Recall

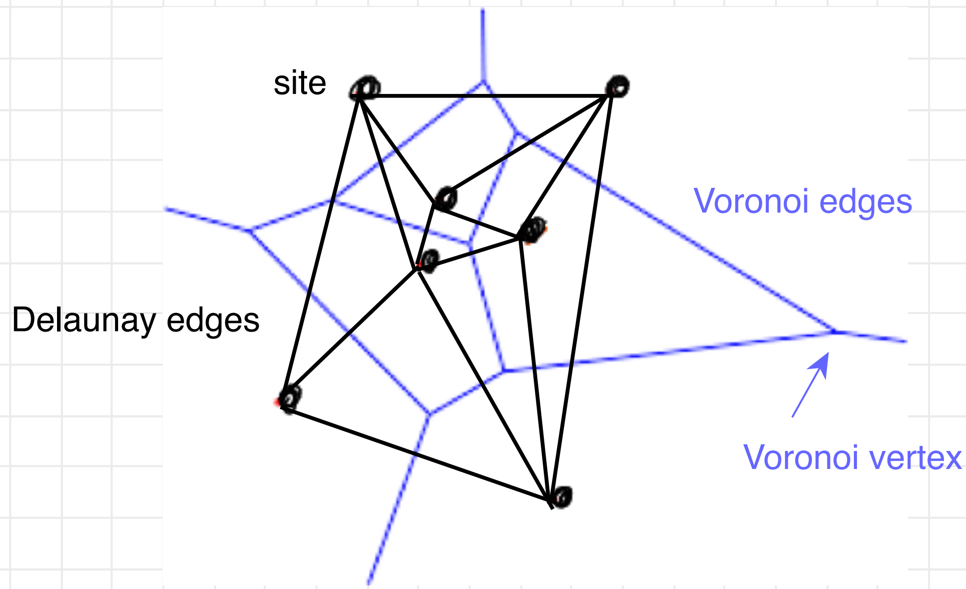
Voronoi diagram

Given points $P = \{p_1, \ldots, p_n\}$ in the plane, the ***Voronoi region*** of p_i is

$$V(p_i) = \{x \in \mathbb{R}^2 : d(x, p_i) \leq d(x, p_j) \forall j \neq i\}$$

p_i is called a ***site***.

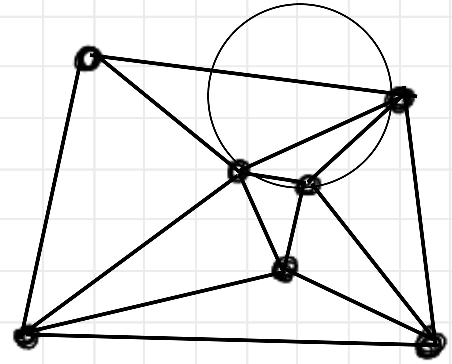
The ***Voronoi diagram*** $\mathcal{V}(P)$ consists of all the Voronoi regions

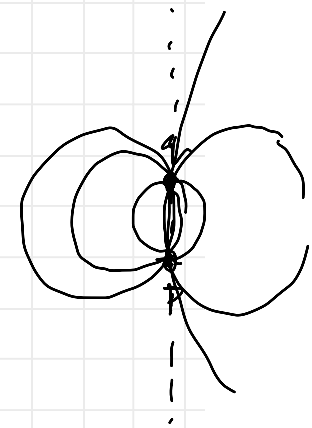


Given points $P = \{p_1, \ldots, p_n\}$ in the plane, the ***Delaunay triangulation*** $\mathcal{D}(P)$ is a graph with vertices $p_1, \ldots, p_n$ and edge (p_i, p_j) iff $V(p_i)$ and $V(p_j)$ share an edge.

$\mathcal{D}(P)$ is the ***planar dual*** of $\mathcal{V}(P)$

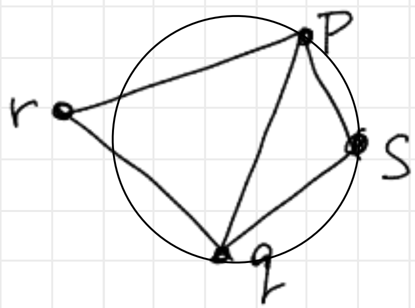
Recall Delaunay triangulation
and empty circle property: (p,q) is an edge of the Delaunay triangulation iff there is an empty circle through p and q.





An algorithmically more useful characterization:

Lemma. A triangulation is Delaunay iff every edge e=(p,q) is legal.



Definition. edge e=(p,q) is ***legal*** if either:
- e is on the convex hull or
- e is interior with triangles pqr and pqs, and r is not in Circle(pqs)

Note: r in Circle(p,q,s) iff s in Circle(p,q,r)

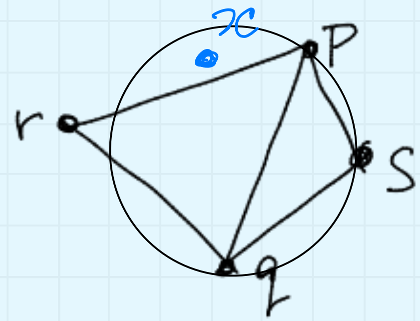
Note that this is a condition about ALL edges, not a single edge:

edge e is Delaunay (∃ an empty circle through its endpoints) ⇒ e is legal
$\not\Leftarrow$

Lemma. A triangulation is Delaunay iff every edge e=(p,q) is legal.

Proof.

$\Rightarrow$ prove e illegal $\Rightarrow$ e not Delaunay
(no empty circle)



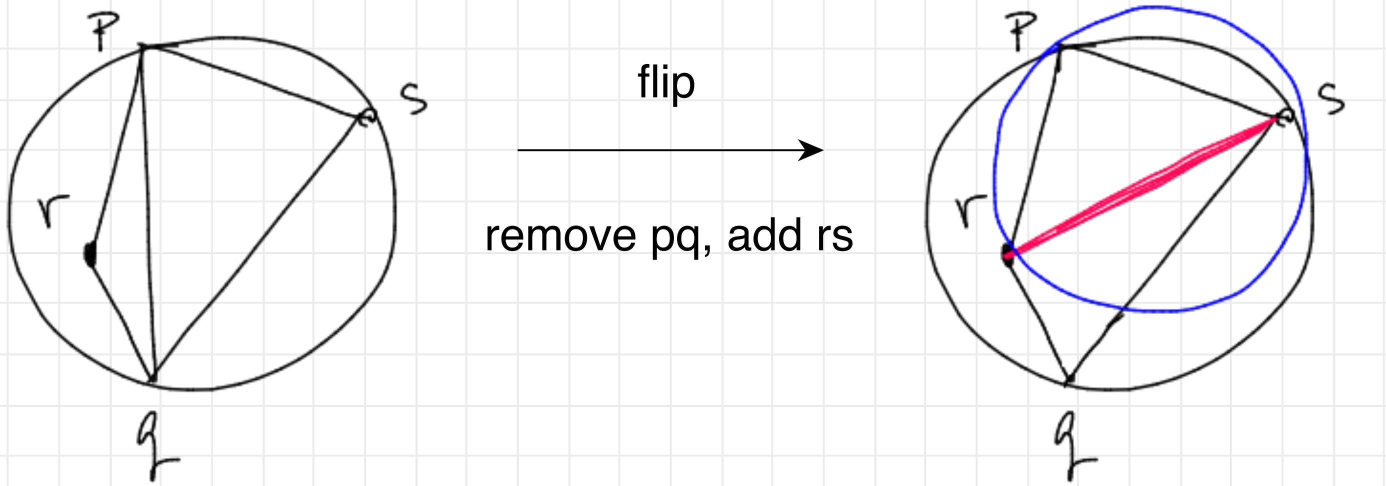
(p,q) is legal

r inside Circle(p,q,s)
$\Rightarrow$ every circle through pq contains r or s.
So pq not a Delaunay edge.

$\Leftarrow$ Suppose triangulation T not Delaunay. Find an illegal edge.
$\exists$ Δpqs and site x inside Circle(p,q,s)
Pick Δpqs and site x to minimize distance x to Δpqs
Consider the triangle pqr on the other side of pq
(exists since pq not on convex hull)
If r inside Circle(p,q,s) then pq is illegal.
Consider Δpqr and site x. x is inside circle and distance x to Δpqr is smaller.
Contra. to how we chose the Δ.
$\therefore$ pq is illegal.

What to do with an illegal edge (p,q)

Edge Flip



Claim: (r,s) is a legal edge.

change Circle (p s q) to Circle (p s r)
It shrinks away from q
So q outside Circle (p s r)
So r s is legal.

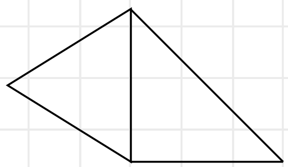
EX. If pq was the only illegal edge, do we get Delaunay triangulation?

Flipping illegal edges makes global improvements in a triangulation:
the **Angle Vector**.

For any triangulation T of a set of points, the ***angle vector*** A(T) is the list of angles of the triangles sorted min to max.

example

T



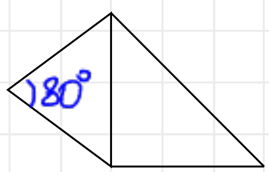
A(T) = (45, 45, 60, 60, 60, 90)

The angle vector always has length 3t where t is the number of triangles.

We compare two angle vectors ***lexicographically*** (dictionary ordering)

example

T'



A(T') = (45, 45, 50, 50, 80, 90)

A(T) = (45, 45, 60, 60, 60, 90)

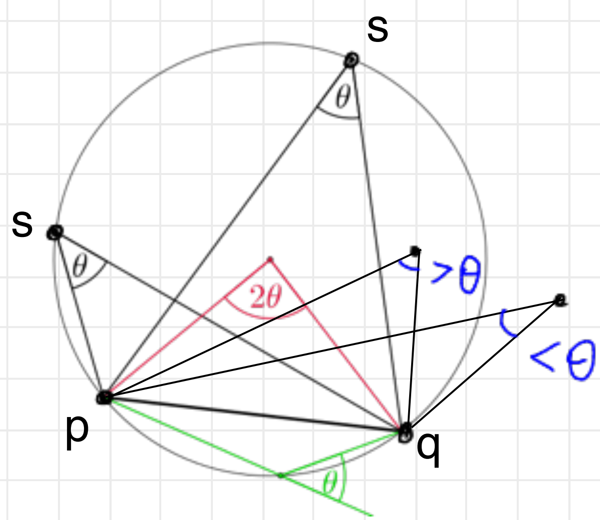
so A(T) > A(T')

Lemma. Flipping an illegal edge increases the angle vector lexicographically.

thus, flipping illegal edges does not cycle and we eventually get the Delaunay triangulation

We need:

Thales Theorem. For pq a chord of a circle, angle psq is constant for s on an arc of the circle. For s inside, the angle is bigger. For s outside the angle is smaller.

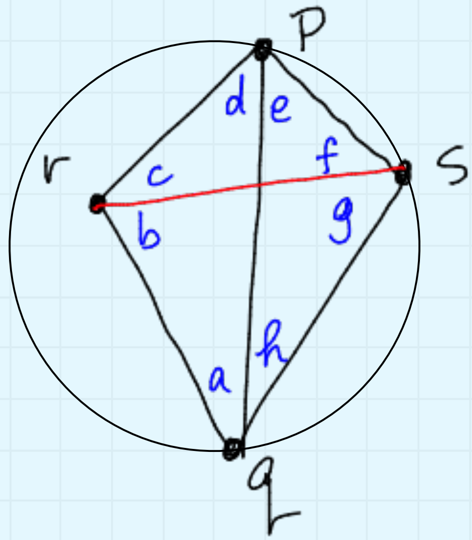


W https://en.wikipedia.org/wiki/Inscribed_angle#Theorem

Actually, Thales considered pq to be a diameter. The generalization is in Euclid.

Lemma. Flipping an illegal edge increases the angle vector lexicographically.

Proof.



only angles shown here change

old angles: $a, d, c+b, e, h, f+g$

new angles: $c, f, d+e, b, g, a+h$
(after flip)

some comparisons:

$c > h$ – Thales on chord ps

$b > e$ – " " chord qs

$f > a$, $g > d$ — chord qr

($f > a$: chord pr)

want: smallest new angle > $\underbrace{\text{smallest old angle}}_{a,\,d,\,e,\text{ or } h}$ (because $c+b > h+e$, $f+g > a+d$)

Every new angle is larger than some old angle

$c > h$ $\quad d+e > d \quad a+h > a$

$f > a$ $\quad b > e$

$g > d$

$\therefore$ new min > old min.

Thus, flipping illegal edges **always** gets you to the Delaunay triangulation,
and the Delaunay triangulation has the lexicographically maximum angle vector.

Consequences:

Theorem. The Delaunay triangulation maximizes the minimum angle.

Algorithm to find the Delaunay triangulation: find ANY triangulation and then flip illegal edges until there are none left.

Not the best algorithm. (Best is $O(n \log n)$ from last day)

How many flips does this take?

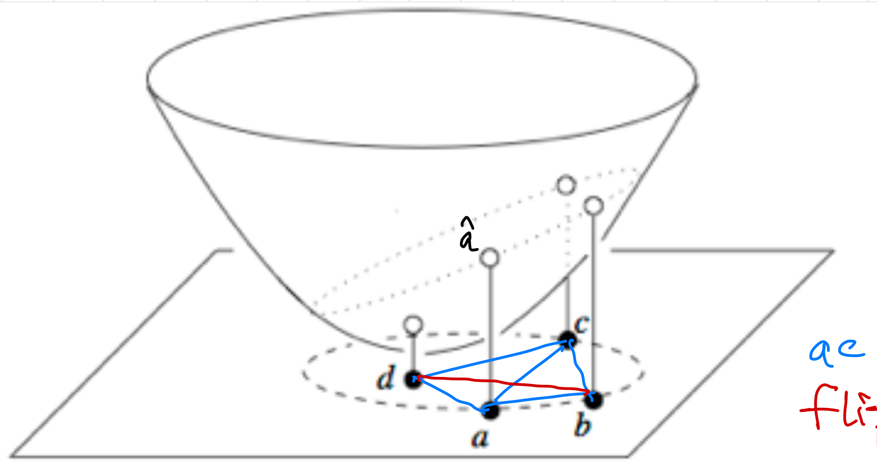
We will prove that no edge reappears.

There are $O(n^2)$ edges

$\therefore$ at most $O(n^2)$ flips.

Lemma. No edge reappears when we flip illegal edges.

Proof. Recall connection between Delaunay triangulation and convex hull



Flip illegal edge (a,c) to (d,b).

ac is illegal
flip to db

d inside Circle(abc) $\implies$

want to show ac never reappears as we flip.

$\hat{a}$ $\hat{b}$ $\hat{c}$ $\hat{d}$ form a tetrahedron

initial triangulation $\rightarrow$ $\hat{a}\hat{c}\hat{b}$ $\hat{a}\hat{d}\hat{c}$ are on top of tetrahedron

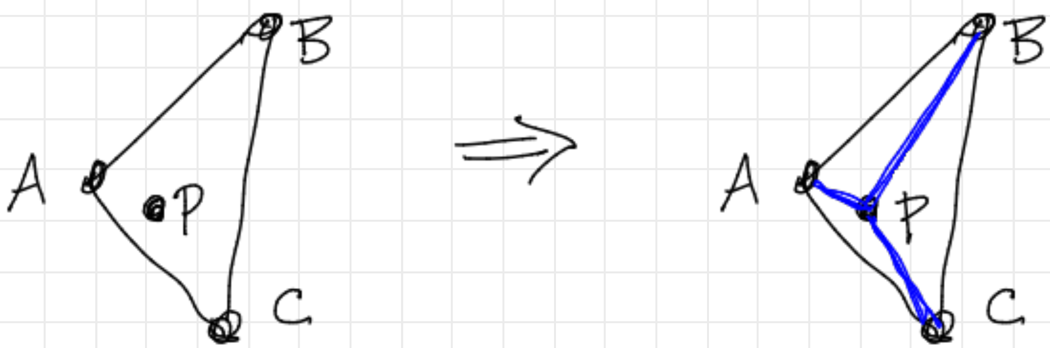
final triang. $\rightarrow$ $\hat{d}\hat{b}\hat{c}$, $\hat{d}\hat{a}\hat{b}$ are on bottom of tetrahedron

So segment $\hat{a}\hat{c}$ disappears above (in 3D) forever.

[Randomized] Incremental Delaunay triangulation algorithm.

Add points one by one, maintaining the Delaunay triangulation.
To add a new point p:

Find the current triangle ABC containing p.



Join p to A, B, C

Then flip illegal edges until there are none left.

Issues and details:

1. what if p is outside the current convex hull?

2. how to limit testing for illegal edges

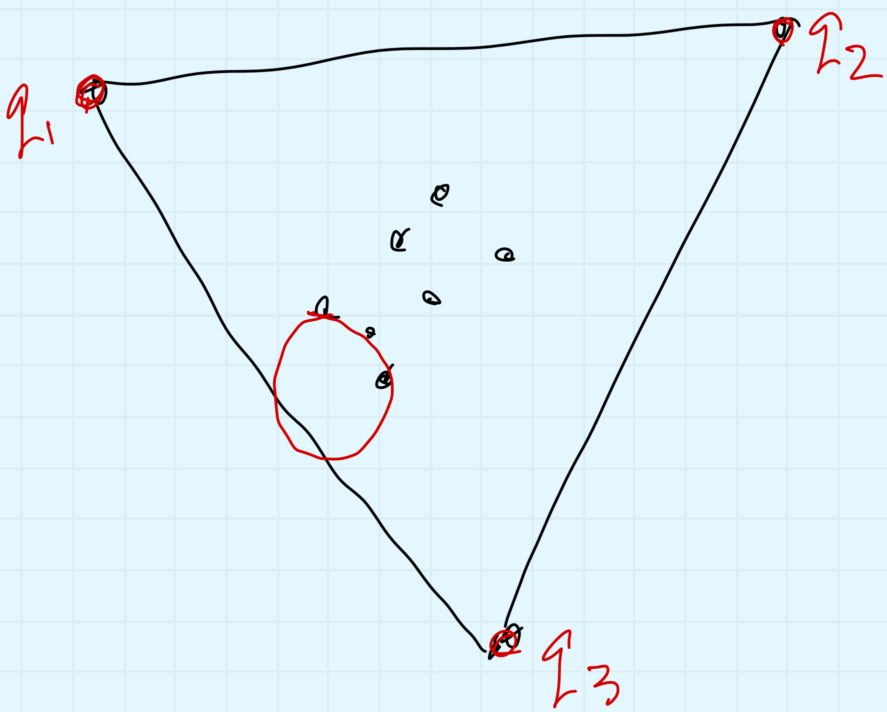
3. how to find the triangle containing p

[Randomized] Incremental Delaunay triangulation algorithm.

Issues and details.

1. what if p is outside the current convex hull?

start by adding a very large outer △

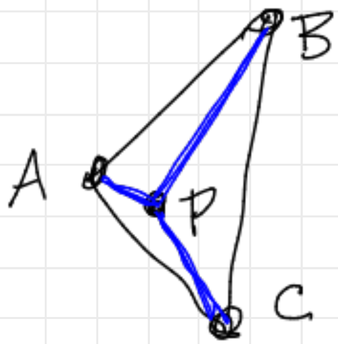


s.t. any triangle thru 3 input points has q_1, q_2, q_3 outside.

[Randomized] Incremental Delaunay triangulation algorithm.

Issues and details.

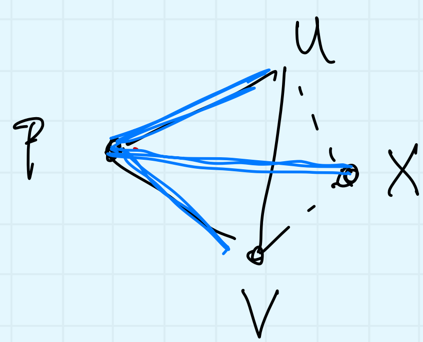
2. how to limit testing for illegal edges after adding point p



Call Test(A,B), Test(B,C), Test(C,A)

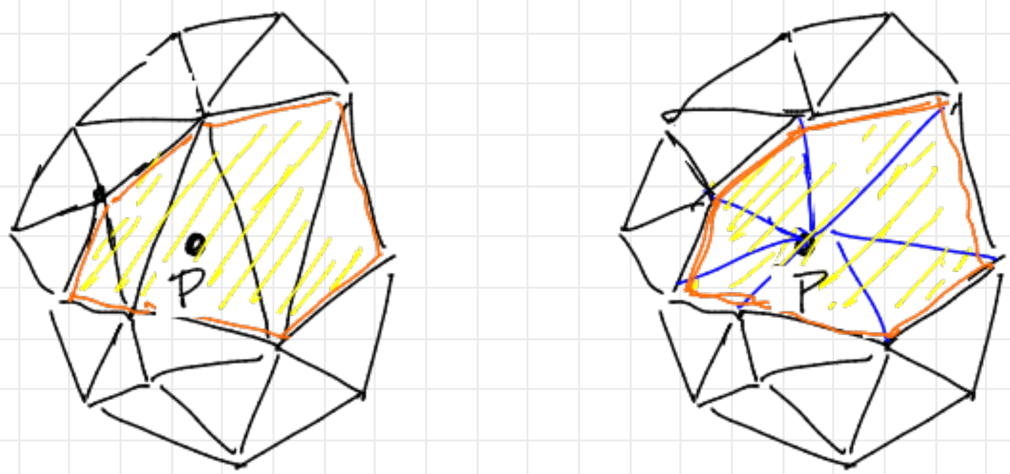
where Test(U,V) is a recursive routine
to fix edge UV in triangle UVp

Test(U,V)



if UV is illegal, flip to pX
and call Test(UX), Test(VX).

Changes produced by this Test update:

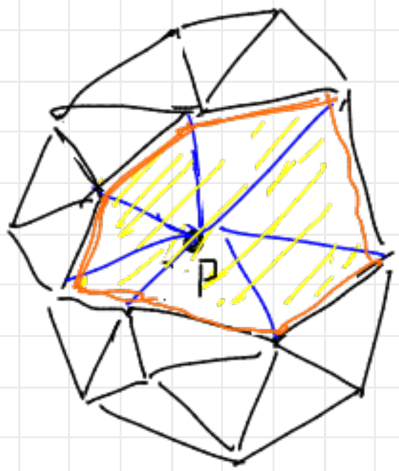


Some region is retriangulated via a star at p.
All the new edges are incident to p.

Correctness. Why is this limited Test and retriangulation sufficient?

- all the tests and flips we do are correct
- the only issue is that we do not test all the edges to check if they are illegal

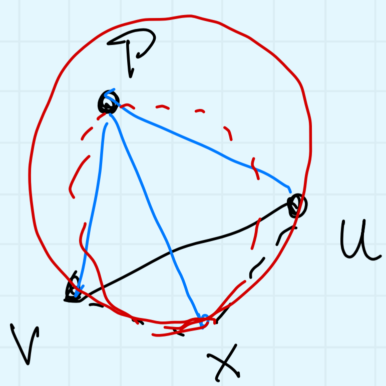
Correctness. Why is this limited Test and retriangulation sufficient?



Claim. Edges not incident to p are legal.

red edges — we tested.
outside edges — nothing changes.

Lemma. Any edge we add (incident to p) is legal. In fact, Delaunay.



we flipped to pX
because p inside Circle(uXv)
and p is only point inside
because we had Delaunay before adding p.
shrink to empty circle through X and p.

[Randomized] Incremental Delaunay triangulation algorithm.

Analysis of expected run time when points are inserted in random order.
Note: we are still ignoring how to find which triangle contains p (and its runtime).

Lemma. The expected time to insert one point is O(1).

Proof.

time spent on Test(u v)
= O(# edges incident to p) at the end.

So we want expected degree of p in Del. triangulation of $p_1\ p_2 \dots p_i = p$

avg. degree in Del. triang. (planar graph) is O(1)
so backwards analysis gives
expected time to insert p is O(1)

Total: O(n)

expected # triangles over course of alg. is O(n).

[Randomized] Incremental Delaunay triangulation algorithm.

Final issue: How to find the triangle containing p.
The method is easy, the analysis is not.
Note: it is this part of the algorithm that causes the O(n log n) expected behaviour.

The idea is like Kirkpatrick's Point Location.
Maintain the history of triangles and changes to them. Then "trace" point p_i through the changes.

Two possible triangle updates

Keep blue arrows. — expected space is O(n) because O(n) expected triangles.
Each update to locate p costs O(1)

[Randomized] Incremental Delaunay triangulation algorithm.

Final issue: How to find the triangle containing p.

How to “trace” p:

- initially (with one big triangle) p is in the big triangle
- at each update, the triangle containing p points to 2 or 3 new ones — check which one contains p

This completes the description of the algorithm.

Analysis of expected work to trace p_i

Can prove it is O(log i). Then total expected work to add all points is

$$O(\sum_i \log i) = O(n \log n)$$

First idea: charge work of tracing p_i to each triangle T in the sequence that contains p_i

Better idea: charge work to Delaunay triangles that appear in the sequence.

Can show that the expected work for triangles of $D(\{p_1, \ldots p_j\})$ is $O(3 / j)$

$$O(\sum_{j=1}^{i-1} \frac{3}{j}) = O(\log i)$$

Harmonic series

There is a lovely backwards analysis involved. For details, see [CGAA].

What primitive operations are needed for this algorithm?

Given 4 points, A, B, C, D, is D inside Circle(A,B,C)?

Use the mapping from last day

$$(x, y) \longrightarrow (x, y, z = x^2 + y^2)$$

Then the test becomes: is $\hat{D}$ below the plane through $\hat{A}$, $\hat{B}$, $\hat{C}$?

This is a Sidedness test in 3D, and can be decided with a few multiplications, additions, subtractions.

Summary

- a randomized incremental algorithm for the Delaunay triangulation
- the idea of flipping illegal edges to get to the Delaunay triangulation
- the Delaunay triangulation maximizes the angle vector

References

- [CGAA] Chapter 9.
- [Zurich notes] Chapter 5.