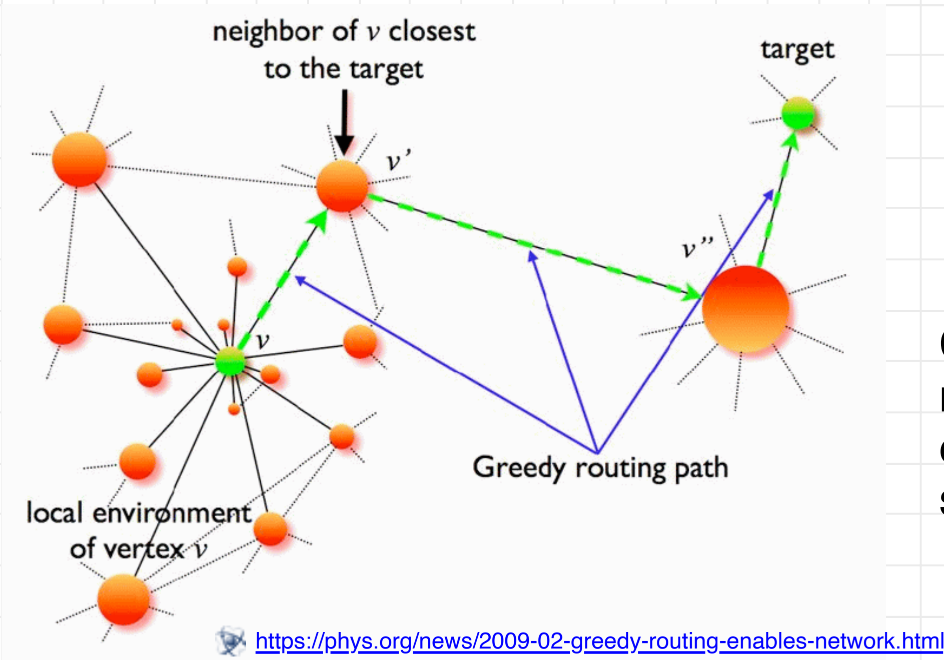
The Delaunay triangulation may have unbounded degree

There are bounded degree planar spanners:

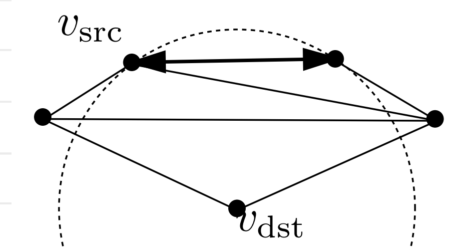
- 20-spanner of degree 4
- 6-spanner of degree 6

Routing. Given a network whose nodes are points in the plane, find a path from source to target just using local information (= coordinates of nodes).

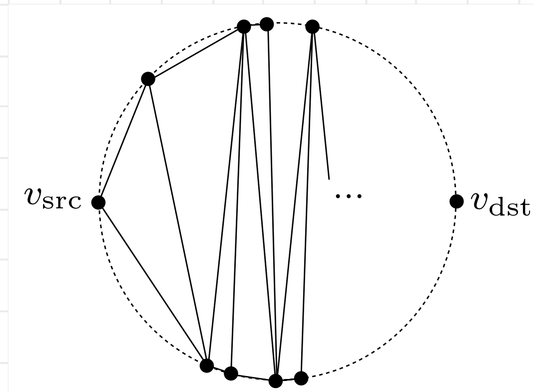
Greedy routing.
From the current node, go to the neighbour that is closest to the target.



Failure example



Condition for success: for every pair of nodes, u, v, there is a ***greedy path*** s.t. each successive node on the path is strictly closer to v.

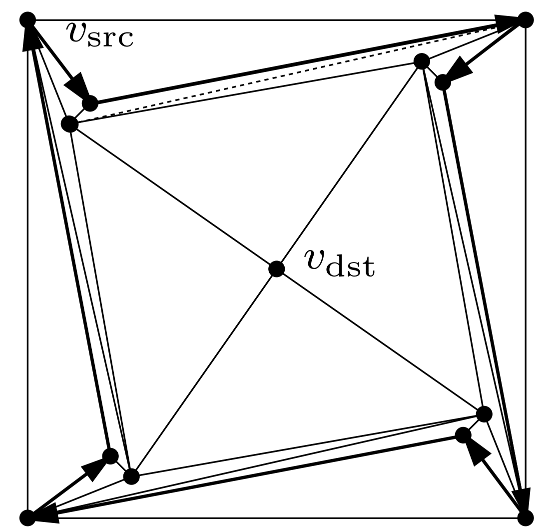


Greedy routing always succeeds for the Delaunay triangulation (the Voronoi path is greedy).
But the path that is found may be long.

Routing. Given a network whose nodes are points in the plane, find a path from source to target just using local information (= coordinates of nodes).

Compass routing.
From the current node, go to the neighbour in the best direction.

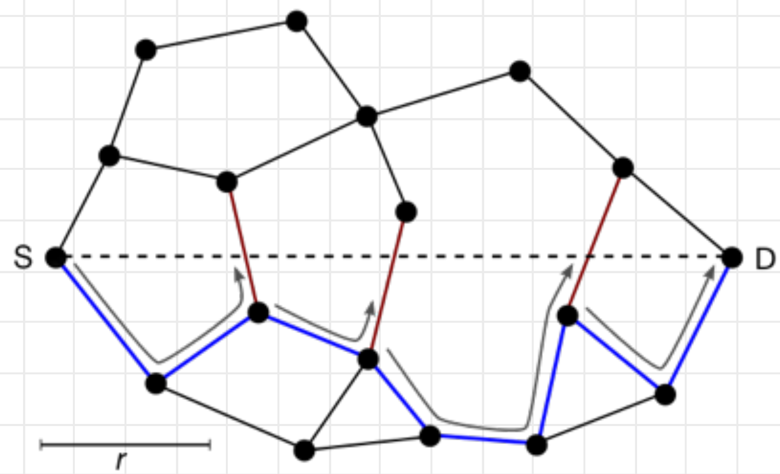
compass routing fails on this triangulation



Compass routing always succeeds for Delaunay triangulations (needs proof).

Routing. Given a network whose nodes are points in the plane, find a path from source to target just using local information (= coordinates of nodes).

Face routing.
Take the face containing the start of the segment SD. Walk around it to the intersection point with SD. Hop to the next face and repeat.
Works for any planar graph.



Bose, Prosenjit, Pat Morin, Ivan Stojmenović, and Jorge Urrutia. "Routing with guaranteed delivery in ad hoc wireless networks." *Wireless networks* 7, no. 6 (2001): 609-616. https://doi.org/10.1023/A:1012319418150

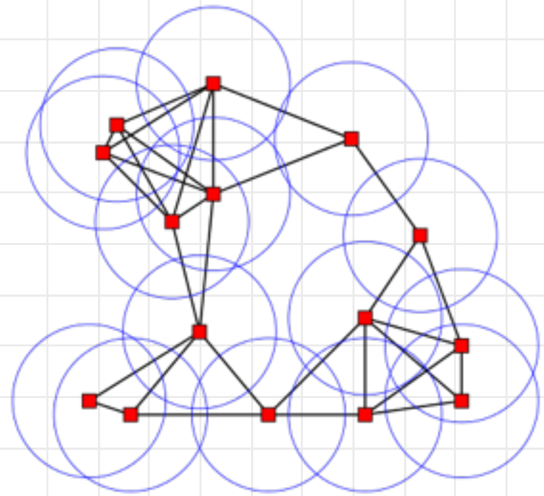
Routing in "ad hoc wireless networks"

A wifi node is a point in the plane.
All nodes have the same range, a disc.
Two nodes can communicate iff their disks intersect.

This is called a ***unit disc graph***, UDG(P) of point set P.

Routing in a unit disc graph:

We can find a planar subgraph and use face routing.

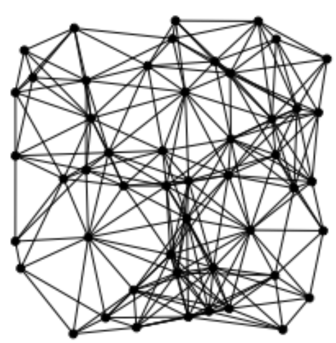


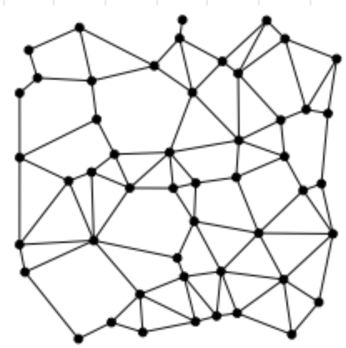
W https://en.wikipedia.org/wiki/Unit_disk_graph

Recall the ***Gabriel Graph***, GG(P) — has an edge (u,v) if the circle with diameter uv is empty of other points. GG(P) is planar (it's a subgraph of the Delaunay triangulation).

Claim. If UDG(P) is connected then UDG(P) ∩ GG(P) is connected — because it contains MST(P).

UDG(P)





UDG(P) ∩ GG(P)

planar and connected

Frey, Ingelrest, Simplot-Ryl

Routing in a network

Routing is usually done via a routing table:

R (current node, destination) = next node to go to

The size of routing tables is a barrier to efficiency.

Grand idea: Associate each node with a point in the plane ("virtual coordinates") such that greedy routing works. Then a routing table is not needed! You only need to store the virtual coordinates.

Conjecture. [Papadimitriou and Ratajczak, 2005] Every 3-connected planar graph has a greedy embedding. d https://doi.org/10.1016/j.tcs.2005.06.022

Note: not every 3-connected planar graph has an embedding as a Delaunay triangulation! (If this was true, it would prove the conjecture.)

Routing in a network

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Conjecture. [Papadimitriou and Ratajczak, 2005] Every 3-connected planar graph has a greedy embedding. d https://doi.org/10.1016/j.tcs.2005.06.022

The conjecture was proved in 2010 (by several groups independently) but unfortunately, the number of bits required for the virtual coordinates is too large to make this practical.

Furthermore, the number of bits MUST be large in the worst case:

Angelini, Patrizio, Giuseppe Di Battista, and Fabrizio Frati. "Succinct greedy drawings do not always exist." *Networks* 59, no. 3 (2012): 267-274. d https://doi.org/10.1002/net.21449

There is a solution if you change the metric a bit:

He, Xin, and Huaming Zhang. "On succinct greedy drawings of plane triangulations and 3-connected plane graphs." *Algorithmica* 68, no. 2 (2014): 531-544. d https://doi.org/10.1007/s00453-012-9682-y

Summary so far:

- spanners — sparse graphs approximately preserving distances

- local routing in a given network
  but here we didn't care about the length of the path that was found

Combining spanners and local routing: find a spanner that permits local routing, and s.t. the local route is within a constant of the Euclidean distance

Use the half θ_6-graph

There is a local routing scheme that finds a uv-path with length $\leq t\,|uv|$ for t = 2.886...
And this is a lower bound for any local routing scheme in the half θ_6-graph.
Which is interesting since the graph is a 2-spanner.

Bose, Prosenjit, Rolf Fagerberg, André Van Renssen, and Sander Verdonschot. "Competitive routing in the half-θ 6-graph." In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pp. 1319-1328. Society for Industrial and Applied Mathematics, 2012.

d https://doi.org/10.1137/1.9781611973099.104

A more practical perspective:

Gao, Jie, Leonidas J. Guibas, John Hershberger, Li Zhang, and An Zhu. "Geometric spanners for routing in mobile networks." *IEEE journal on selected areas in communications* 23, no. 1 (2005): 174-185.

d 10.1109/JSAC.2004.837364

Summary

- idea of spanners, some spanner constructions
- idea of local routing, some approaches

References - see papers listed in slides