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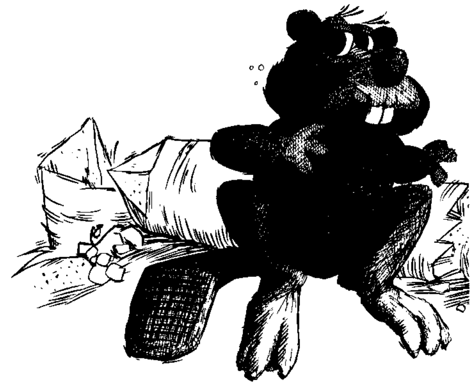

On the Path Length of Binary Trees

Rolf Klein
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Research Report
CS-87-06

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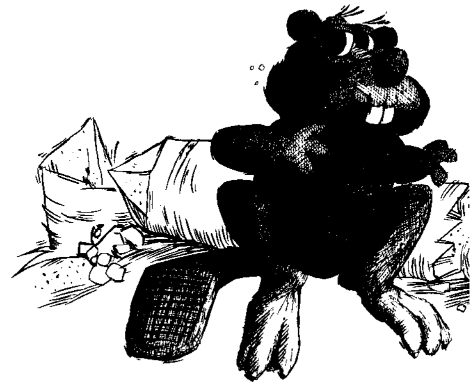

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Abstract

We show that the external path length of a binary tree is closely related to the ratios of means of certain integers and establish the upper bound

$$\text{External Path Length} \leq N(\log_2 N + \triangle - \log_2 \triangle - 0.6623)$$

where N denotes the number of external nodes in the tree and $\triangle$ is the difference in length between a longest and a shortest path. Then we prove that this bound is (almost) achieved if N and $\triangle$ are arbitrary integers that satisfy $\triangle \leq \sqrt{N}$. If $\triangle > \sqrt{N}$, we construct binary trees whose external path length is at least as large as $N(\log_2 N + \phi(N, \triangle)\triangle - \log_2 \triangle - 4)$, where $\phi(N, \triangle) = (1 + \Theta(\frac{\triangle}{N}))^{-1}$.

Keywords: Binary trees, path length, comparison cost, node visit cost, ratio of means.

1 Introduction

The time taken by a search operation in a search tree depends on the length of the path from the root to the node that contains the desired information. More generally, the execution time an algorithm needs to reach a certain state from its initial state is related to the length of the corresponding path in the decision tree. Therefore, the path length of a tree is a cost measure of great importance for the analysis of algorithms.

We consider the *external path length* EPL(T) of an extended binary tree T, that is, the total number of edges along all the paths from the root to the

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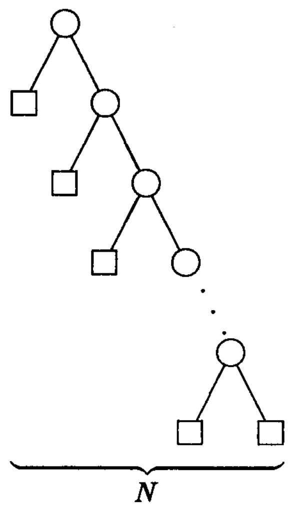

Figure 1: A snake.

external nodes of T. If N denotes the number of external nodes (the *weight* of T) then $\mathrm{EPL}(T)/N$ is just the average length of a path from the root of T to an external node. It is well known that the external path length is a minimum if and only if all paths in T differ in length by at most 1. In this case

$$EPL(T) = N(\log_2 N + 1 + \theta - 2^\theta) \tag{1}$$

holds, where $\theta = \lceil \log_2 N \rceil - \log_2 N \in [0, 1)$; see Knuth [6], p. 194. This formula establishes a lower bound for the external path length. On the other hand, the path length takes its maximum value

$$\frac{N(N+1)}{2} - 1$$

if the tree is a "snake" as shown in Figure 1[1]. Here the shortest path is $N-2$ levels shorter than the longest path.

In this paper we present an upper bound for the external path length in terms of the weight N and the maximal path length difference $\triangle$ (see Figure 2) by proving that

$$EPL(T) \leq N(\log_2 N + \triangle - \log_2 \triangle - \Psi(\triangle)) \tag{2}$$

holds for all binary trees, where

$$\Psi(\triangle) = 0.9139 - o(1) \geq 0.6623$$

and $o(1)$ tends to zero as $\triangle$ tends to infinity. For the tree in Figure 2, for example, we obtain the value 31.56 whereas its actual path length is equal to 30.

[1]This and the following figures have been produced using TreeTEX; see [1]

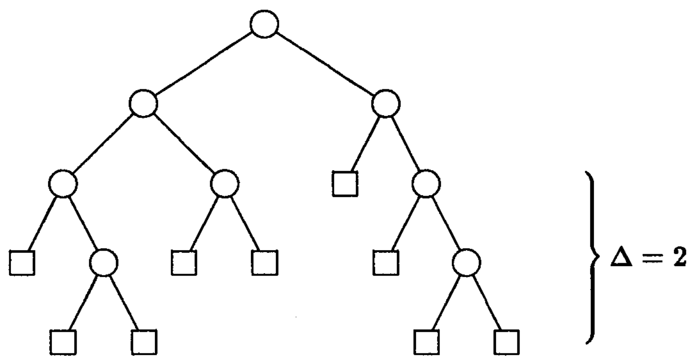

Figure 2: An example tree.

In order to establish this upper bound we first show, via the Kraft Inequality, that the external path length of a binary tree is related to the ratio of the geometric and the harmonic means of the integers 2^{l_i}, where l_i denotes the length of the i-th path. It is also related to the ratio of the arithmetic and the geometric mean of the integers 2^{h-l_i}, where h denotes the height of the tree. Either relation can be used to obtain an upper bound for the path length by applying a general theorem by Specht that imposes an upper bound on the ratio of means of arbitrary real numbers. However, we shall also give a direct proof for our result, in order to keep this paper self-contained.

In Section 4 we discuss the tightness of our upper bound. If $\triangle = 1$, then expression (2) is exactly equal to the maximum value the lower bound in (1) can take. If $\triangle$ and N are integers of arbitrary, independent orders of magnitude, we can build a tree whose path length is greater than

$$N(\log_2 N + \triangle - \log_2 \triangle - 4)$$

if $\triangle \leq \sqrt{N}$, and greater than

$$N\left(\log_2 N + \frac{1}{1+\Theta(\frac{\triangle}{N})}\triangle - \log_2 \triangle - 4\right)$$

if $\triangle > \sqrt{N}$. This shows that the upper bound for the external path length obtained here is tight if $\triangle \leq \sqrt{N}$ and quite sharp if $\triangle > \sqrt{N}$.

2 Path length and ratios of means

Let T be an extended binary tree. We count the *level* of nodes starting with level 0 at the root. The *access path* to a node at level j is of *length* j, because it consists of j edges. The *height* h of the tree T is the maximum level number or, equivalently, the length of a longest path in T. A node at level i is said to be at *height $h - i$ with respect to T*. Furthermore, $N = weight(T)$ denotes the number of external nodes of T. Finally, we let

$$EPL(T) = \sum_{i=1}^{N} l_i$$

where l_i is the length of the path to the i-th external node.

First, we recall the definition of means. Let $a_1, \ldots, a_N$ and $q_1, \ldots, q_N$ be sequences of positive real numbers such that $q_1 + q_2 + \ldots + q_N = 1$. Then

$$M_N^{[-1]}(a,q) = (\sum_{i=1}^{N} \frac{q_i}{a_i})^{-1}$$

is the *weighted harmonic mean*,

$$M_N^{[0]}(a,q) = \prod_{i=1}^{N} a_i^{q_i}$$

is the *weighted geometric mean*, and

$$M_N^{[1]}(a,q) = \sum_{i=1}^{N} q_i a_i$$

is the *weighted arithmetic mean* of the numbers $a_1, \ldots, a_N$ with weights $q_1, \ldots, q_N$.

Lemma 2.1 *Let T be a binary tree of weight N whose paths to the external nodes are of length $l_1, \ldots, l_N$. Let $a_i = 2^{l_i}$ and $q_i = \frac{1}{N}, 1 \leq i \leq N$. Then*

$$\frac{M^{[0]}(a,q)}{M^{[-1]}(a,q)} = \frac{2^{\frac{EPL(T)}{N}}}{N}$$

Proof: By Kraft's Theorem (usually referred to as the Kraft Inequality, see [3]) there exists a binary tree whose paths to the external nodes are of length $l_1, \ldots, l_N$ if and only if

$$\sum_{i=1}^{N} 2^{-l_i} = 1$$

Hence,

$$\begin{aligned} M_N^{[-1]}(a,q) &= \left(\sum_{i=1}^{N} \frac{2^{-l_i}}{N}\right)^{-1} \\ &= N \end{aligned}$$

Furthermore,

$$\begin{aligned} M^{[0]}(a,q) &= \left(\prod_{i=1}^{N} 2^{l_i}\right)^{\frac{1}{N}} \\ &= 2^{\frac{1}{N}EPL(T)} \end{aligned}$$

□

We can compute the external path length and the height of a binary tree of weight N if we know only the heights $h_1, \ldots, h_N$ of the external nodes in the tree. This leads to

Lemma 2.2 *Let T be a binary tree of weight N whose external nodes are of height $h_1, \ldots, h_N$ in T. Let $b_i = 2^{h_i}$ and $q_i = \frac{1}{N}, 1 \leq i \leq N$. Then*

$$\frac{M_N^{[1]}(b,q)}{M_N^{[0]}(b,q)} = \frac{2^{\frac{EPL(T)}{N}}}{N}$$

Proof: After multiplying by 2^h, the Kraft Inequality becomes

$$2^h = \sum_{i=1}^{N} 2^{h-l_i} = \sum_{i=1}^{N} 2^{h_i}$$

Therefore,

$$EPL(T) = Nh - \sum_{i=1}^{N} h_i$$

$$\begin{aligned} \frac{EPL(T)}{N} &= \log_2\left(\sum_{i=1}^{N} 2^{h_i}\right) - \log_2\left(2^{\frac{1}{N}\sum_{i=1}^{N} h_i}\right) \\ &= \log_2\left(\sum_{i=1}^{N} b_i\right) - \log_2\left(\prod_{i=1}^{N} b_i^{\frac{1}{N}}\right) \end{aligned}$$

and

$$2^{\frac{EPL(T)}{N}} = \frac{N M_N^{[1]}(b,q)}{M_N^{[0]}(b,q)}$$

□

Inequalities involving means were first studied by the Pythagoreans and Euclid, and many interesting results have been obtained since. For example, it is well known that

$$M_N^{[-1]}(a,q) \leq M_N^{[0]}(a,q) \leq M_N^{[1]}(a,q)$$

holds, for any sequences of numbers a_i and weights q_i. Either inequality, combined with the corresponding Lemma above, yields immediately

$$EPL(T) \geq N \log_2 N$$

In the next section we will use an upper bound for the ratios of these means discovered by Specht [7] in order to derive a new upper bound for the external path length of binary trees.

3 An upper bound for the external path length

Throughout this paper, $\triangle(T)$ denotes the difference between the length of a longest path of T and the length of a shortest path to an external node. We also refer to $\triangle$ as to the *thickness of the fringe* of T.

Theorem 3.1 *Let T be a binary tree of weight N whose fringe is of thickness $\triangle$. Then*

$$EPL(T) \leq N(\log_2 N + \triangle - \log_2 \triangle - \Psi(\triangle))$$

where

$$\begin{aligned} \Psi(\triangle) &= \log_2 e - \log_2 \log_2 e - \frac{\triangle}{2^{\triangle} - 1} - \log_2 \left(1 - \frac{1}{2^{\triangle}}\right) \\ &= 0.91392867 - o(1) \\ &\geq 0.66229950 \end{aligned}$$

and e denotes the basis of the natural logarithm.

Proof: (first version) By Lemma 2.1,

$$\frac{2^{\frac{EPL(T)}{N}}}{N} = \frac{M_N^{[0]}(a,q)}{M_N^{[-1]}(a,q)}$$

where $a_i = 2^{l_i}$, l_i = length of the path to the i-th external node, and $q_i = \frac{1}{N}$, for $i = 1, \ldots, N$. By a theorem by Specht (Satz 1, (5.4) in [7]) we have

$$\frac{M_N^{[0]}(a,q)}{M_N^{[-1]}(a,q)} \leq \left(\frac{\frac{1}{B} - 1}{-\ln B}\right) e^{\left(-1 - \frac{\ln B}{\frac{1}{B} - 1}\right)} \tag{3}$$

if $B = \frac{M}{m}$ is such that $m \leq a_1, \ldots, a_N \leq M$. If $l = \min_i l_i$, then $B = \frac{2^h}{2^l} = 2^{\triangle}$ will do. The above exponential term equals

$$e^{-1} B^{1 + \frac{1}{B-1}}$$

whereas the left hand factor is equal to

$$\frac{1-\frac{1}{B}}{(\log_2 e)^{-1}\triangle}$$

observing that $\ln x = (\log_2 e)^{-1}\log_2 x$ holds for the natural logarithm. Taking logs yields

$$\begin{aligned}\frac{EPL(T)}{N} - \log_2 N \quad \leq \quad & \log_2\left(1-\frac{1}{2^\triangle}\right) + \log_2\log_2 e - \log_2\triangle \\ & -\log_2 e + \left(1+\frac{1}{2^\triangle-1}\right)\triangle\end{aligned}$$

In order to complete the proof we note that the function $\log_2\left(1-\frac{1}{2^\triangle}\right) + \frac{\triangle}{2^\triangle-1}$ takes its maximum value among all integer arguments $\triangle \geq 1$, if $\triangle = 2$. $\square$

Another proof for the theorem used in the above proof was given by Cargo and Shisha in [2]. In addition, they showed for which values of $a_1,\ldots,a_N$ and B the inequality (3) becomes an equality. However, we now give a direct proof of Theorem 3.1.

Proof: (second version) We want to determine the maximum value of $EPL(T) = \sum_{i=1}^N l_i$ under the condition that $\sum_{i=1}^N 2^{-l_i} = 1$ (the Kraft Inequality), where $\max_i l_i - \min_i l_i = \triangle$. To this end, we let $l_i = X_0 + (\sin X_i)^2\triangle$. Here X_0 denotes the (unknown) length of a shortest path in T. The value of $(\sin X_i)^2$ oscillates in [0,1] as X_i varies in $\Re$, thereby leading to a total path length l_i that lies between X_0 and $X_0+\triangle$.
We consider the function

$$f(X_0,X_1,\ldots,X_N) = \sum_{i=1}^N (X_0 + (\sin X_i)^2\triangle)$$

under the condition that $g(X_0,X_1,\ldots,X_N) = 0$, where

$$g(X_0,X_1,\ldots,X_N) = \sum_{i=1}^N 2^{-(X_0+(\sin X_i)^2\triangle)} - 1$$

For each constrained maximum $p = (a_0,a_1,\ldots,a_N)$ of f there must be a real number λ such that all partial derivatives of $f-\lambda g$ vanish in p, by the Lagrange Multiplier Theorem (see [4], for example). This means

$$0 = \frac{\partial(f-\lambda g)}{\partial X_0}(p) = N + \lambda\ln 2\sum_{i=1}^N 2^{-(a_0+(\sin a_i)^2\triangle)} = N + \lambda\ln 2$$

and

$$0 = \frac{\partial(f - \lambda g)}{\partial X_i}(p) = 2\sin a_i \cos a_i \triangle \left(1 + \lambda \frac{\ln 2}{2^{a_0 + (\sin a_i)^2 \triangle}}\right)$$

for $i = 1, \ldots, N$. The latter equalities imply

$$(\sin a_i)^2 \in \{0, 1\} \text{ or } N = 2^{a_0 + (\sin a_i)^2 \triangle}$$

due to the first equality. Therefore,

$$l_i = a_0 + (\sin a_i)^2 \triangle \in \{a_0, a_0 + \triangle, \log_2 N\}$$

for $i = 1, \ldots, N$. In order to determine how often each of these three values occurs we consider the constrained maxima of

$$f(X, V, W, R) = V^2 X + W^2(X + \triangle) + R^2 \log_2 N$$

subject to the conditions $g_1(X, V, W, R) = 0$ and $g_2(X, V, W, R) = 0$, where

$$g_1(X, V, W, R) = V^2 \frac{1}{2^X} + W^2 \frac{1}{2^{X+\triangle}} + \frac{R^2}{N} - 1$$

represents the Kraft Inequality and

$$g_2(X, V, W, R) = V^2 + W^2 + R^2 - N$$

is because we are considering trees of weight N. Again, for each maximum $p = (x, v, w, r)$ of f_1 there must be real numbers λ and μ such that the partial derivatives of the function $f - \lambda g_1 - \mu g_2$ with respect to the variables X, V, W, and R vanish at p. This means

$$\begin{aligned} 0 &= v^2 + w^2 + \lambda \ln 2 \left(v^2 \frac{1}{2^x} + w^2 \frac{1}{2^{x+\triangle}}\right) \\ &= (N - r^2)\left(1 + \frac{\lambda \ln 2}{N}\right) \end{aligned} \tag{4}$$

due to the constraint conditions, and

$$0 = 2v\gamma(x) \tag{5}$$
$$0 = 2w\gamma(x + \triangle) \tag{6}$$
$$0 = 2r\gamma(\log_2 N) \tag{7}$$

where

$$\gamma(Z) = Z - \lambda \frac{1}{2^Z} - \mu$$

If $r^2 = N$, then according to g_1, $v = w = 0$ and f takes the value $N \log_2 N$ at p—the minimum! Therefore, we must have $\lambda = \frac{-N}{\ln 2}$, due to (4).

The function $\gamma(Z)$ takes its unique minimum if $Z = \log_2 N$, because $\frac{d\gamma}{dZ}(Z) = 1 - \frac{N}{2^Z}$. Hence, $\gamma(y) = \gamma(\log_2 N)$ implies $y = \log_2 N$, for arbitrary real numbers y. If we assume $r \neq 0$ then (7) implies $\gamma(\log_2 N) = 0$.Therefore, due to (5) and (6), v or w must be equal to zero because $\triangle > 0$. Assume $v = 0$ and $w \neq 0$. Then $\gamma(x + \triangle) = \gamma(\log_2 N)$ implies $x + \triangle = \log_2 N$ and again, f takes its minimum at p, a contradiction. The same holds if we assume $v \neq 0$ and $w = 0$ or $v = w = 0$.

Therefore, r must be equal to zero. This yields $v \neq 0$ and $w \neq 0$ (otherwise f would take a minimum), hence $\gamma(x) = 0 = \gamma(x + \triangle)$. So,

$$x + \frac{N}{\ln 2}\frac{1}{2^x} = x + \triangle + \frac{N}{\ln 2}\frac{1}{2^x + \triangle}$$

or

$$x = \log_2\left(\frac{N}{\triangle \ln 2}\right) + \log_2\left(1 - \frac{1}{2^\triangle}\right) \tag{8}$$

The constraint conditions now read as $v^2 + w^2 = N$ and $v^2 2^\triangle + w^2 = \frac{N}{\triangle \ln 2}(2^\triangle - 1)$, the solution of these linear equations being

$$v^2 = N\left(\frac{1}{\triangle \ln 2} - \frac{1}{2^\triangle - 1}\right) \tag{9}$$

$$w^2 = N\left(\frac{2^\triangle}{2^\triangle - 1} - \frac{1}{\triangle \ln 2}\right) \tag{10}$$

Now combining (8) and (10) yields

$$\begin{aligned} f(p) &= v^2 x + w^2(x + \triangle) \\ &= Nx + w^2\,\triangle \\ &= N\,(\log_2 N - \log_2 \ln 2 - \log_2 \triangle \\ &\quad + \log_2\left(1 - \frac{1}{2^\triangle}\right) + \triangle\frac{2^\triangle}{2^\triangle - 1} - \frac{1}{\ln 2}\right) \end{aligned}$$

Therefore, for all binary trees T of weight N and fringe thickness $\triangle$ we have

$$EPL(T) \leq f(p) = N(\log_2 N + \triangle - \log_2 \triangle - \Psi(\triangle))$$

□

We observe a certain similarity between the formula in Theorem 3.1 and the tight upper bound for the path length of AVL trees recently obtained by Klein and Wood [5], caused by the term $\log_2 \triangle$. Namely, for each AVL tree T we have

$$\triangle \leq \frac{1}{2}h(T) \leq \frac{1}{2}1.4404\log_2 N$$

which makes Theorem 3.1 read as

$$EPL(T) \leq N(1.7202 \log_2 N - \log_2 \log_2 N) + O(N)$$

This bound is bigger than the tight upper bound

$$1.4404N(\log_2 N - \log_2 \log_2 N) + O(N)$$

in [5], but the presence of the term $\log_2 \log_2 N$ in the above equation is surprising!

Equations (8), (9), and (10) in the above proof seem to indicate how, for given integers $\triangle$ and N, a binary tree of maximal external path length has to look. Namely, $N\left(\frac{1}{\triangle \ln 2} - \frac{1}{2^{\triangle}-1}\right)$ external nodes should appear at level $x = \log_2\left(\frac{N}{\triangle \ln 2}\right) + \log_2\left(1 - \frac{1}{2^{\triangle}}\right)$, and the rest of them at level $x + \triangle$. But these numbers are reals, not integers, a difficulty that in this case cannot be overcome by rounding! For example, if $N = 320000$ and $\triangle = 14427$ then these formulae yield that $31.99989\ldots$ external nodes should be located at level $4.999995\ldots$. But there is no 2-level-tree that has 32 external nodes at level 5, because it couldn't have any internal node at level 5. Moreover, there is no 2-level-tree at all, if $N < 2^{\triangle}$! Nevertheless, in the next paragraph we will construct binary trees whose external path length comes very close to the upper bound established in Theorem 3.1.

4 Binary trees of high external path length

In order to investigate how close to reality the upper bound established in Theorem 3.1 is we have to allow the parameters N and $\triangle$ to vary independently. If $\triangle = 1$, then $\Psi(\triangle) = \log_2 e - \log_2 \log_2 e = (1 + \ln\ln 2)/\ln 2$, and our upper bound takes the form

$$N(\log_2 N + 1 - (1 + \ln\ln 2)/\ln 2)$$

This is exactly the maximum value of the lower bound $N(\log_2 N + 1 + \theta - 2^{\theta})$ for the external path length, for $\theta = -(\ln\ln 2)/\ln 2$, see formula (1) in Section 1, and Knuth [6], p. 194.

Next we consider the case $2^{\triangle} \leq N$.

Lemma 4.1 *Let $\triangle = 2^a \geq 1$. Then for each integer $s \geq 0$, there exists a binary tree T of weight $N = \Theta(2^{\triangle+s})$ whose fringe is of thickness $\triangle$ such that*

$$EPL(T) \geq N(\log_2 N + \triangle - \log_2 \triangle - 2)$$

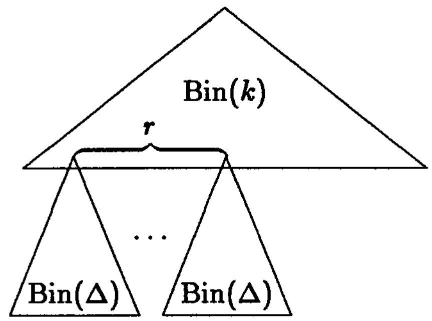

Figure 3: The tree $T_1(r, k, \triangle)$.

Proof: Consider the tree $T = T_1(r, k, \triangle)$ displayed in Figure 3, a complete binary tree of height k, r of whose external nodes are the roots of complete binary trees of height $\triangle$.
Since a complete binary tree $Bin(h)$ of height h has 2^h external nodes and external path length $h2^h$ we have

$$weight(T) = N = 2^k + (2^{\triangle} - 1)r$$

and

$$\begin{aligned} EPL(T) &= k(2^k - r) + r(k + \triangle)2^{\triangle} \\ &= kN + r \triangle 2^{\triangle} \end{aligned}$$

Now let $k = \triangle - \log_2 \triangle + s = 2^a - a + s$ and $r = \triangle 2^{k-\triangle} = 2^s \geq 1$. Then

$$EPL(T) = kN + \triangle^2 2^k$$

and

$$\begin{aligned} N &= \triangle 2^k \frac{\triangle 2^{\triangle} + 2^{\triangle} - \triangle}{\triangle 2^{\triangle}} \\ &= \Theta(\triangle 2^k) \\ &= \Theta(2^{\triangle + s}) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{EPL(T)}{N} &= k + \triangle \frac{\triangle 2^{\triangle}}{\triangle 2^{\triangle} + 2^{\triangle} - \triangle} \\ &\geq k + \triangle - 1 \end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned} \log_2 N &= k + \log_2 \triangle + \log_2 \frac{\triangle 2^{\triangle} + 2^{\triangle} - \triangle}{\triangle 2^{\triangle}} \\ &\leq k + \log_2 \triangle + 1 \end{aligned} \tag{12}$$

Combining equations (11) and (12) completes the proof. □

In the above construction we could build a 2-level-tree because $2^{\triangle} \leq N$. But if the fringe grows thicker (in relation to the weight) we have to place the external nodes at more than 2 levels in the tree. This tends to keep the maximal possible external path length smaller than the value of the upper bound. However, if $\triangle \leq \sqrt{N}$, then the difference from $EPL(T)/N$ is only a small additive constant, as the following lemma shows.

Lemma 4.2 *Let $\triangle = 2^a \geq 1$. Then, for each integer k in $[1, \triangle - a]$, there exists a binary tree T of weight $N = \Theta(\triangle 2^k)$ whose fringe is of thickness $\triangle$ such that*

A. $EPL(T) \geq N(\log_2 N + \triangle - \log_2 \triangle - 4)$

holds if $\triangle \leq \sqrt{N}$ and

B. $EPL(T) \geq N\left(\log_2 N + \frac{1}{1+\Theta(\frac{\triangle}{N})} \triangle - \log_2 \triangle - 4\right)$

holds otherwise.

Proof: We consider the tree $T = T_2(k, s, t)$ shown in Figure 4, a complete binary tree of height $k \geq 1$ in one of whose external nodes a "snake" of length s originates that leads to another complete binary tree of height t. Clearly, $\triangle = s + t$, $N = 2^k + s - 1 + 2^t$ and

$$\begin{aligned} EPL(T) &= k(2^k - 1) + sk + \frac{s(s+1)}{2} + (k + s + t)2^t \\ &\geq kN + \triangle 2^t \end{aligned}$$

Now let $t = k + a$ and $s = \triangle - t$. Then $2^t = \triangle 2^k$ and

$$N = (\triangle + 1)2^k + s - 1 \tag{13}$$

Therefore

$$\frac{EPL(T)}{N} \geq k + \frac{\triangle^2 2^k}{(\triangle + 1)2^k + s - 1}$$

Because of

$$s - 1 = \triangle - t - 1 = 2^a - 1 - t \leq 2^{2a-t} 2^{t-a} = \triangle^2 2^{-t} 2^k \tag{14}$$

we obtain

$$\begin{aligned} \frac{EPL}{N} &\geq k + \triangle \frac{\triangle}{\triangle + 1 + \triangle^2 2^{-t}} \\ &\geq k + \triangle \frac{1}{1 + \triangle 2^{-t}} - 1 \end{aligned} \tag{15}$$

The latter inequality can be verified easily by crossmultiplying. Because

$$\left(1+\frac{1}{\triangle}\right)2^t \le \left(1+\frac{1}{\triangle}\right)2^t + s = N+1 \le \left(1+\frac{1}{\triangle}\right)2^t + \triangle$$

we have $\frac{N}{2^t} \le 1 + \frac{1}{\triangle} + \frac{\triangle}{2^t} \le 1 + \frac{1}{\triangle} + \frac{1}{2}$ and

$$N = \Theta(2^t) = \Theta(\triangle 2^k)$$

The former, applied to (15), yields

$$\frac{EPL(T)}{N} \ge k + \triangle \frac{1}{1+\Theta(\frac{\triangle}{N})} - 1 \tag{16}$$

where $\Theta\left(\frac{\triangle}{N}\right) = \frac{3}{2}\frac{\triangle}{N} + \frac{1}{N}$. If $\triangle^2 \le N$, then

$$\left(1+\frac{3}{2}\frac{\triangle}{N}+\frac{1}{N}\right)^{-1} \ge 1 - \frac{3}{2\triangle}$$

hence

$$\frac{EPL(T)}{N} \ge k + \triangle - \frac{5}{2} \tag{17}$$

On the other hand, equation (13) yields

$$\begin{aligned} \log_2 N &= \log_2\left(\triangle 2^k\left(1+\frac{2^k+s-1}{\triangle 2^k}\right)\right) \\ &\le \log_2 \triangle + k + \frac{3}{2} \end{aligned} \tag{18}$$

because

$$\begin{aligned} 1+\frac{2^k+s-1}{\triangle 2^k} &\le 1+\frac{1+\triangle^2 2^{-t}}{\triangle} \\ &\le \frac{5}{2} \end{aligned}$$

according to (14). Now assertions A and B follow by combining (17) and (16) with (18), correspondingly. □

Lemma 4.2 covers the case where $2^\triangle > N$ holds (by orders of magnitude). According to assertion A, the upper bound for EPL established in Theorem 3.1 is tight up to a small additive $O(N)$ term if $\triangle \le \sqrt{N}$. If $\triangle$ is increased beyond $\sqrt{N}$, then the coefficient

$$\rho = \frac{1}{1+\Theta(\frac{\triangle}{N})}$$

of $\triangle$ in B begins to decrease. However, as long as $\triangle = O(N^\alpha)$ holds for some real number $\alpha > 1$ the value of ρ still comes arbitrarily close to 1 for large integers N. Only if $\triangle = \Theta(N)$ is the decreasing of ρ substantial — but bounded, nevertheless. In fact, in the extreme case where the tree T is a snake (see Figure 1) we have $\triangle = N - 2$ and

$$\begin{aligned} EPL(T) &= \frac{N(N+1)}{2} - 1 \\ &= N\frac{1}{2}\triangle + O(N) \\ &= N\left(\log_2 N + \frac{1}{2}\triangle - \log_2 \triangle\right) + O(N) \end{aligned}$$

This indicates that there is a difference between the world of reals where the upper bound of Theorem 3.1 is tight for all values of N and $\triangle$ and the real world of trees — but only a small one! (See the end of Section 3.)

5 Concluding remarks

We have used the relationship between the external path length of a binary tree and the ratio of means of certain integers to derive an upper bound for the path length in terms of the weight N and the thickness of the fringe, $\triangle$, namely

$$EPL(T) \leq N(\log_2 N + \triangle - \log_2 \triangle + O(1))$$

Then we have constructed binary trees that have a high external path length to show that this bound is tight if $\triangle \leq \sqrt{N}$ and reasonably sharp otherwise.

The result obtained here raises a number of interesting problems for further research. Does our result extend to weighted binary trees? To multiway trees? What does a tight upper bound look like in the case $\triangle > \sqrt{N}$? And finally, how much better a bound can be established if more information about the fringe is available?

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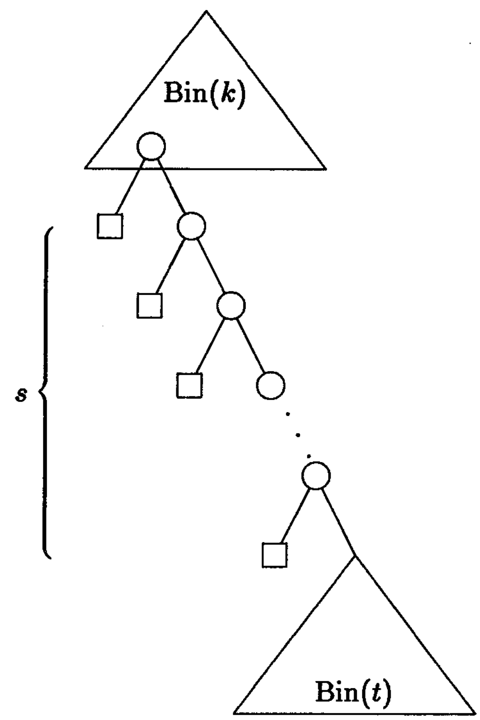

Figure 4: The tree $T_2(k, s, t)$.