An Efficient Algorithm for Computing the ith Letter of $\varphi^n(a)$

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Abstract

Let Σ be a finite alphabet, and let $\varphi : \Sigma^* \to \Sigma^*$ be a homomorphism, i.e., a mapping satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Sigma^*$. Let $a \in \Sigma$, and let $i \geq 1$, $n \geq 0$ be integers. We give the first efficient algorithm for computing the ith letter of $\varphi^n(a)$. Our algorithm runs in time polynomial in the size of the input, i.e., polynomial in $\log n$, $\log i$, and the description size of φ. Our algorithm can be easily modified to give the distribution of letters in the prefix of length i of $\varphi^n(a)$. There are applications of our algorithm to computer graphics and biological modelling.

1 Introduction

Let Σ be a finite alphabet. A *homomorphism* is a map φ from Σ^* to Σ^* such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Sigma^*$. Let $a \in \Sigma$; we define $\varphi^0(a) = a$, and $\varphi^i(a) = \varphi(\varphi^{i-1}(a))$ for $i \geq 1$.

For $x \in \Sigma^*$, and $a \in \Sigma$, let $|x|$ denote the length of x, and let $|x|_a$ denote the number of occurrences of the letter a in x. We define the the *depth* d of a homomorphism φ to be the cardinality of its domain Σ, and the *width* w of a homomorphism φ to be the maximum value of $|\varphi(a)|$ over all $a \in \Sigma$.

Consider the following problem:

Given a homomorphism $\varphi : \Sigma^* \to \Sigma^*$, integers $n \geq 0$ and $i \geq 1$, and a letter $a \in \Sigma$, efficiently calculate the ith letter of $\varphi^n(a)$.

In this paper, we present the first algorithm which solves this problem in time bounded by a polynomial in the *size* of the input data. More precisely, the running time of our algorithm is polynomial in $\log n$, $\log i$, w, and d. Our model of computation is the familiar "naive bit complexity" model; see, for example, [1]. In this model, adding together two n-bit integers uses $O(n)$ bit operations, while multiplying two n-bit integers uses $O(n^2)$ bit operations.

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Figure 1: Geometric interpretations of strings produced by D0L-systems [7].

Let p be the prefix of $\varphi^n(a)$ of length i. Our algorithm is easily modified to return the distribution of the first i letters in $\varphi^n(a)$ as the Parikh vector $\mathbf{p} = (m_b)_{b\in\Sigma}$, where $m_b = |p|_b$.

Before proceding further, we discuss some properties of the letters of Σ with respect to a homomorphism φ. A letter b is *accessible* from a if $\varphi^n(a) = w_1 b w_2$ for some strings $w_1, w_2 \in \Sigma^*$ and for some $n \geq 0$. A letter a is *mortal* if $\varphi^n(a) = \epsilon$ for some $n > 0$. We say a is *immortal* if it is not mortal.

Iterated homomorphisms have been previously studied in the form of deterministic context free Lindenmayer systems, or *D0L-systems* [13]. A D0L-system is a 3-tuple $G = \{\Sigma, \varphi, z\}$ where Σ is a finite alphabet, φ is a set of production rules, and $z \in \Sigma^*$ is the initial word or *axiom*. The *language* of a D0L-system G is defined as $L_G = \{\varphi^n(z) \ : \ n \geq 0\}$.

A word of a D0L-language can be interpreted as instructions in a "turtle language" to draw an image [12]. Many have used D0L-systems to generate fractals and to model biological systems such as the branching structure of a tree [2, 7, 8]; see Figure 1. In this context, our new algorithm allows us to *efficiently calculate the structure of small twigs, without having to calculate the structure of the entire tree.*

Another application of our algorithm is the efficient computation of the ith letter of a fixed point of a homomorphism, which was stated as an open problem in [10]. Let a be a letter such that $\varphi(a) = ax$, where x is a string containing at least one immortal letter. Then φ has a unique fixed point starting with a, of the form

$$\varphi^\omega(a) = a\, x\, \varphi(x)\, \varphi^2(x)\, \varphi^3(x) \cdots .$$

We can efficiently compute the ith letter of such a fixed point by using binary search to determine which factor the ith letter lies in, and then using our algorithm to find the appro-

priate letter within a factor. Such a binary search uses an efficient subroutine for computing $|x\varphi(x)\varphi^2(x)\cdots\varphi^n(x)|$, which is given in the algorithm SUPERPOWER below in Section 5.3.

This paper is based on results in the second author's M.Math. thesis [11].

2 Previous Work

Several authors have discussed the complexity of decision problems involving D0L-systems. For example, Jones and Skyum [4] gave a polynomial time algorithm for solving the D0L membership problem: given G and x, is $x \in G$? Their result is somewhat orthogonal to ours, for in their situation the input is x, z, and φ, and they want to decide if there exists n with $x = \varphi^n(z)$. Note that $|x|$ could be as large as $w^n|z|$, where w is the width of φ. This quantity is *doubly exponential* in $\log n$. Later, Jones and Skyum [5] gave a DSPACE($\log^2 n$) algorithm to solve the D0L membership problem.

The *growth function* f_G of a D0L-system G is defined by $f_G(n) = |\varphi^n(z)|$. Salomaa [9] showed that the growth function of any D0L-system is either polynomial or exponential. With respect to a homomorphism φ, we say a string $x \in \Sigma^*$ *grows polynomially* if $|\varphi^n(x)| = O(n^c)$ for some constant c. Otherwise, we say x *grows exponentially.* There is an efficient algorithm to determine if a letter grows exponentially [13].

3 The BASICTREEDESCENT Subroutine

We first discuss a simple method to calculate the ith letter of $\varphi^n(a)$, called BASICTREEDESCENT. This algorithm is similar to one found by Jones and Skyum [5]. While its running time is *not* necessarily polynomial in $\log n$, $\log i$, w, and d, we use BASICTREEDESCENT to calculate the ith letter of $\varphi^n(a)$ for the cases in which n *is* polynomial in $\log i$, w, and d.

For an integer n and a letter $a \in \Sigma$, we view the sequence of words a, $\varphi(a)$, $\varphi^2(a)$, $\cdots$, $\varphi^n(a)$ as labels of the levels of an ordered tree. More precisely, the *derivation tree* of $\varphi^n(a)$ is the ordered tree of height n, with a as its root, such that every non-leaf node b has children labeled with the letters of $\varphi(b)$. Instead of computing $\varphi^n(a)$ by repeatedly applying φ to each successive string, we calculate the ith letter of $\varphi^n(a)$ by descending n levels down the derivation tree to the appropriate letter. The only difficulty lies in determining which subtree to descend.

Let LENGTH(φ, n, a, i) be a procedure which returns the value $|\varphi^n(a)|$ and let $\varphi(a) = z = z_1z_2\cdots z_{|\varphi(a)|}$. BASICTREEDESCENT uses LENGTH to compute $|\varphi^{n-1}(z_1)|, |\varphi^{n-1}(z_2)|, \cdots$ and by adding, computes $|\varphi^{n-1}(z_1)|, |\varphi^{n-1}(z_1z_2)|, \cdots$ until a value t is determined such that

$$|\varphi^{n-1}(z_1z_2\cdots z_{t-1})| < i \leq |\varphi^{n-1}(z_1z_2\cdots z_t)|.$$

Once t has been calculated, the algorithm adjusts the values of n, a, and i to $n-1$, z_t, and $i - |\varphi^{n-1}(z_1z_2\cdots z_{t-1})|$ respectively and thus descends one level in the derivation tree. Of course, if $n = 0$, the algorithm simply returns a.

During the execution of the Basic Tree Descent algorithm, we descend exactly n levels. Each time we do so, we calculate the length of at most w strings. Since the running time is at least linear in n, BasicTreeDescent is only useful to us when n itself is bounded by a polynomial in $\log i, w$, and d.

4 Our Improved Algorithm

The idea behind our new algorithm is to avoid descending n levels in the derivation tree for the cases where n is not bounded by a polynomial in $\log i$, w, and d, by exploiting the recursive nature of the homomorphism. Basically, we descend the derivation tree until we encounter a letter for the second time and then shortcut to the node where this repeated letter occurs last. We continue descending, taking shortcuts whenever possible until we reach the ith letter of the bottom level.

4.1 Finding a Repeated Letter

We discuss some notation concerning the path taken down the derivation tree. Let a_0 be the root of the derivation tree. Let a_j be the letter encountered after descending j levels and set x_j, y_j such that $\varphi(a_{j-1}) = x_j a_j y_j$. Therefore, after descending l levels, the sequence of letters encountered during the tree descent is $a_0, a_1, \cdots, a_l$ and the sequences of strings $x_1, x_2, \cdots, x_l$ and $y_1, y_2, \cdots, y_l$ describe the branches of the tree which were not followed.

We begin our algorithm by descending the derivation tree until we encounter some letter a_l for the second time, where l is the number of levels we have descended thus far. Let $l - q$ be the level containing the previous occurrence of a_l. We need only descend d levels before encountering such a letter a_l since $|\Sigma| = d$. More precisely, $a_l = a_{l-q}$ for some $l \leq d$.

By descending l levels, we have calculated new values for n, a, and i, which yield the same results, namely $i - |\varphi^{n-1}(x_1)\varphi^{n-2}(x_2)\cdots\varphi^{n-l}(x_l)|$, $n - l$, and a_l. We have also discovered strings x and y and an integer q such that $\varphi^q(a) = xay$. Specifically, x and y are $\varphi^{q-1}(x_{l-q+1})\cdots\varphi(x_{l-1})x_l$, and $y_l\varphi(y_{l-1})\cdots\varphi^{q-1}(y_{l-q+1})$ respectively.

4.2 Jumping Ahead

Once a repeated letter a is found, the remainder of the algorithm calculates the last occurrence of the letter a in the tree descent and then either invokes the BasicTreeDescent algorithm or restarts our algorithm with new values of n and i.

Setting s and r such that $n = sq+r$ and $r < q$, we use the strings x, y, and the integers q, r, and s to describe $\varphi^n(a)$. We know $\varphi^q(a) = x\,a\,y$ and $\varphi^{2q}(a) = \varphi^q(x)\,x\,a\,y\,\varphi^q(y)$. Continuing this way, we see that

$$\varphi^{sq}(a) = \varphi^{(s-1)q}(x)\cdots\varphi^q(x)\,x\,a\,y\,\varphi^q(y)\cdots\varphi^{(s-1)q}(y).$$

Finally, applying φ^r to both sides yields

$$\varphi^n(a) = \underbrace{\varphi^{(s-1)q+r}(x)\cdots\varphi^{q+r}(x)\,\varphi^r(x)}_{X_1}\,\underbrace{\varphi^r(a)}_{X_2}\,\underbrace{\varphi^r(y)\,\varphi^{q+r}(y)\cdots\varphi^{(s-1)q+r}(y)}_{X_3}.$$

Let GENERICLENGTH be a procedure which returns $\sum_{t\le j<s}|\varphi^{qj+r}(x)|$. Rather than using the word x itself as input, we use the Parikh vector $\mathbf{x}$ to indicate the distribution of letters in x, that is, $\mathbf{x} = (m_j)_{1\le j\le d}$, where $m_j = |x|_{a_j}$.

Using GENERICLENGTH$(\varphi, \mathbf{x}, q, r, s, t, i)$ and LENGTH(φ, n, a, i), we determine which of X_1, X_2, or X_3 the ith letter falls in and we handle these three cases separately.

Case 1: If x contains an exponentially growing letter, then Lemma 9 below states that $n = O(d\log i)$ and therefore, we can safely call BASICTREEDESCENT.

Otherwise, x grows polynomially. Using GENERICLENGTH, we do a binary search for a value of t such that the ith letter lies in the subword $\varphi^{tq+r}(x)$, more precisely,

$$|\varphi^{(s-1)q+r}(x)\cdots\varphi^{(t+1)q+r}(x)| < i \le |\varphi^{(s-1)q+r}(x)\cdots\varphi^{tq+r}(x)|.$$

Once t is found, we set $n \leftarrow (t+1)q + r$ and $i \leftarrow i - |\varphi^{(s-1)q+r}(x)\cdots\varphi^{(t+1)q+r}(x)|$. As we prove in Lemma 10 below, adjusting the values of i and n this way lets us descend to the point in the derivation tree where the letter a occurs last.

Case 2: Since we have already calculated the length of X_1, we need only set $i \leftarrow i - |X_1|$, set $n \leftarrow r$, and then call BASICTREEDESCENT. We call BASICTREEDESCENT, since now $n = r < q \le d$.

Case 3: Similarly to Case 1, we use GENERICLENGTH to do a binary search for a value t, such that the ith letter lies in the the subword $\varphi^{tq+r}(y)$, that is,

$$|\varphi^r(y)\varphi^{q+r}(y)\cdots\varphi^{(t-1)q+r}(y)| < i' \le |\varphi^r(y)\varphi^{q+r}(y)\cdots\varphi^{tq+r}(y)|,$$

where $i' = i - |X_1X_2|$. Once the value of t is found, we rewrite $\varphi^n(a)$ as

$$\varphi^{(s-1)q+r}(x)\cdots\varphi^{(t+1)q+r}(x)\varphi^{(t+1)q+r}(a)\varphi^{(t+1)q+r}(y)\cdots\varphi^{(s-1)q+r}(y).$$

Since we wish to reduce i by the number of letters to the *left* of the substring $\varphi^{(t+1)q+r}(a)$, we set $i \leftarrow i - |\varphi^{(s-1)q+r}(x)\cdots\varphi^{(t+1)q+r}(x)|$. and $n \leftarrow (t+1)q + r$.

If y is polynomially growing, then Lemma 10 below shows that the letter a is not encountered again. Hence, we continue descending the tree until the next shortcut is taken or until the bottom is reached.

Otherwise, y grows exponentially. By Lemma 9 below, $n = O(d\log i)$ and therefore we call BASICTREEDESCENT.

4.3 Outline of Our Algorithm

The following pseudocode finds the ith letter of $\varphi^n(a)$. We assume that $1 \le i \le |\varphi^n(a)|$.

```
procedure FIND(φ, n, a, i);
a_0 ← a;

for (l ← 1 to ∞)   //executes for ≤ d iterations
    z ← φ(a) = z_1 z_2 ··· z_{|φ(a)|};
    find smallest index t such that i > |φ^{n-1}(z_1 z_2 ··· z_{t-1})|;
    i ← i − |φ^{n-1}(z_1 z_2 ··· z_{t-1})|;   n ← n − 1;   a ← z_t;
    x_l ← z_1 ··· z_{t-1};   a_l ← z_t;   y_l ← z_{t+1} ··· z_{|φ(a)|};

    if (n ≤ d) then return BASICTREEDESCENT(φ, n, a, i);
    else if (a_j = a_l for some j < l) then
        q ← l − j;   s ← ⌊n/q⌋;   r ← n − sq;
        x ← Parikh vector of φ^{q-1}(x_{l-q+1}) ··· φ(x_{l-1}) x_l;
        y ← Parikh vector of y_l φ(y_{l-1}) ··· φ^{q-1}(y_{l-q+1});
        S_1 ← GENERICLENGTH(φ, x, q, r, s, 0, i);   //Length of X_1.
        S_2 ← LENGTH(φ, r, a, i − S_1);   //Length of X_2.

        if (i ≤ S_1) then   //Case 1
            if (x grows exponentially) then return BASICTREEDESCENT(φ, n, a, i);
            use GENERICLENGTH to do a binary search for t such that
                |φ^{(s-1)q+r}(x) ··· φ^{(t+1)q+r}(x)| < i ≤ |φ^{(s-1)q+r}(x) ··· φ^{tq+r}(x)|
            i ← i− GENERICLENGTH(φ, x, q, r, s, t + 1, i);
            n ← (t + 1)q + r;
            return FIND(φ, n, a, i);
        else if (i ≤ S_1 + S_2) then   //Case 2
            return BASICTREEDESCENT(φ, r, a, i − S_1);
        else   // Case 3
            use GENERICLENGTH to do a binary search for t such that
                |φ^r(y)φ^{q+r}(y) ··· φ^{(t-1)q+r}(y)| < i − S_1 − S_2 ≤ |φ^r(y)φ^{q+r}(y) ··· φ^{tq+r}(y)|
            i ← i− GENERICLENGTH(φ, x, q, r, s, t + 1, i);
            n ← (t + 1)q + r;
            if (y grows exponentially) then return BASICTREEDESCENT(φ, n, a, i);
            else return FIND(φ, n, a, i);
```

5 LENGTH and GENERICLENGTH

Before moving on to the analysis of FIND, we discuss details concerning the subroutines it calls. In particular, we show how GENERICLENGTH computes $\sum_{t \le < s} |\varphi^{jq+r}(a)|$, how LENGTH computes $|\varphi^n(a)|$, as well as providing a run-time estimate for these routines.

5.1 LENGTH

Let $\Sigma = \{a_1, a_2, \cdots a_d\}$. We define the *incidence matrix* of φ as $M(\varphi) = (m_{ij})_{1 \le i,j \le d}$, where $m_{ij} = |\varphi(a_j)|_{a_i}$. By multiplying the incidence matrix of φ by itself k times, we get a matrix whose m_{ij} entry equals the number of a_i's in $\varphi^k(a_j)$, that is, $M(\varphi)^k = M(\varphi^k)$ for $k \ge 0$.

Let $\mathbf{a}$ be the Parikh vector of a. Hence $M(\varphi)\mathbf{a}$ is equal to the Parikh vector of $\varphi(a)$. The sum of the values in $M(\varphi)\mathbf{a}$ equals the length of $\varphi(a)$, that is,

$$|\varphi(a)| = \mathbf{e}^T M(\varphi)\mathbf{a},$$

where $\mathbf{e}$ is the column vector whose entries are all 1's.

We take some measures to avoid unnecessary calculations which may increase the running time of LENGTH.

The well-known *binary method of exponentiation* (e.g., [1]) provides a method for raising a matrix to the nth power with $O(\log n)$ matrix multiplications.

Note that, for our purposes, it suffices to report $|\varphi^n(a)| \ge i$ without having to actually calculate $|\varphi^n(a)|$. Accordingly, after every matrix $M(\varphi^k)$ is calculated, we check whether the number of immortal letters in $\varphi^k(a)$ is less than i. Let $\mathbf{v}$ be the vector indicating the immortal letters of Σ. Then we proceed by calculating whether $\mathbf{v}M(\varphi^k)\mathbf{a} < i$. If not, LENGTH stops the calculation and reports that $|\varphi^n(a)| \ge i$.

If a is a slowly growing letter and other letters grow exponentially, then matrix entries may grow too large. Lemma 2 below shows that this problem of quickly growing letters is solved if we restrict Σ to the letters accessible from a,

To summarize, when LENGTH is executed, it constructs the incidence matrix $M(\varphi)$, and then uses the binary method of exponentiation to calculate $M(\varphi^n)$ while taking the precautions noted above. If $M(\varphi^n)$ is finally calculated, then LENGTH returns $|\varphi^n(a)| = \mathbf{e}^T M(\varphi^n)\mathbf{a}$.

We are now ready to formally define LENGTH. Since the binary method of matrix exponentiation will be used for other calculations, we write the method as the subroutine POWER.

procedure POWER$(\varphi, n, \mathbf{a}, i)$
// Returns $M(\varphi^n)$ restricted to the letters accessible from a, making sure that the sum of
// the immortal letters of $M(\varphi^n)\mathbf{a}$ is less than i.

$Y \leftarrow$ the identity matrix;

```
Z ← M(φ) restricted to the letters accessible from a;
v ← column vector indicating the immortal letters of φ;

// calculate Y to be Z^n
while (n > 0) do
    if (n ≡ 1 mod 2) then
        Y ← YZ;
        if (v^T Y a ≥ i) then return NIL;
    n ← ⌊n/2⌋;
    if (n > 0) then
        Z ← Z^2;
        if (v^T Z a ≥ i) then return NIL;

return Y;
```

The following is the pseudocode for the procedure LENGTH. We assume $n \geq 0$, and $i \geq 1$.

```
procedure LENGTH(φ, n, a, i)
// returns |φ^n(a)|, if |φ^n(a)| < i; −1, otherwise.

    e ← column vector of all 1's;
    a ← Parikh vector of a;

    // calculate M(φ^n) restricted to the letters accessible from a
    Z ← POWER(φ, n, a, i);

    // return the result
    if ((Z = NIL) or (e^T Z a ≥ i)) then return −1;
    return e^T Z a;
```

5.2 Analysis of POWER and LENGTH

Before discussing the running time of LENGTH, we prove a theorem about the size of the numbers that POWER deals with. We start with two technical lemmas.

In the following lemma, let Im be the function that takes a string and deletes the mortal letters from it.

Lemma 1 *If φ, a, i, w, and d are defined as before, and $|\mathrm{Im}(\varphi^k(a))| \leq i$ for some integer $k > 0$, then $|\varphi^k(a)| \leq w^d i$.*

Proof. If $k \leq d$, then $|\varphi^k(a)| \leq w^k \leq w^d$ and so the statement is true for any i. Hence we assume $k > d$.

Applying φ^d to a mortal letter produces the empty string. Then applying φ^d to $\varphi^{k-d}(a)$ is equivalent to applying φ^d to the immortal letters of $\varphi^{k-d}(a)$. More precisely,

$$\varphi^k(a) = \varphi^d(\mathrm{Im}(\varphi^{k-d}(a)).$$

On the other hand, applying φ^d to an immortal letter produces a string of size at most w^d. Thus,

$$|\varphi^k(a)| \leq w^d|\mathrm{Im}(\varphi^{k-d}(a))|.$$

Notice $|\mathrm{Im}(\varphi^{k-d}(a))| \leq |\mathrm{Im}(\varphi^k(a))| \leq i$. Therefore,

$$|\varphi^k(a)| \leq w^d i.$$

■

Lemma 2 *If φ, a, i, w, and d are defined as before, and b is accessible from a, then $|\varphi^k(b)| \leq w^d|\varphi^k(a)|$.*

Proof. Since b is a letter accessible from a, then for some $l \leq d$, $\varphi^l(a)$ contains a b. Hence $\varphi^k(b)$ is a substring of $\varphi^k(\varphi^l(a))$ and therefore,

$$|\varphi^k(b)| \leq |\varphi^l(\varphi^k(a))|.$$

Applying φ^l to any string increases the size by at most a factor of w^l. Hence,

$$|\varphi^k(b)| \leq w^l|\varphi^k(a)|.$$

And since $l \leq d$,

$$|\varphi^k(b)| \leq w^d|\varphi^k(a)|.$$

■

We now prove our bounds on the entries of incidence matrices calculated by POWER.

Lemma 3 *If $M(\varphi^l)$ is a matrix calculated during the execution of* POWER*, then each entry of $M(\varphi^l)$ is less than $dw^{4d}i^2$.*

Proof. If $l = 1$ then each entry of $M(\varphi^l)$ is bounded by w.

Otherwise $M(\varphi^l)$ is calculated by multiplying $M(\varphi^j)$ and $M(\varphi^k)$ where $\mathbf{v}^T M(\varphi^j)\mathbf{a} \leq i$ and $\mathbf{v}^T M(\varphi^k)\mathbf{a} \leq i$. By Lemma 1, $\mathbf{e}^T M(\varphi^k)\mathbf{a} \geq w^d i$, and then by Lemma 2, the sum of any column of $M(\varphi^k)$ is less than $w^{2d}i$. Also, the sum of any row of $M(\varphi^j)$ is less than $dw^{2d}i$. Therefore, each entry in $M(\varphi^l)$ is bounded by $w^{2d}i \times dw^{2d}i = dw^{4d}i^2$. ■

Finally, we discuss the running time of LENGTH.

Theorem 4 *The procedures* POWER *and* LENGTH *run in* $O((\log n)d^3(d\log w + \log i)^2)$ *bit operations.*

Proof. The calculation of $\mathbf{e}^T Z\mathbf{a}$ in LENGTH takes less time than the matrix multiplications in POWER so we restrict our attention to the running time of POWER.

By Lemma 3, no entry of Y and Z in POWER is greater than $dw^{4d}i^2$. Hence, we need to do $O(\log n)$ multiplications of $d \times d$ matrices with entries of size $\leq O(d\log w + \log i)$ bits. The total time taken by the matrix multiplications is $O((\log n)d^3(d\log w + \log i)^2)$.

In POWER, the checks which are done after each matrix multiplication involve at most d^2 multiplications and additions, which requires only $O(d^2(d\log w + \log i)^2)$ bit operations for each check.

Hence, both POWER and LENGTH run in $O((\log n)d^3(d\log w + \log i))$ bit operations. ■

5.3 SUPERPOWER

Before we discuss how GENERICLENGTH returns the value of $\sum_{t\leq j<s} |\varphi^{qj+r}(x)|$. We describe an instrumental idea used in GENERICLENGTH: the ability to calculate matrices of the form $\mathbf{C}_n = \mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{n-1}$ efficiently. An inductive argument shows that if $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}$, then $\mathbf{B}^n = \begin{bmatrix} \mathbf{A}^n & \mathbf{O} \\ \mathbf{C}_n & \mathbf{I} \end{bmatrix}$.

The following subroutine called SUPERPOWER uses the binary method of exponentiation to compute $\mathbf{B}^n$ and hence $\mathbf{C}_{n+1}$ in $\log n$ matrix multiplications.

procedure SUPERPOWER$(\mathbf{A}, n, \mathbf{x}, i)$
// returns $\mathbf{C}_{n+1} = \mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^n$, making sure that the sum of the entries of $\mathbf{C}_{n+1}\mathbf{x}$
// is less than i

```
    B ← [A O; I I];
    D ← the identity matrix of the same order as B;
    x' ← [x; 0];
    e' ← column vector of 1's of the same order as x';

    // calculate D to be B^n
    while (n > 0) do
        if (n ≡ 1 mod 2) then
            D ← DB;
            if (e'^T D x' ≥ i) then return NIL;
        n ← ⌊n/2⌋;
        if (n > 0) then
```

$\mathbf{B} \leftarrow \mathbf{B}^2$;
if $(\mathbf{e}'^T\mathbf{B}\mathbf{x}' \geq i)$ then return NIL;

// $\mathbf{C}_n$ is in the lower left quadrant of $\mathbf{D}$ and
// $\mathbf{A}^n$ is in the upper left quadrant of $\mathbf{D}$
return $(\mathbf{C}_n + \mathbf{A}^n)$;

We prove a lemma concerning the size of the numbers that SUPERPOWER deals with.

Lemma 5 *If $\varphi : \Sigma^* \to \Sigma^*$ is a homomorphism of depth d and width w, and $\mathbf{A} = M(\varphi^k)$ restricted to the letters accessible from a non-empty word x for some k, and $\mathbf{B}^l$ is a matrix calculated by* SUPERPOWER$(\mathbf{A}, n, \mathbf{x}, i)$, *then each entry of $\mathbf{B}^l$ is less than $2dw^{8dk}i^2$.*

Proof. The matrix $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{C} & \mathbf{I} \end{bmatrix}$, is the same as the incidence matrix for the homomorphism $\varphi' : \Sigma'^* \to \Sigma'^*$, where $\Sigma' = \{a_1, a_2, \cdots, a_d, a'_1, a'_2, \cdots, a'_d\}$, and φ' is defined by: $\varphi'(a_j) = \varphi^k(a_j)$, and $\varphi'(a'_j) = a_j a'_j$ for all $j \leq d$.

SUPERPOWER computes $M(\varphi'^n)$ similar to the way POWER computes $M(\varphi^n)$ but with a few differences, namely, there is no regard for which letters are immortal or for which letters are accessible from x. We address these issues first.

Since $\mathbf{e}'^T\mathbf{B}^l\mathbf{x}' = \mathbf{e}^T\mathbf{C}_{l+1}\mathbf{x}$, then $\mathbf{e}'^T\mathbf{B}^l\mathbf{x}$ is non-decreasing with respect to l. Therefore, SUPERPOWER correctly returns NIL when $\mathbf{e}'^T\mathbf{B}^l\mathbf{x}' \geq i$ without any problems caused by the deletion of mortal letters.

By our assumption, the set of letters $\{a_1, a_2, \cdots, a_d\}$ is accessible from x and we see that the set $\{a'_1, a'_2, \cdots, a'_d\}$ is not. However, the entries in $\mathbf{B}^l$ corresponding to these inaccessible letters are always either 1 or 0 and hence are not in any danger of growing too large.

Now that the above concerns are handled, an argument analogous to Lemma 3 shows that each entry of $\mathbf{B}^l$ is less than $d'w'^{4d'}i^2$, where w' and d' are the width and depth of φ' respectively. Since $w' \leq w^k$ and $d' = 2d$, then each entry of $\mathbf{B}^l$ is less than $(2d)(w^k)^{4(2d)}i^2 = 2dw^{8dk}i^2$. ■

5.4 GENERICLENGTH

GENERICLENGTH begins by constructing $M(\varphi)$. Using POWER, it calculates $\mathbf{Y} = M(\varphi^{tq+r})$, and $\mathbf{A} = M(\varphi^q)$. SUPERPOWER is then used with $\mathbf{A}$ to compute the matrix $\mathbf{C}_n = M(\varphi^0) + M(\varphi^q) + \cdots + M(\varphi^{(s-t-1)q})$. Finally, we calculate $\mathbf{e}^T\mathbf{Y}\mathbf{C}_n\mathbf{x}$ to obtain the value $\sum_{t \leq j < s} |\varphi^{jq+r}(x)|$.

Because we are stopping the subroutines when a matrix becomes too large, it is necessary to handle the case where $t = 0$ and $s = 1$ separately since the matrix $M(\varphi^q)$ may have larger entries than desired. If this is the case, then we simply calculate $\mathbf{Y} = M(\varphi^r)$, and return the value of $\mathbf{e}^T\mathbf{Y}\mathbf{x}$.

```
procedure GenericLength(φ, x, q, r, s, t, i)
// returns Σ_{t≤j<s} |φ^{jq+r}(x)|, if Σ_{t≤j<s} |φ^{jq+r}(x)| < i.
// returns −1, otherwise.
    e ← column vector of all 1's;

    if (s − 1 < t) then
        return 0;

    else if (s − 1 = t = 0) then
        Y ← Power(φ, r, x, i);                  //M(φ^r)
        if (Y = nil) then return −1;

    else
        Y ← Power(φ, tq + r, x, i);             //M(φ^{tq+r})
        if (Y = nil) then return −1;
        A ← Power(φ, q, x, i);                  //M(φ^q)
        if (A = nil) then return −1;
        C ← SuperPower(A, s − t − 1, x, i);//M(φ^{(0)q}) + ··· + M(φ^{(s−t−1)q})
        if (C = nil) then return −1;
        Y ← YC_n;                               //M(φ^{tq+r}) + ··· + M(φ^{(s−1)q+r})

    if (e^T Yx ≥ i) then return −1;
    return e^T Yx;
```

We end this section by proving a theorem about the running time of GenericLength.

Theorem 6 GenericLength *runs in* $O((\log n)d^3(d^2 \log w + \log i)^2)$ *time.*

Proof. The longest possible execution of GenericLength calculates the matrix Y by using SuperPower when $t < s$ and $s \neq 1$. Recall that $sq + r = n$ and hence $qt + r < n$. Therefore, by Theorem 4, the running time of Power called within GenericLength is $O((\log n)d^3(d \log w + \log i)^2)$.

An argument analogous to Theorem 4 shows that SuperPower runs in

$$O((\log n)d^3(dk \log w + \log i)^2)$$

bit operations. But $k = q \leq d$.

Hence the running time of GenericLength takes worst case

$$O((\log n)d^3(d^2 \log w + \log i)^2)$$

bit operations. ■

6 Correctness and Analysis of the New Algorithm

We first prove the correctness of FIND, and then discuss its running time.

Theorem 7 FIND(φ, n, a, i) *returns the ith letter of $\varphi^n(a)$.*

Proof. We proceed by induction on n. When $n < d$, the algorithm calls BASICTREEDESCENT thus returning the correct answer. Otherwise, we assume that our algorithm works for values less than n. It remains to show that every time the value of n is reduced that a and i are modified in such a way that the ith letter of $\varphi^n(a)$ is the same.

The first part of the procedure descends the derivation tree one level at a time. Each time n is decreased by 1, the values for a, and i are z_t and $i - |\varphi^{n-1}(z_1 z_2 \cdots z_{t-1})|$ respectively, since the ith letter of $\varphi^n(a)$ lies in the subtree of z_t.

When a repeated letter a is found, we calculate the values for q, r, s, x and y such that for any $t < s$,

$$\varphi^n(a) = \varphi^{(s-1)q+r}(x) \cdots \varphi^{tq+r}(x)\varphi^{tq+r}(a)\varphi^{tq+r}(y) \cdots \varphi^{(s-1)q+r}(y).$$

Accordingly, for both Case 1 and Case 3, $i \leftarrow i - |\varphi^{(s-1)q+r}(x) \cdots \varphi^{(t+1)q+r}(x)|$, and $n \leftarrow (t+1)q + r$.

Therefore, each time n is reduced, the values for a and i are also modified correctly. Hence FIND returns the ith letter of $\varphi^n(a)$. ■

Theorems 4 and 6 already tell us the running times of the subroutines LENGTH and GENERICLENGTH.

Lemma 8 *The procedure* BASICTREEDESCENT *uses $O(nw(\log n)d^3(d \log w + \log i)^2)$ bit operations.*

Proof. We obtain the result by multiplying the running time of LENGTH by nw. ■

We need to show that whenever BASICTREEDESCENT is called, n is polynomial in $\log i$, w, or d. In our algorithm, we call BASICTREEDESCENT when $n \leq d$, when $n = r < q \leq d$, or when x or y grows exponentially. The following lemma shows that when x or y grows exponentially, then n is a polynomial in d and $\log i$.

Lemma 9 *Let $a \in \Sigma$, and $u, v \in \Sigma^*$ such that $\varphi^q(a) = uv$ for some $q < d$. If the ith letter of $\varphi^n(a)$ lies in the substring $\varphi^{n-q}(v)$, and u contains an exponentially growing letter b, then $n = O(d \log i)$.*

Proof. If $\varphi^d(b)$ contains less than 2 exponentially growing letters, then $\varphi^j(b)$ contains at most one exponentially growing letter for all $j \geq 0$, which is impossible. Hence there are at least 2 exponentially growing letters in $\varphi^d(u)$. Therefore $|\varphi^n(u)|$ grows at least as fast as c^n where $c = 2^{1/d}$.

Since, $c^{n-q} \leq |\varphi^{n-q}(u)| < i$, then we know $(n-q)\log c < \log i$. Hence $n \leq (\log i)/(\log c) + q$. But $1/\log c = O(d)$ and $q < d$. Consequently, $n = O(d \log i)$. ■

We apply Lemma 9 to Case 1 by setting $u = x$ and $v = ay$. For Case 3, we know a grows exponentially whenever y does, Hence, we apply Lemma 9 with $u = xa$ and $v = y$.

We have running times for the subroutines that FIND calls, but FIND also calls itself. The following shows that FIND recursively calls itself at most d times by showing that each of the d letters can be the repeated letter of our algorithm at most one time.

Lemma 10 *Let q, r, and t be integers such that $r < q$, and $\varphi^q(a) = xay$ for some $x, y \in \Sigma^*$ and $a \in \Sigma$. If x grows polynomially and $i \leq |\varphi^{tq+r}(x)|$, or if y grows polynomially and $i > |\varphi^{tq+r}(xa)|$, then descending the derivation tree of $\varphi^{(t+1)q+r}(a)$ to the ith letter encounters the letter a exactly once at the root.*

Proof. Consider the case where x grows polynomially and $i \leq |\varphi^{tq+r}(x)|$. Descending q levels takes us into the subword x. Hence, our descent takes an earlier branch than the path to the letter a in xay at level q.

Since no a is encountered before the tree descent leaves the path to the a at level q, a second a can only be encountered after leaving this path. But encountering another a would imply that a and hence x is exponentially growing, a contradiction.

Otherwise y grows polynomially and $i > |\varphi^{tq+r}(xa)|$. Similarly, we descend q levels into the subword y. If our descent encounters an a after leaving the path to the letter a in xay at level q, then we know y grows exponentially, another contradiction. ■

Finally, we state our main result: the running time of FIND is polynomial.

Theorem 11 *The number of bit operations used by* FIND *is*

$$O((d + \log i)w(\log n)d^4(d \log w + \log i)^2 + (\log n)^2 d^4(d^2 \log w + \log i)^2).$$

Proof. Let L be the running time of LENGTH, and G, the running time of GENERICLENGTH.

At worst case, the "for" loop of FIND executes d times, calculating the length of the strings of subtrees w times, before one of the three cases is reached. Handling the cases takes at worst case $O(\log nG + L)$ time. This gives one instantiation of FIND a running time of

$$O(wdL + (\log n)G).$$

We know FIND recursively calls itself at most d times before reaching the base case and calling BASICTREEDESCENT with $n < d \log i$. The running time of BASICTREEDESCENT then is

$$O(nwL) = O(d(\log i)L).$$

Hence the total running time of FIND is

$$O(d(wdL + (\log n)G) + wd(\log i)L).$$

Rearranging and substituting for G and L gives us the final running time. ■

Table 1: Performance of FIND and BASICTREEDESCENT in seconds

			FIND $n = 1000$			FIND $n = 1000000$			BTD $n = 1000$		
Σ	d	w	i=100	10^4	10^6	i=100	10^4	10^6	i=100	10^4	10^6
exp	5	2	0.17	0.25	0.33	0.19	0.27	0.41	3.87	4.40	4.71
		10	0.47	0.75	1.17	0.42	0.59	0.86	9.90	10.35	10.65
	30	2	7.67	16.58	19.27	7.59	16.83	19.93	164.97	214.80	263.64
		10	7.24	10.89	14.13	7.19	10.78	14.09	151.58	200.73	211.40
poly	5	2	0.11	0.22	0.36	0.17	0.19	0.27	4.31	4.15	3.93
		10	0.26	0.30	0.53	0.40	0.41	0.46	9.75	18.19	42.05
	30	2	5.08	5.76	6.32	5.45	6.04	6.35	20.24	23.65	25.47
		10	4.79	4.96	6.27	5.33	5.62	5.68	59.30	116.15	76.08
mix	5	2	0.08	0.12	0.19	0.14	0.15	0.03	4.68	3.11	3.17
		10	0.18	0.23	0.26	0.24	0.33	0.34	7.82	8.34	7.54
	30	2	2.67	3.09	3.36	3.34	3.70	4.04	45.52	47.97	49.54
		10	6.67	9.39	11.98	6.78	9.53	12.10	151.26	161.09	159.70

The most significant difference between the running times of FIND and BASICTREEDESCENT is that there is only a $(\log n)^2$ term in the running time of FIND whereas there is a factor of n in the running time of BASICTREEDESCENT.

The upper bound we obtained in Theorem 11 overstates the actual running time obtained in practice. For most typical input data, FIND is rarely called more than once. It makes sense, then, to implement these procedures and analyze the actual running time of FIND for varying n, i, d, and w.

Table 1 reports the performance of FIND with $n = 1000$, and $n = 1000000$. The running times of BASICTREEDESCENT with $n = 1000$ are included for comparison. Each row of the table corresponds to a homomorphism with varying values of w and d and with alphabets of exponentially growing letters, polynomially growing letters, or both. For each homomorphism, we calculated the 100th, the 10000th, and the 1000000th letter of each string.

These results show a number of details concerning the performance of FIND on various inputs. The running times of both FIND and BASICTREEDESCENT have a high dependence on d. In practice however, d is not very large (typically less than 10 letters).

Notice that BASICTREEDESCENT does not perform as well as FIND does when $n = 1000$. For even larger n, the BASICTREEDESCENT would have an even worse performance compared with FIND. For example, obtaining the ith letter of $\varphi^{1000000}(a)$ using BASICTREEDESCENT would have taken about a day for each calculation. This shows that our new algorithm enables us to calculate letters of strings produced by homomorphisms which were not obtainable before, particularly those produced with large values of n.

7 Extensions

Our algorithm may be easily extended to calculate how many times each letter in Σ occurs in the prefix of $\varphi^n(a)$ of length i. The procedures LENGTH and GENERICLENGTH compute the lengths of strings which precede the ith letter of $\varphi^n(a)$. In the process of calculating these lengths, we already compute the Parikh vector of these strings. Computing the total distribution of the letters which precede the ith letter entails only summing the corresponding Parikh vectors each time we reduce i. By subtracting two Parikh vectors for prefixes of different lengths, we can also efficiently compute the letter distributions of subwords.

8 Open Problem

It would be interesting to extend these results to efficiently compute the ith letter of strings produced by the iterated application of a finite-state transducer. An example of such a string is the well-known Kolakoski sequence [6]:

$$\mathbf{K} = \texttt{1221121221221121122121}\cdots$$

which has the property that the sequence of run-lengths of $\mathbf{K}$ is the same as $\mathbf{K}$ itself. A procedure to efficiently calculate the letter distributions for prefixes of these strings would help in studying the long range distribution of such sequences [3].

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