CIRCULAR PERMUTATION GRAPHS*

by

D. Rotem[+] and J. Urrutia[§]

Research Report CS-79-32

Department of Computer Science

University of Waterloo
Waterloo, Ontario, Canada

[+] Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

[§] Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

* Some of the results of this paper appear in J. Urrutia's Ph.D. thesis (in preparation).

This research was supported by the National Sciences and Engineering Research Council of Canada under grant numbers A8142 and A3055.

A B S T R A C T

A new class of graphs called circular permutation graphs is introduced and characterized. A circular permutation diagram for a permutation $P(1),\ldots,P(n)$ consists of two circles C_1 and C_2, the numbers $1',2',\ldots,n'$ and $P(1),\ldots,P(n)$ on C_1 and C_2 respectively and a set of n chords $\bar{1},\bar{2},\ldots,\bar{n}$ connecting i to i'. A graph G is a circular permutation graph if there is a labelling of $V(G)$ with $\{1,\ldots,n\}$ such that i is adjacent to j iff $\bar{i}$ and $\bar{j}$ intersect. Circular permutation graphs generalize permutation graphs ([1],[6]) and are embedded in the set of comparability graphs [3]. The characterization leads to a recognition algorithm which requires $O(\Delta\cdot|E|)$ steps where Δ is the maximum degree of a vertex. Also, an algorithm for finding a maximum independent set in a CPG which requires $O(n^2 \log_2 n)$ steps is presented.

Keywords: Permutation graphs, comparability graphs, longest increasing subsequence, partial order.

CR Categories: 5.32

Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ of unordered pairs of vertices. We consider here only graphs with no multiple edges or self loops. For two vertices $x, y \in V(G)$, we denote $x \overset{G}{-} y$ if $(x,y) \in E(G)$ otherwise $x \overset{G}{\not -} y$. The set $\Gamma_G(v) = \{x \in V(G) \mid v \overset{G}{-} x\}$. A graph D is a directed graph if its edge set consists of ordered pairs $\langle x,y \rangle$. We denote $x \overset{D}{\rightarrow} y$ if $\langle x,y \rangle \in E(D)$. For a vertex $v \in D$, $\Gamma_D^-(v) = \{x \in V(D) \mid x \overset{D}{\rightarrow} v\}$ and $\Gamma_D^+(v) = \{x \in V(D) \mid v \overset{D}{\rightarrow} x\}$. The cardinality of a set S is denoted by $|S|$.

A permutation diagram $D(P)$ for a permutation $P = \langle P(1), P(2), \ldots, P(n) \rangle$ on $\{1,2,\ldots,n\}$, consists of two parallel lines L_1 and L_2. The numbers $1', 2', \ldots, n'$ appear on L_1 in increasing order and $P(1), P(2), \ldots, P(n)$ appear on L_2 according to their order in P, the number i' on L_1 is joined to i on L_2 with a segment $\bar{i}$, for $1 \le i \le n$. (See Figure 1).

A graph $G(P)$ with n vertices represents $D(P)$ if there exists a labelling of its vertex set with $\{1,2,\ldots,n\}$ such that $i \overset{G(P)}{-\!-\!-} j$ if and only if $\bar{i}$ intersects $\bar{j}$ in $D(P)$. A graph G which represents at least one permutation diagram is called a Permutation Graph (PG).

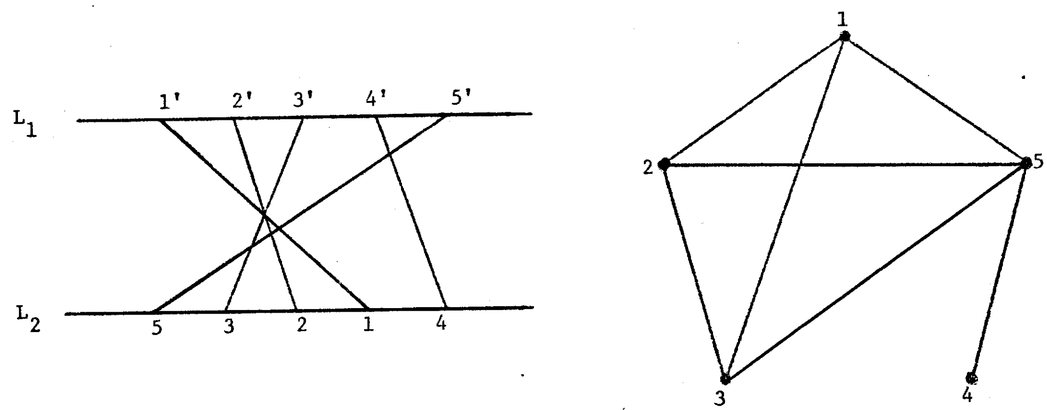


Figure 1. A permutation diagram and a graph for P = <5,3,2,1,4>.

The class of PG was studied in Pnueli et al [6] and Even et al [1] where they were applied to model and solve problems concerning memory allocation and circuit layout. It has also been shown in [6] that PG can be characterized as those graphs where both the graph and its complement are comparability graphs. A graph G is a comparability graph, ([3],[5],[6], also called transitively orientable graph) if its edges can be oriented such that for $x,y,z \in V(G)$ $x \overset{G}{\to} y$ and $y \overset{G}{\to} z$ implies $x \overset{G}{\to} z$. It can also be shown that PG represent partial orders of dimension 2 except for the complete graph which is a PG and represents a partial order of dimension 1 .

In this paper a new class of graphs which generalize PG is introduced and characterized, these are called Circular Permutation Graphs.

A circular permutation diagram $C(P)$ for a permutation P on $\{1,2,\ldots,n\}$ consists of two cocentric circles C_1 and C_2, C_1 contained in C_2. The numbers $1',2',\ldots,n'$ and $P(1),P(2),\ldots,P(n)$ appear on C_1 and C_2 respectively in the anticlockwise direction. We now choose a set of n chords $\bar{1},\bar{2},\ldots,\bar{n}$, totally contained in the annular region between C_1 and C_2 such that chord $\bar{i}$ joins i' on C_1 to i on C_2 and for $i \neq j$, $\bar{i}$ and $\bar{j}$ intersect each other at most once (Figure 2). As in the case of PG, a graph G represents a circular permutation diagram $C(P)$ if its vertex set $V(G)$ can be labelled $\{1,2,\ldots,n\}$ such that $i \overset{G}{-} j$ if and only if $\bar{i}$ intersects $\bar{j}$ in $C(P)$. A graph G is called a Circular Permutation Graph (CPG) if it represents at least one circular permutation diagram.

In Section 2 a characterization of CPG is given which leads to a recognition algorithm which requires $O(\Delta \cdot |E(G)|)$ steps where Δ is the maximum degree of a vertex. A representation of CPG using a defining permutation is given in Section 3 and is used in an algorithm for finding a maximum independent set.

§1 Preliminaries

In [6] it was shown that if H is a PG its corresponding permutation diagram can be constructed as follows: We first construct a permutation P (we denote by $P^{-1}(\ell)$ the position of the element ℓ in P) as follows:

1. Find transitive orientations $\vec{H}$ and $\vec{H}^C$ for H and its complement H^C.
2. Construct the tournament $T_1 = \vec{H} \cup \vec{H}^C$ and label V(H) by $1,2,\ldots,n$ such that a vertex v is labelled by $|\Gamma_T^-(v)| + 1$.
3. Reverse the orientation $\vec{H}^C$ to obtain $\overleftarrow{H}^C$ and consider the tournament $T_2 = \vec{H} \cup \overleftarrow{H}^C$. For a vertex labelled ℓ in step 2, let $P^{-1}(\ell) = |\Gamma^+_{T_2}(\ell)| + 1$. The permutation diagram can be now constructed using $P = \langle P(1),\ldots,P(n)\rangle$.

It is shown in [6] that $i \overset{G}{-} j$ if and only if $(i - j)(P^{-1}(i) - P^{-1}(j)) < 0$, or in words i and j form an 'inversion' in P . A PG G can therefore be represented by a permutation P obtained by this construction.

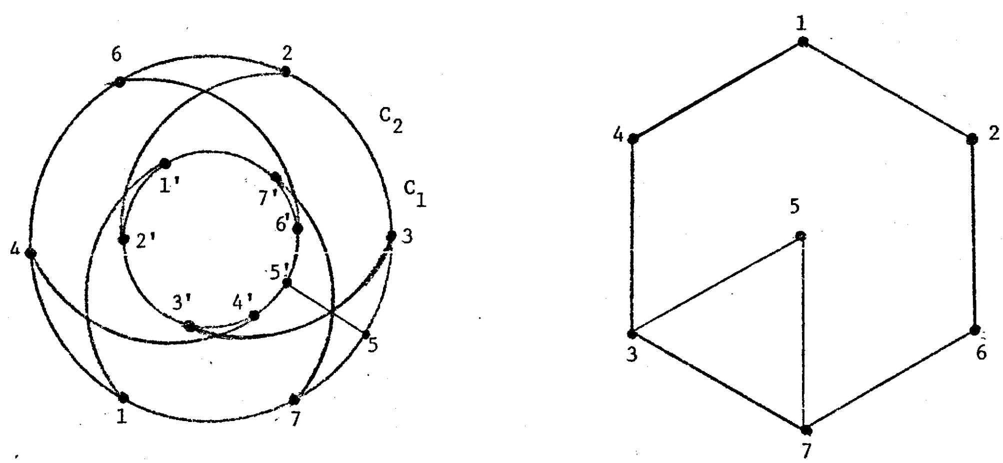


Figure 2. A circular permutation diagram and its corresponding graph.

Clearly, every PG is also a CPG since every permutation diagram can be transformed into a circular permutation diagram where L_1 and L_2 are mapped into the cocentric circles C_1 and C_2 .

However, a graph G which is a CPG is also a PG, if and only if there exists a circular permutation diagram C represented by G , such that it is possible to draw a chord $\bar{v}$ with endpoints on C_1 and C_2 which does not intersect the chords $\bar{1},\ldots,\bar{n}$ of C . In this case, the circular permutation diagram C can be opened along $\bar{v}$ to form a permutation diagram. From this it follows that a CPG which contains an isolated vertex is a PG . (See Figure 3)

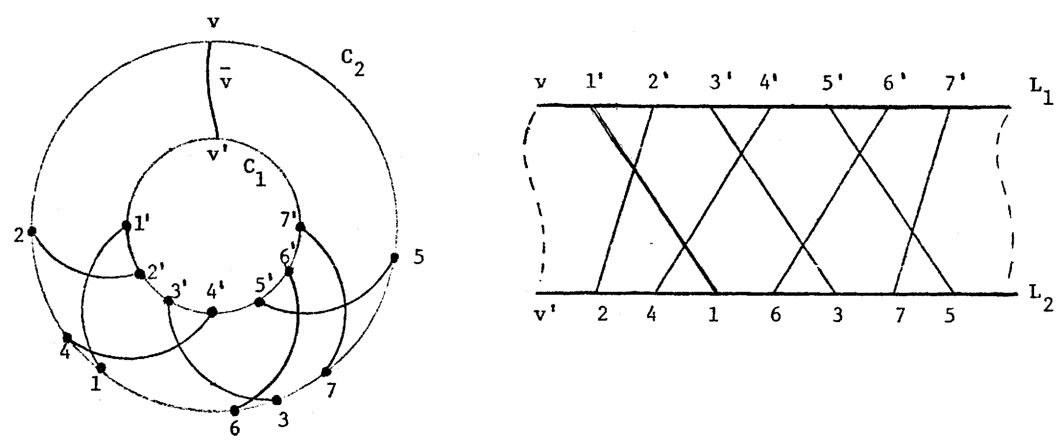
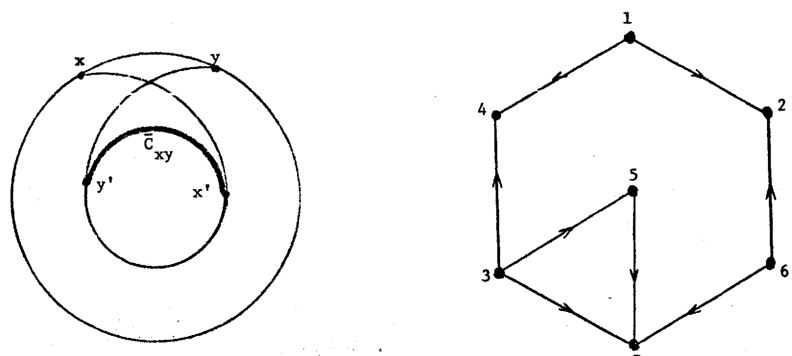


Figure 3. A circular permutation diagram opened along $\bar{v}$.

Examples of a CPG which is not a PG are all the even cycles C_{2n} with $n \geq 3$. Since every permutation diagram can be transformed into a circular permutation diagram, in what follows a permutation diagram will be taken to mean the circular permutation diagram obtained by the above mapping of L_1 and L_2 into C_1 and C_2 .

§2 Characterization of Circular Permutation Graphs

Given a circular permutation diagram C and its representing graph G , we can induce an orientation in G using C as follows: For every pair of intersecting chords $\bar{x}$ and $\bar{y}$ in C with endpoints x', y' on C_1 , consider the region which is bounded by the arc $\bar{C}_{xy}$ on C_1 , $\bar{x}$ and $\bar{y}$ (See Fig.4). We direct $x \overset{G}{\to} y$ if x' precedes y' on $\bar{C}_{xy}$ in the anticlockwise direction. In this case we say that $\bar{x}$ intersects $\bar{y}$ anticlockwise or equivalently $\bar{y}$ intersects $\bar{x}$ clockwise. This induced orientation of G is denoted by $\vec{G}$. A chord $\bar{x}$ is a source (sink) in C if all other chords which intersect it do so clockwise(anticlockwise).



Orientation of the edge (x,y).

Orientation of G of Fig. 2

Figure 4.

Lemma 1: A CPG is a comparability graph.

Proof: We show that $\vec{G}$ is transitive. Let $x,y,z \in V(G)$ such that $x \overset{\vec{G}}{\to} y$, $y \overset{\vec{G}}{\to} z$. Since $\bar{x}$ intersects $\bar{y}$ anticlockwise and $\bar{z}$ intersects $\bar{y}$ clockwise, then $\bar{x}$ must intersect $\bar{z}$, and x' precedes z' in the anticlockwise direction on $\bar{C}_{xz}$, which implies $x \overset{\vec{G}}{\to} z$. □

Remark 1: It can be seen that for the special case where G is a PG , the permutation diagram D(P) obtained by the construction of [6] as described in the introduction, induces in G the orientation $\vec{G}$ which is exactly the one used in the construction of the tournaments T_1 and T_2.

The operation 'switch' on a vertex $v \in V(G)$, which plays an important role in the characterization of CPG, is defined as follows: Connect v to all vertices x such that $x \in V(G) - \Gamma_G(v)$ and delete the edges (v,y) , $y \in \Gamma_G(v)$. Let G_v be the graph obtained from G by 'switching' $v \in V(G)$, then clearly $(G_v)_v = G$. Similarly, for a set $S \subset V(G)$ the graph G_S is obtained from G by 'switching' all vertices of S one at a time. It is easy to see that G_S is uniquely defined independent of the order in which the members of S are 'switched', also $(G_S)_S = G$. In what follows $\Gamma_G(v)$ is simply called $\Gamma(v)$.

We can now state the characterization of CPG in the following theorem.

Theorem : Let v be a vertex in G , then G is a CPG if and only if

(a) G is a comparability graph

(b) $G_{\Gamma(v)}$ is a PG .

Before proceeding with the proof, we develop some more properties of the 'switch' operation which are given in the following lemmas.

For a directed graph D, we define the 'switch' operation only with respect to sources or sinks of D. Let v be a source or a sink, we first 'switch' v as defined previously ignoring the directions in D, then D_v is oriented such that v becomes a source of D_v if it was a sink of D and vice versa, all other directions of edges in D_v are the same as in D.

<u>Lemma 2</u>: Let D be a transitive image of a comparability graph G and let v be a source or a sink in D, then D_v is a transitive image of G_v.

<u>Proof</u>: Let v be a source in D, and assume that there exist three vertices $x, y, z \in D_v$ such that $x \overset{D_v}{\rightarrow} y$, $y \overset{D_v}{\rightarrow} z$ and $x \overset{D_v}{\not\rightarrow} z$. One of x, y or z is v, and since v is a sink it can only be that $v = z$ and we cannot have $z \overset{D_v}{\rightarrow} x$. Therefore $x \overset{D_v}{\not\rightarrow} z$ and $z \overset{D}{\rightarrow} x$. But since $x \overset{D}{\rightarrow} y$ and D is transitive we have $z \overset{D}{\rightarrow} y$ which contradicts $y \overset{D_v}{\rightarrow} z$ by the definition of the 'switch' operation for directed graphs. Similar arguments can be used when v is a sink of D. $\square$

<u>Lemma 3</u>: If G is a comparability graph then $G_{\Gamma(v)}$ is also a comparability graph.

<u>Proof</u>: Let D be a transitive orientation of G, and v a vertex in G (not necessarily a source or a sink in D). Assume $|\Gamma(v)| = \ell$ and $|\Gamma_D^-(v)| = k$. There exists a vertex $y_1 \in \Gamma_D^-(v)$ which is a source in D since $\Gamma_D^-(v)$ induces an acylic subgraph of D, and all

edges with exactly one endpoint $x \in \Gamma_D^-(v)$ are outdirected from x . Label this vertex by 1. Similarly in $D - \{1\}$, there exists a source $y_2 \in \Gamma_D^-(v)$ we label this vertex by 2.

By using this elimination of sources procedure iteratively k times where i is a source of $D - \{1,2,\ldots,i-1\}$ in $\Gamma_D^-(v)$, we can label the vertices of $\Gamma_D^-(v)$ from 1 to k. We then find a sink of $D - \{1,\ldots,k\}$ in $\Gamma_D^+(v)$ and label it $k + 1$. Proceeding in a similar way as for $\Gamma_D^-(v)$, we can label the vertices of $\Gamma_D^+(v)$ with $k+1,k+2,\ldots,\ell$ by choosing a sink of $D - \{1,2,\ldots,k+s\}$ in $\Gamma_D^+(v)$ and label it $k + s + 1$ $(k + s + 1 \le \ell)$. We now show that for $0 \le i < k$, $i + 1$ is a source in $D^i = (D^{i-1})_i$ where $D^0 = D$, (D^i is obtained by 'switching' i in D^{i-1}), and for $k \le i < \ell$, $i + 1$ is a sink in D^i . If this is shown, then since for $0 \le i \le \ell$ D^i is a transitive image of $(G_S)_i$ where $S = \{1,2,\ldots,i-1\}$ (by succesive applications of Lemma 2), it follows that D^ℓ is a transitive image of $G_{T(v)}$.

To see that $1 \le i \le k$ is a source in D^{i-1} we use induction on i . For $i = 1$, vertex 1 is a source in $D^0 = D$ by its choice in the source elimination process. Assume that i is a source in D^{i-1} for $i = m < k$. Consider the graph D^i , and assume $i + 1 \overset{D^i}{\text{—}} j$ where $j \le i$ and $j \in \Gamma_D^-(v)$. Then the edge connecting $i + 1$ to j was introduced when we 'switched' j in D^{j-1}, and since j was a source in D^{j-1} it follows that it became a sink in D^j and therefore $i + 1 \overset{D_j}{\to} j$. This edge is not changed again in the construction of $D^{j+1},\ldots,D^i$ therefore $i + 1 \overset{D_i}{\to} j$. For all other vertices $v \in D^i$ we have $i + 1 \overset{D_i}{\rightarrow} v$ by the source elimination labelling (i.e. $i + 1$ is a source in $D - \{1,2,\ldots,i\}$), it follows that $i + 1$ is a source in D^i. Note that since D is a transitive graph, for every vertex $x \in \Gamma_D^-(v)$ and every vertex $y \in \Gamma_D^+(v)$ we have $x \overset{D}{\rightarrow} v \overset{D}{\rightarrow} y$ which implies $x \overset{D}{\rightarrow} y$. Therefore in D^k, no edge exists between $\Gamma_D^-(v)$ and $\Gamma_D^+(v)$, and $k + 1$ which is a sink in D is also a sink in D^k. We can now apply a similar induction argument to show that $i + 1$ is a sink in D^i for $k \leq i < \ell$, which completes the proof. □

We now define the 'switch' operation for chords in a circular permutation diagram which will be proved to be closely related to the 'switch' operation in directed graphs.

For a chord $\bar{v}$ in C with endpoints v' on C_1 and v on C_2 we say that $\bar{v}$ can be 'switched' to the chord $\bar{v}'$ if $\bar{v}'$ joins v' and v, and intersects exactly those chords which do not intersect $\bar{v}$ (See Figure 5), it can also be seen that $\bar{v}'$ does not intersect any chord more than once. Let $C(\bar{v})$ denote the diagram obtained from C by 'switching' $\bar{v}$. Then from the above definition it follows that if G represents C then G_v represents $C(\bar{v})$.

Let G be a CPG which represents C.

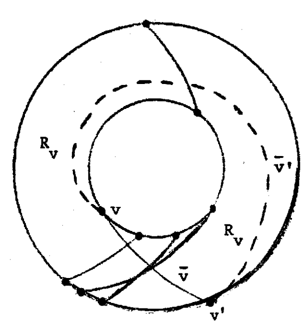


Figure 5. Switching the chord $\bar{v}$.

Lemma 4: A chord $\bar{v}$ can be 'switched' in C , if and only if it is a source or a sink in C .

Proof: If $\bar{v}$ is a source in C , then all chords which intersect $\bar{v}$ in C do so clockwise. Those chords divide the annular region between C_1 and C_2 into connected regions one of which contains the points v' and v on its boundary, we call this region R_v (See Fig. 5). The chord $\bar{v}'$ can be drawn such that it is totally contained in R_v and does not intersect any chord which intersects $\bar{v}$. To see that $\bar{v}'$ intersects all chords which $\bar{v}$ does not intersect, we consider the closed curve γ obtained by the union of $\bar{v}$ and $\bar{v}'$. Clearly any chord $\bar{x}$ in C intersects γ , and therefore if $\bar{x}$ does not intersect γ in $\bar{v}$ it must do so in $\bar{v}'$, and therefore $\bar{v}$ can be switched. Similarly it can be shown that $\bar{v}$ can be switched if it is a sink.

Conversely, assume that $\bar{v}$ is not a source or a sink, then there exist two chords $\bar{x}$ and $\bar{y}$ which cross $\bar{v}$ clockwise and anti-clockwise respectively. Therefore $\bar{x}$ intersects $\bar{y}$ and v' is an interior point on the arc $\bar{C}_{xy}$. Therefore any chord which joins v' and v must cross at least one of $\bar{x}$ or $\bar{y}$ contrary to the definition of the 'switch' operation. □

We note that $\bar{v}'$ can be switched in $C(\bar{v})$ back to $\bar{v}$, therefore $\bar{v}'$ is a source or a sink in $C(\bar{v})$. Clearly $\bar{v}'$ is a source in $C(\bar{v})$ if and only if it was a sink in C . From this and Lemma 4 we conclude:

Lemma 5: If G is a CPG which represents C and $D = \vec{G}$, then G represents $C(\bar{v})$ and $\vec{G}_v = D_v$. □

In words, the transitive orientation induced by $C(\bar{v})$ on G_v is equal to the orientation obtained by 'switching' v in the directed graph D .

We are now in a position to prove the characterization theorem.

<u>Proof:</u> <u>The 'if' part</u>

Since G is a CPG there exists a diagram C which is represented by G, and by Lemma 1 C induces a transitive orientation $\vec{G}$ on G which proves (a).

For any vertex $v \in \vec{G}$ with $|\Gamma(v)| = \ell$ we can order and label the vertices of $\Gamma(v)$ from $1,2,\ldots,\ell$ using the source and sink elimination procedure of Lemma 3 . A vertex i can be switched in G if it is a source or a sink (by definition of 'switching' in a directed graph) and by Lemma 4 its corresponding chord $\bar{i}$ can be switched in C, where $C(\bar{i})$ is represented by G_i by Lemma 5 . We can therefore 'switch' the chords $\bar{1},\bar{2},\ldots,\bar{\ell}$ in this order in C to obtain a diagram $C(\Gamma(v))$ which is represented by $G_{\Gamma(v)}$. Therefore $G_{\Gamma(v)}$ is a CPG , and v is an isolated vertex in it because all vertices in $\Gamma(v)$ have been 'switched' in G , by the remarks in the introduction $G_{\Gamma(v)}$ is a PG which proves (b).

<u>The 'only if' part</u>

Suppose that G is a comparability graph and for $v \in V(G)$, $G_{\Gamma(v)} = P$ is a PG . Let D be a transitive image of G . We then label the elements of $\Gamma(v)$ from 1 to ℓ as in Lemma 3 and 'switch' them in D according to this order to get $D_{\Gamma(v)}$ which by Lemma 3 is a transitive image of $P = G_{\Gamma(v)}$. Using the construction of Pnueli et al [6] where $H = G_{\Gamma(v)}$ and $\vec{H} = D_{\Gamma(v)}$ we can get a permutation diagram $C(\Gamma(v))$ for $G_{\Gamma(v)}$ which induces in it the orientation $D_{\Gamma(v)}$ (see Remark 1 after Lemma 1).

Since $D_{\Gamma(v)}$ was obtained from D by switching $1,\ldots,\ell$, we can now switch the vertices $\ell,\ell-1,\ldots,1$ in $D_{\Gamma(v)}$ in this order to obtain D . Similarly, in $C(\Gamma(v))$, we can switch the chords $\bar{\ell}$, $\bar{\ell}-1,\ldots,\bar{1}$ in this order and obtain by Lemmas 4 and 5 a diagram C which is represented by G . Therefore G is a CPG. □

<u>Remark 2</u>: Note that in the 'only if' part we chose any transitive orientation D of G and obtained a diagram C for G such that $\vec{G}$ (the orientation induced by C in G) is equal to D . We therefore conclude that for every transitive orientation D of a CPG G there exists a diagram C , which induces in G the orientation D.

Based on the theorem, we have a recognition algorithm for a CPG which runs as follows:

<u>Step 1</u>: Check that G is a comparability graph using the algorithm of [4]. This requires $O(\Delta\cdot|E|)$ steps. In case of a negative answer 'stop', G is not a CPG .

<u>Step 2</u>: Choose an arbitrary vertex v and obtain $G_{\Gamma(v)}$ by 'switching' the neighbours of v . This requires $O(n^2)$ steps.

<u>Step 3</u>: Check that $G^{C}_{\Gamma(v)}$ (the complement of $G_{\Gamma(v)}$) is a comparability graph. If the answer is negative G is not a CPG, otherwise G is a CPG since $G_{\Gamma(v)}$ is a PG by Lemma 3 and the characterization of PG in [6].

Clearly the running time is dominated by $O(\Delta\cdot|E|)$.

§3 Permutation Representation and Finding a Maximum Independent Set

Consider two cocentric circles C_1 and C_2 with the numbers $1',\ldots,n'$ on C_1 and $P = \langle P(1),\ldots,P(n)\rangle$ on C_2. This information does not uniquely define a CPG G since i' on C_1 can be joined to i on C_2 in different ways. Therefore a permutation $P = \langle P(1),\ldots,P(n)\rangle$ read anticlockwise from C_2 of a circular permutation diagram C, cannot define by itself the graph G which represents C. However, if we fix chord $\bar{1}$ as a reference chord, we can construct a permutation P_1 by reading the endpoints of the chords on C_2 starting from the endpoint 1 proceeding anticlockwise i.e. $P_1(1) = 1$. We then mark in P_1 all elements which are endpoints of chords which intersect $\bar{1}$. Clearly this requires at most n extra bits. Let $M_1 = \langle i_1,\ldots,i_\ell\rangle$ be the subsequence of marked elements in P_1. (See Figure 6(a)).

Lemma 6: Given a permutation $P_1 = \langle 1,P(2),\ldots,P(n)\rangle$, and the subsequence M_1, then for $i < j$, $i \overset{G}{-} j$ if and only if:

(a) $i, j \in P_1 - M_1$ and $P_1^{-1}(i) > P_1^{-1}(j)$

or

(b) $i, j \in M_1$ and $P_1^{-1}(i) > P_1^{-1}(j)$

or

(c) $i \in M_1$ and $j \in P_1 - M_1$ and $P_1^{-1}(i) < P_1^{-1}(j)$

or

$j \in M$ and $i \in P_1 - M_1$ and $P_1^{-1}(i) < P_1^{-1}(j)$

<u>Proof</u>: (a) Since $i < j$, i' precedes j' on C_1 moving from $1'$ anticlockwise. Suppose $i \overset{G}{-} j$, then on C_2 j must precede i anticlockwise therefore $P_1^{-1}(j) < P_1^{-1}(i)$. Conversely, if $i \overset{G}{\not-} j$, i precedes j on C_2 and $P_1^{-1}(i) < P_1^{-1}(j)$.

Cases (b) and (c) can be proved using similar arguments. □

For an illustration see Figure 6(a).

We can get a representing permutation P_i for G using an arbitrary chord $\bar{i} \neq \bar{1}$ as a reference chord by simply relabelling the endpoints of the chords on C_1 and C_2. The chord $\bar{i}$ must have its endpoints relabelled $1'$ on C_1 and 1 on C_2, the endpoints of $\bar{i} - 1$ are relabelled n' and n, and in general the chord $\bar{k}$ will now have endpoints $(k - i + 1)' \bmod n$ and $(k - i + 1) \bmod n$ for $k \neq i - 1$. (See Figure 6(b)). We then obtain P_i as we obtained P_1 previously.

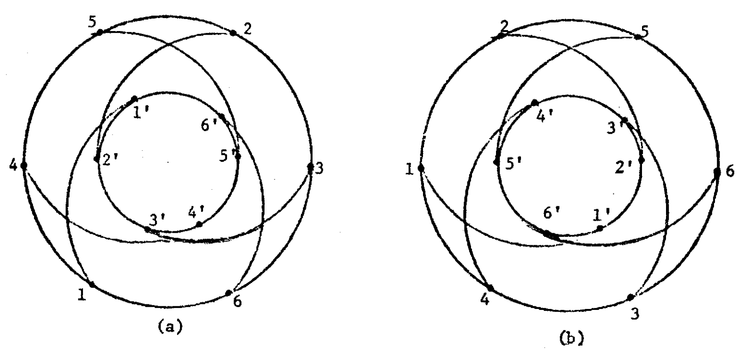


Figure 6. $P_1 = <1,6,3,2,5,4>$ and

$P_4 = <1,4,3,6,5,2>$

Given a diagram for a CPG G we can find a maximum independent set in G using the permutations $P_1,\ldots,P_n$ as follows. We observe that an independent set in G which contains vertex i has all its elements in $G - \Gamma(i)$. Therefore it corresponds to an increasing subsequence in $P_i - M_i$ by Lemma 6 , where an increasing subsequence in a permutation P is a sequence of elements $x_1<x_2<\ldots<x_k$ with $P^{-1}(x_1)<P^{-1}(x_2)<\ldots<P^{-1}(x_k)$.

Let I_i be a maximum independent set which includes vertex i . Then the elements of I_i correspond to a longest increasing subsequence in $P_i - M_i$. A longest increasing subsequence can be found in $O(n \log_2 n)$ steps using an algorithm by Fredman [2]. The maximum independent set in G is $\max_i\{|I_i| \ / \ i = 1,2,\ldots,n\}$ and can be found in $O(n^2 \log_2 n)$ steps since constructing each P_i requires only $O(n)$ steps. This compares favourably with Golumbic's algorithm [4] for finding a maximum independent set in a comparability graph which requires $O(|V| \cdot |E|)$ steps.

From a computational point of view it is useful to be able to find $P_1,\ldots,P_n$ for a CPG without explicitly constructing the diagram. This can be done by using the directed graph $D_{\Gamma(v)}$ used in the recognition algorithm. We recall that $D_{\Gamma(v)}$ is a permutation graph which represents $C(\Gamma(v))$ where v is an isolated vertex in $D_{\Gamma(v)}$. In order to construct a diagram for G , we only 'switch' chords in $C(\Gamma(v))$. This operation clearly does not change the positions of the endpoints of the chords on C_1 or C_2 . Therefore we can construct a permutation P_1 which represents G , by labelling the chord $\bar{v}$ by $\bar{1}$ and finding $\langle P(2),\ldots,P(n)\rangle$ on $\{2,\ldots,n\}$ which represents $D_{\Gamma(v)} - v$ using the algorithm of [6] .

Then let $P_1 = \langle 1, P(2), \ldots, P(n) \rangle$ where M_1 is the sequence of endpoints of chords intersecting $\bar{v}$ (i.e. vertices in $\Gamma(v)$) . To get P_i $i \neq 1$ from P_1 we proceed as follows:

(a) Find the elements in P_1 which represent endpoints of chords which intersect $\bar{i}$. This requires $n - 1$ comparisons using the rules of Lemma 6. Call those elements M_i' .

(b) We relabel the elements of P_1 such that i becomes 1 , $i + 1$ becomes 2 etc. and $i - 1$ becomes n, as we did previously to obtain P_i when the diagram was given. This can be done by changing P_1 to the permutation P_1' defined as follows: For $k = 1, 2, \ldots, n$

$$P_1'(k) = (P_1(k) - i + 1) \bmod n \quad \text{if} \quad P_1(k) \neq i - 1$$

$$P_1'(k) = n \quad \text{if} \quad P_1(k) = i - 1$$

In P_i the set of marked elements M_i consists of exactly the relabelled elements of M_i' .

(c) Read cyclically P_1' from position $P_1^{-1}(i)$ (which in P_1' contains the number 1) to obtain P_i. This is equivalent to reading the permutation P_1' from C_2 starting at the endpoint 1 (previously in P_1 this endpoint was labelled i), i.e. for $v = 1, \ldots, n$ $P_i(k) = P_1'(\ell)$ where $\ell = (P_1'^{-1}(1) + k - 1) \bmod n$. For example consider the cycle of length 6 (See Figure 6).

$$P_1 = (1, 6, 3, 2, 5, 4)$$

$$M_1 = (2, 4) \quad .$$

We construct P_4 as follows:

Step (a) $M_4' = (1, 3)$

Step (b) $P_1' = (4, 3, 6, 5, 2, 1)$

$M_4 = (4, 6)$ and finally

Step (c) $P_4 = (1, 4, 3, 6, 5, 2)$.

In this case by symmetry $P_2 = P_4 = P_6$ and $P_1 = P_3 = P_5$. The maximum independent set is of size 3 since $|I_i| = 3$ for $i = 1,\dots,6$, e.g. $I_4 = \langle 1, 3, 5 \rangle$. □

Summary

A new class of graphs called circular permutation graphs was introduced. It was shown that this class generalizes the class of permutation graphs which were studied in [6], and it is embedded in the class of comparability graphs . The characterization uses properties of permutation graphs and comparability graphs and is of the same complexity as the known recognition algorithms for these types of graphs. Also it was shown that CPG can be represented as permutations with marked elements, this representation was used in an efficient algorithm for determining a maximum independent set.

References

[1] Even, S., Pnueli, A., Lempel, A. : Permutation graphs and transitive graphs. J. ACM 19, 400-410 (1972).

[2] Fredman, M. L. : On Computing the length of longest increasing subsequences. Discrete Math. 11, (1975), 29-35.

[3] Gilmore, P. C., Hoffman, A. J.: A characterization of comparability graphs and of interval graphs, Can. J. Math 16, 539-548 (1964).

[4] Golumbic, M.C. : The Complexity of Comparability Graph : Recognition and Coloring, Computing 18, 199-208, (1977).

[5] Golumbic, M.C. : Comparability graphs and a new matroid, J. Comb. Th., Series B 22, 68-90 (1977).

[6] Pnueli, A, Lempel, A., Even, S.: Transitive orientation of graphs and identification of permutation graphs, Can. J. Math. 23, 160-175 (1971).

CIRCULAR PERMUTATION GRAPHS*

by

D. Rotem[+] and J. Urrutia[§]

Research Report CS-79-32

Department of Computer Science

University of Waterloo
Waterloo, Ontario, Canada

[+] Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

[§] Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

* Some of the results of this paper appear in J. Urrutia's Ph.D. thesis (in preparation).

This research was supported by the National Sciences and Engineering Research Council of Canada under grant numbers A8142 and A3055.

ABSTRACT

A new class of graphs called circular permutation graphs is introduced and characterized. A circular permutation diagram for a permutation $P(1),\ldots,P(n)$ consists of two circles C_1 and C_2, the numbers $1',2',\ldots,n'$ and $P(1),\ldots,P(n)$ on C_1 and C_2 respectively and a set of n chords $\bar{1},\bar{2},\ldots,\bar{n}$ connecting i to i'. A graph G is a circular permutation graph if there is a labelling of $V(G)$ with $\{1,\ldots,n\}$ such that i is adjacent to j iff $\bar{i}$ and $\bar{j}$ intersect. Circular permutation graphs generalize permutation graphs ([1],[6]) and are embedded in the set of comparability graphs [3]. The characterization leads to a recognition algorithm which requires $O(\Delta\cdot|E|)$ steps where Δ is the maximum degree of a vertex. Also, an algorithm for finding a maximum independent set in a CPG which requires $O(n^2 \log_2 n)$ steps is presented.

Keywords: Permutation graphs, comparability graphs, longest increasing subsequence, partial order.

CR Categories: 5.32

Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ of unordered pairs of vertices. We consider here only graphs with no multiple edges or self loops. For two vertices $x, y \in V(G)$, we denote $x \overset{G}{-} y$ if $(x,y) \in E(G)$ otherwise $x \overset{G}{\not -} y$. The set $\Gamma_G(v) = \{x \in V(G) \mid v \overset{G}{-} x\}$. A graph D is a directed graph if its edge set consists of ordered pairs $<x,y>$. We denote $x \overset{D}{\rightarrow} y$ if $<x,y> \in E(D)$. For a vertex $v \in D$, $\Gamma_D^-(v) = \{x \in V(D) \mid x \overset{D}{\rightarrow} v\}$ and $\Gamma_D^+(v) = \{x \in V(D) \mid v \overset{D}{\rightarrow} x\}$. The cardinality of a set S is denoted by $|S|$.

A permutation diagram $D(P)$ for a permutation $P = <P(1),P(2),\ldots,P(n)>$ on $\{1,2,\ldots,n\}$, consists of two parallel lines L_1 and L_2. The numbers $1',2',\ldots,n'$ appear on L_1 in increasing order and $P(1),P(2),\ldots,P(n)$ appear on L_2 according to their order in P, the number i' on L_1 is joined to i on L_2 with a segment $\bar{i}$, for $1 \le i \le n$. (See Figure 1).

A graph $G(P)$ with n vertices represents $D(P)$ if there exists a labelling of its vertex set with $\{1,2,\ldots,n\}$ such that $i \overset{G(P)}{-\!-} j$ if and only if $\bar{i}$ intersects $\bar{j}$ in $D(P)$. A graph G which represents at least one permutation diagram is called a Permutation Graph (PG).

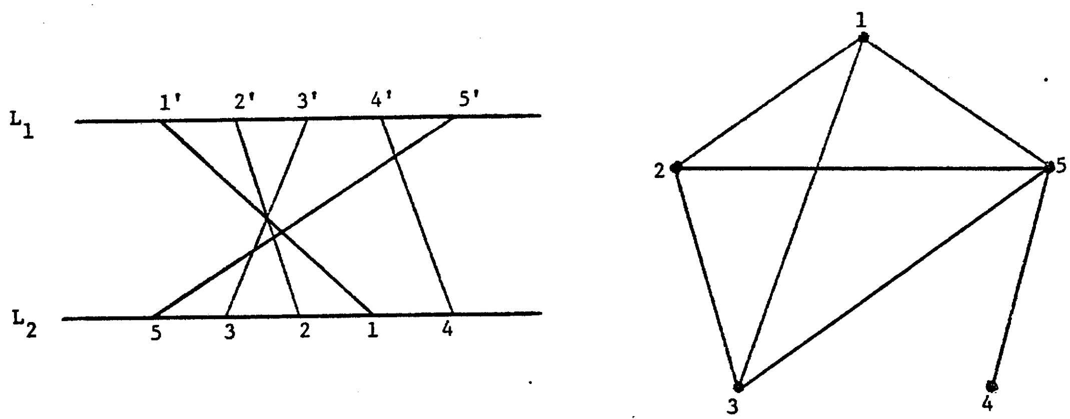


Figure 1. A permutation diagram and a graph for P = <5,3,2,1,4>.

The class of PG was studied in Pnueli et al [6] and Even et al [1] where they were applied to model and solve problems concerning memory allocation and circuit layout. It has also been shown in [6] that PG can be characterized as those graphs where both the graph and its complement are comparability graphs. A graph G is a comparability graph, ([3],[5],[6], also called transitively orientable graph) if its edges can be oriented such that for $x,y,z \in V(G)$ $x \overset{G}{\rightarrow} y$ and $y \overset{G}{\rightarrow} z$ implies $x \overset{G}{\rightarrow} z$. It can also be shown that PG represent partial orders of dimension 2 except for the complete graph which is a PG and represents a partial order of dimension 1 .

In this paper a new class of graphs which generalize PG is introduced and characterized, these are called Circular Permutation Graphs.

A circular permutation diagram $C(P)$ for a permutation P on $\{1,2,\ldots,n\}$ consists of two cocentric circles C_1 and C_2, C_1 contained in C_2. The numbers $1',2',\ldots,n'$ and $P(1),P(2),\ldots,P(n)$ appear on C_1 and C_2 respectively in the anticlockwise direction. We now choose a set of n chords $\bar{1},\bar{2},\ldots,\bar{n}$, totally contained in the annular region between C_1 and C_2 such that chord $\bar{i}$ joins i' on C_1 to i on C_2 and for $i \neq j$, $\bar{i}$ and $\bar{j}$ intersect each other at most once (Figure 2). As in the case of PG, a graph G represents a circular permutation diagram $C(P)$ if its vertex set $V(G)$ can be labelled $\{1,2,\ldots,n\}$ such that $i \overset{G}{-} j$ if and only if $\bar{i}$ intersects $\bar{j}$ in $C(P)$. A graph G is called a Circular Permutation Graph (CPG) if it represents at least one circular permutation diagram.

In Section 2 a characterization of CPG is given which leads to a recognition algorithm which requires $O(\Delta\cdot|E(G)|)$ steps where Δ is the maximum degree of a vertex. A representation of CPG using a defining permutation is given in Section 3 and is used in an algorithm for finding a maximum independent set.

§1 Preliminaries

In [6] it was shown that if H is a PG its corresponding permutation diagram can be constructed as follows: We first construct a permutation P (we denote by $P^{-1}(\ell)$ the position of the element ℓ in P) as follows:

1. Find transitive orientations $\vec{H}$ and $\vec{H}^C$ for H and its complement H^C.
2. Construct the tournament $T_1 = \vec{H} \cup \vec{H}^C$ and label $V(H)$ by $1,2,\ldots,n$ such that a vertex v is labelled by $|\Gamma_T^-(v)| + 1$.
3. Reverse the orientation $\vec{H}^C$ to obtain $\overleftarrow{H}^C$ and consider the tournament $T_2 = \vec{H} \cup \overleftarrow{H}^C$. For a vertex labelled ℓ in step 2, let $P^{-1}(\ell) = |\Gamma_{T_2}^+(\ell)| + 1$. The permutation diagram can be now constructed using $P = \langle P(1),\ldots,P(n)\rangle$.

It is shown in [6] that $i \overset{G}{-} j$ if and only if $(i - j)(P^{-1}(i) - P^{-1}(j)) < 0$, or in words i and j form an 'inversion' in P. A PG G can therefore be represented by a permutation P obtained by this construction.

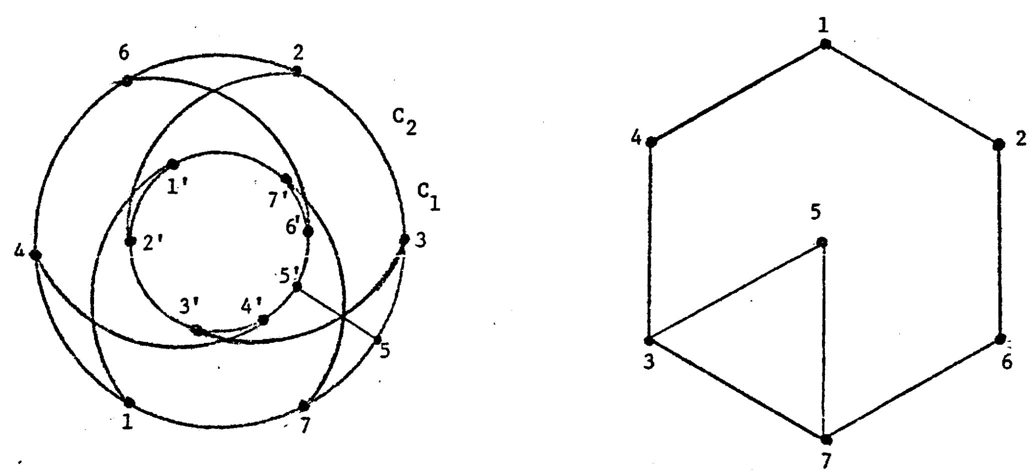


Figure 2. A circular permutation diagram and its corresponding graph.

Clearly, every PG is also a CPG since every permutation diagram can be transformed into a circular permutation diagram where L_1 and L_2 are mapped into the cocentric circles C_1 and C_2 .

However, a graph G which is a CPG is also a PG, if and only if there exists a circular permutation diagram C represented by G , such that it is possible to draw a chord $\bar{v}$ with endpoints on C_1 and C_2 which does not intersect the chords $\bar{1},\ldots,\bar{n}$ of C . In this case, the circular permutation diagram C can be opened along $\bar{v}$ to form a permutation diagram. From this it follows that a CPG which contains an isolated vertex is a PG . (See Figure 3)

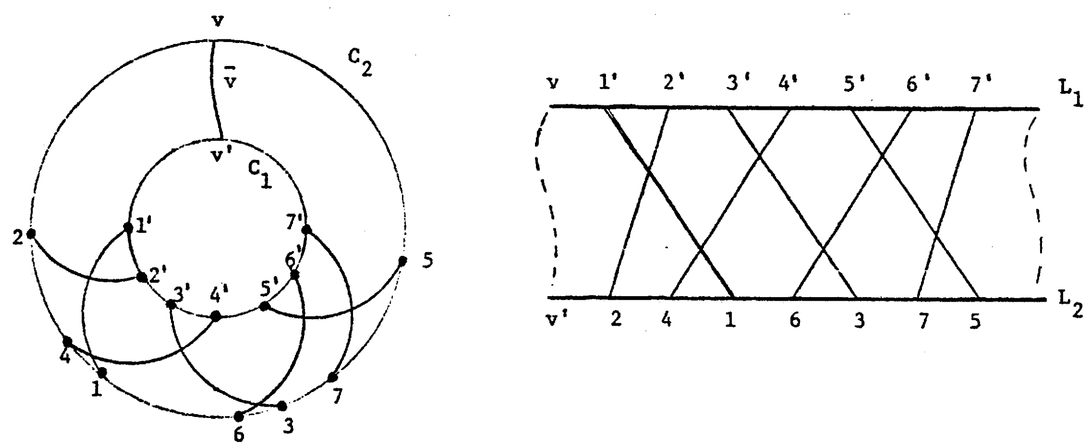
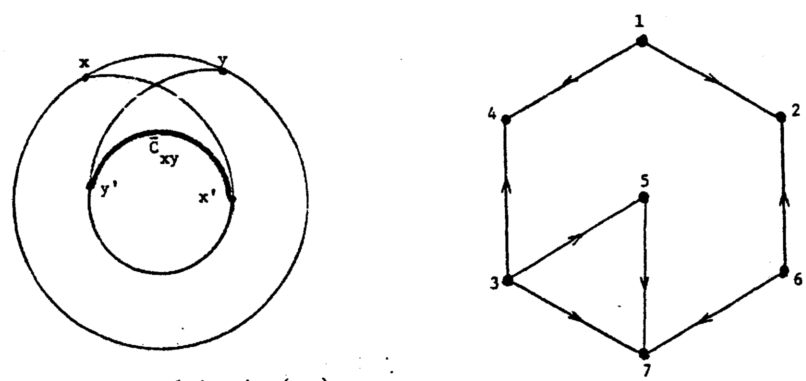


Figure 3. A circular permutation diagram opened along $\bar{v}$.

Examples of a CPG which is not a PG are all the even cycles C_{2n} with $n \geq 3$. Since every permutation diagram can be transformed into a circular permutation diagram, in what follows a permutation diagram will be taken to mean the circular permutation diagram obtained by the above mapping of L_1 and L_2 into C_1 and C_2 .

§2 Characterization of Circular Permutation Graphs

Given a circular permutation diagram C and its representing graph G , we can induce an orientation in G using C as follows: For every pair of intersecting chords $\bar{x}$ and $\bar{y}$ in C with endpoints x', y' on C_1 , consider the region which is bounded by the arc $\bar{C}_{xy}$ on C_1 , $\bar{x}$ and $\bar{y}$ (See Fig. 4). We direct $x \overset{G}{\rightarrow} y$ if x' precedes y' on $\bar{C}_{xy}$ in the anticlockwise direction. In this case we say that $\bar{x}$ intersects $\bar{y}$ anticlockwise or equivalently $\bar{y}$ intersects $\bar{x}$ clockwise. This induced orientation of G is denoted by $\vec{G}$. A chord $\bar{x}$ is a source (sink) in C if all other chords which intersect it do so clockwise(anticlockwise).



Orientation of the edge (x,y).

Orientation of G of Fig. 2

Figure 4.

Lemma 1: A CPG is a comparability graph.

Proof: We show that $\vec{G}$ is transitive. Let $x,y,z \in V(G)$ such that $x \overset{\vec{G}}{\rightarrow} y$, $y \overset{\vec{G}}{\rightarrow} z$. Since $\bar{x}$ intersects $\bar{y}$ anticlockwise and $\bar{z}$ intersects $\bar{y}$ clockwise, then $\bar{x}$ must intersect $\bar{z}$, and x' precedes

z' in the anticlockwise direction on $\bar{C}_{xz}$, which implies $x \overset{\vec{G}}{\to} z$. □

Remark 1: It can be seen that for the special case where G is a PG , the permutation diagram D(P) obtained by the construction of [6] as described in the introduction, induces in G the orientation $\vec{G}$ which is exactly the one used in the construction of the tournaments T_1 and T_2.

The operation 'switch' on a vertex $v \in V(G)$, which plays an important role in the characterization of CPG, is defined as follows: Connect v to all vertices x such that $x \in V(G) - \Gamma_G(v)$ and delete the edges (v,y) , $y \in \Gamma_G(v)$. Let G_v be the graph obtained from G by 'switching' $v \in V(G)$, then clearly $(G_v)_v = G$. Similarly, for a set $S \subset V(G)$ the graph G_S is obtained from G by 'switching' all vertices of S one at a time. It is easy to see that G_S is uniquely defined independent of the order in which the members of S are 'switched', also $(G_S)_S = G$. In what follows $\Gamma_G(v)$ is simply called $\Gamma(v)$.

We can now state the characterization of CPG in the following theorem.

Theorem : Let v be a vertex in G , then G is a CPG if and only if

(a) G is a comparability graph

(b) $G_{\Gamma(v)}$ is a PG .

Before proceeding with the proof, we develop some more properties of the 'switch' operation which are given in the following lemmas.

For a directed graph D , we define the 'switch' operation only with respect to sources or sinks of D . Let v be a source or a sink, we first 'switch' v as defined previously ignoring the directions in D , then D_v is oriented such that v becomes a source of D_v if it was a sink of D and vice versa, all other directions of edges in D_v are the same as in D .

Lemma 2: Let D be a transitive image of a comparability graph G and let v be a source or a sink in D , then D_v is a transitive image of G_v .

Proof: Let v be a source in D , and assume that there exist three vertices $x, y, z \in D_v$ such that $x \overset{D_v}{\rightarrow} y$, $y \overset{D_v}{\rightarrow} z$ and $x \overset{D_v}{\not\rightarrow} z$. One of x, y or z is v , and since v is a sink it can only be that $v = z$ and we cannot have $z \overset{D_v}{\rightarrow} x$. Therefore $x \overset{D_v}{\not\rightarrow} z$ and $z \overset{D}{\rightarrow} x$. But since $x \overset{D}{\rightarrow} y$ and D is transitive we have $z \overset{D}{\rightarrow} y$ which contradicts $y \overset{D_v}{\rightarrow} z$ by the definition of the 'switch' operation for directed graphs. Similar arguments can be used when v is a sink of D. □

Lemma 3: If G is a comparability graph then $G_{\Gamma(v)}$ is also a comparability graph.

Proof: Let D be a transitive orientation of G , and v a vertex in G (not necessarily a source or a sink in D). Assume $|\Gamma(v)| = \ell$ and $|\Gamma_D^-(v)| = k$. There exists a vertex $y_1 \in \Gamma_D^-(v)$ which is a source in D since $\Gamma_D^-(v)$ induces an acylic subgraph of D , and all edges with exactly one endpoint $x \in \Gamma_D^-(v)$ are outdirected from x. Label this vertex by 1. Similarly in $D - \{1\}$, there exists a source $y_2 \in \Gamma_D^-(v)$ we label this vertex by 2.

By using this elimination of sources procedure iteratively k times where i is a source of $D - \{1,2,\ldots,i-1\}$ in $\Gamma_D^-(v)$, we can label the vertices of $\Gamma_D^-(v)$ from 1 to k. We then find a sink of $D - \{1,\ldots,k\}$ in $\Gamma_D^+(v)$ and label it $k + 1$. Proceeding in a similar way as for $\Gamma_D^-(v)$, we can label the vertices of $\Gamma_D^+(v)$ with $k+1,k+2,\ldots,\ell$ by choosing a sink of $D - \{1,2,\ldots,k+s\}$ in $\Gamma_D^+(v)$ and label it $k + s + 1$ $(k + s + 1 \le \ell)$. We now show that for $0 \le i < k$, $i + 1$ is a source in $D^i = (D^{i-1})_i$ where $D^0 = D$, (D^i is obtained by 'switching' i in D^{i-1}), and for $k \le i < \ell$, $i + 1$ is a sink in D^i. If this is shown, then since for $0 \le i \le \ell$ D^i is a transitive image of $(G_S)_i$ where $S = \{1,2,\ldots,i-1\}$ (by succesive applications of Lemma 2), it follows that D^ℓ is a transitive image of $G_{\Gamma(v)}$.

To see that $1 \le i \le k$ is a source in D^{i-1} we use induction on i. For $i = 1$, vertex 1 is a source in $D^0 = D$ by its choice in the source elimination process. Assume that i is a source in D^{i-1} for $i = m < k$. Consider the graph D^i, and assume $i + 1 \overset{D^i}{—} j$ where $j \le i$ and $j \in \Gamma_D^-(v)$. Then the edge connecting $i + 1$ to j was introduced when we 'switched' j in D^{j-1}, and since j was a source in D^{j-1} it follows that it became a sink in D^j and therefore $i + 1 \overset{D_j}{\rightarrow} j$. This edge is not changed again in the construction of $D^{j+1},\ldots,D^i$ therefore $i + 1 \overset{D_i}{\rightarrow} j$. For all other vertices $v \in D^i$ we have $i+1 \overset{D_i}{\to} v$ by the source elimination labelling (i.e. $i+1$ is a source in $D - \{1,2,\ldots,i\}$), it follows that $i+1$ is a source in D^i . Note that since D is a transitive graph, for every vertex $x \in \Gamma_D^-(v)$ and every vertex $y \in \Gamma_D^+(v)$ we have $x \overset{D}{\to} v \overset{D}{\to} y$ which implies $x \overset{D}{\to} y$. Therefore in D^k , no edge exists between $\Gamma_D^-(v)$ and $\Gamma_D^+(v)$, and $k+1$ which is a sink in D is also a sink in D^k . We can now apply a similar induction argument to show that $i+1$ is a sink in D^i for $k \le i < \ell$, which completes the proof. □

We now define the 'switch' operation for chords in a circular permutation diagram which will be proved to be closely related to the 'switch' operation in directed graphs.

For a chord $\bar{v}$ in C with endpoints v' on C_1 and v on C_2 we say that $\bar{v}$ can be 'switched' to the chord $\bar{v}'$ if $\bar{v}'$ joins v' and v , and intersects exactly those chords which do not intersect $\bar{v}$ (See Figure 5), it can also be seen that $\bar{v}'$ does not intersect any chord more than once. Let $C(\bar{v})$ denote the diagram obtained from C by 'switching' $\bar{v}$. Then from the above definition it follows that if G represents C then G_v represents $C(\bar{v})$.

Let G be a CPG which represents C .

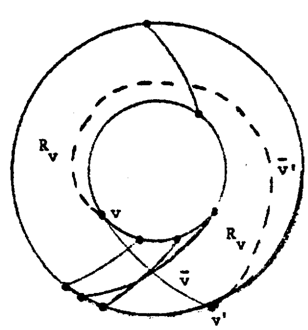


Figure 5. Switching the chord $\bar{v}$.

<u>Lemma 4</u>: A chord $\bar{v}$ can be 'switched' in C , if and only if it is a source or a sink in C .

<u>Proof</u>: If $\bar{v}$ is a source in C , then all chords which intersect $\bar{v}$ in C do so clockwise. Those chords divide the annular region between C_1 and C_2 into connected regions one of which contains the points v' and v on its boundary, we call this region R_v (See Fig. 5). The chord $\bar{v}'$ can be drawn such that it is totally contained in R_v and does not intersect any chord which intersects $\bar{v}$. To see that $\bar{v}'$ intersects all chords which $\bar{v}$ does not intersect, we consider the closed curve γ obtained by the union of $\bar{v}$ and $\bar{v}'$. Clearly any chord $\bar{x}$ in C intersects γ , and therefore if $\bar{x}$ does not intersect γ in $\bar{v}$ it must do so in $\bar{v}'$, and therefore $\bar{v}$ can be switched. Similarly it can be shown that $\bar{v}$ can be switched if it is a sink.

Conversely, assume that $\bar{v}$ is not a source or a sink, then there exist two chords $\bar{x}$ and $\bar{y}$ which cross $\bar{v}$ clockwise and anti-clockwise respectively. Therefore $\bar{x}$ intersects $\bar{y}$ and v' is an interior point on the arc $\bar{C}_{xy}$. Therefore any chord which joins v' and v must cross at least one of $\bar{x}$ or $\bar{y}$ contrary to the definition of the 'switch' operation. □

We note that $\bar{v}'$ can be switched in $C(\bar{v})$ back to $\bar{v}$, therefore $\bar{v}'$ is a source or a sink in $C(\bar{v})$. Clearly $\bar{v}'$ is a source in $C(\bar{v})$ if and only if it was a sink in C . From this and Lemma 4 we conclude:

<u>Lemma 5</u>: If G is a CPG which represents C and $D = \vec{G}$, then G represents $C(\bar{v})$ and $\vec{G}_v = D_v$. □

In words, the transitive orientation induced by $C(\bar{v})$ on G_v is equal to the orientation obtained by 'switching' v in the directed graph D .

We are now in a position to prove the characterization theorem.

Proof: The 'if' part

Since G is a CPG there exists a diagram C which is represented by G, and by Lemma 1 C induces a transitive orientation $\vec{G}$ on G which proves (a).

For any vertex $v \in \vec{G}$ with $|\Gamma(v)| = \ell$ we can order and label the vertices of $\Gamma(v)$ from $1,2,\ldots,\ell$ using the source and sink elimination procedure of Lemma 3 . A vertex i can be switched in G if it is a source or a sink (by definition of 'switching' in a directed graph) and by Lemma 4 its corresponding chord $\bar{i}$ can be switched in C, where $C(\bar{i})$ is represented by G_i by Lemma 5 . We can therefore 'switch' the chords $\bar{1},\bar{2},\ldots,\bar{\ell}$ in this order in C to obtain a diagram $C(\Gamma(v))$ which is represented by $G_{\Gamma(v)}$. Therefore $G_{\Gamma(v)}$ is a CPG , and v is an isolated vertex in it because all vertices in $\Gamma(v)$ have been 'switched' in G , by the remarks in the introduction $G_{\Gamma(v)}$ is a PG which proves (b).

The 'only if' part

Suppose that G is a comparability graph and for $v \in V(G)$, $G_{\Gamma(v)} = P$ is a PG . Let D be a transitive image of G . We then label the elements of $\Gamma(v)$ from 1 to ℓ as in Lemma 3 and 'switch' them in D according to this order to get $D_{\Gamma(v)}$ which by Lemma 3 is a transitive image of $P = G_{\Gamma(v)}$. Using the construction of Pnueli et al [6] where $H = G_{\Gamma(v)}$ and $\vec{H} = D_{\Gamma(v)}$ we can get a permutation diagram $C(\Gamma(v))$ for $G_{\Gamma(v)}$ which induces in it the orientation $D_{\Gamma(v)}$ (see Remark 1 after Lemma 1).

Since $D_{\Gamma(v)}$ was obtained from D by switching $1,\ldots,\ell$, we can now switch the vertices $\ell,\ell-1,\ldots,1$ in $D_{\Gamma(v)}$ in this order to obtain D . Similarly, in $C(\Gamma(v))$, we can switch the chords $\bar{\ell}$, $\bar{\ell}-1,\ldots,\bar{1}$ in this order and obtain by Lemmas 4 and 5 a diagram C which is represented by G . Therefore G is a CPG. □

<u>Remark 2</u>: Note that in the 'only if' part we chose any transitive orientation D of G and obtained a diagram C for G such that $\vec{G}$ (the orientation induced by C in G) is equal to D . We therefore conclude that for every transitive orientation D of a CPG G there exists a diagram C , which induces in G the orientation D.

Based on the theorem, we have a recognition algorithm for a CPG which runs as follows:

<u>Step 1</u>: Check that G is a comparability graph using the algorithm of [4]. This requires $O(\Delta\cdot|E|)$ steps. In case of a negative answer 'stop', G is not a CPG .

<u>Step 2</u>: Choose an arbitrary vertex v and obtain $G_{\Gamma(v)}$ by 'switching' the neighbours of v . This requires $O(n^2)$ steps.

<u>Step 3</u>: Check that $G^c_{\Gamma(v)}$ (the complement of $G_{\Gamma(v)}$) is a comparability graph. If the answer is negative G is not a CPG, otherwise G is a CPG since $G_{\Gamma(v)}$ is a PG by Lemma 3 and the characterization of PG in [6]

Clearly the running time is dominated by $O(\Delta\cdot|E|)$.

§3 Permutation Representation and Finding a Maximum Independent Set

Consider two cocentric circles C_1 and C_2 with the numbers $1',\ldots,n'$ on C_1 and $P = \langle P(1),\ldots,P(n)\rangle$ on C_2. This information does not uniquely define a CPG G since i' on C_1 can be joined to i on C_2 in different ways. Therefore a permutation $P = \langle P(1),\ldots,P(n)\rangle$ read anticlockwise from C_2 of a circular permutation diagram C, cannot define by itself the graph G which represents C. However, if we fix chord $\bar{1}$ as a reference chord, we can construct a permutation P_1 by reading the endpoints of the chords on C_2 starting from the endpoint 1 proceeding anticlockwise i.e. $P_1(1) = 1$. We then mark in P_1 all elements which are endpoints of chords which intersect $\bar{1}$. Clearly this requires at most n extra bits. Let $M_1 = \langle i_1,\ldots,i_\ell\rangle$ be the subsequence of marked elements in P_1. (See Figure 6(a)).

Lemma 6: Given a permutation $P_1 = \langle 1,P(2),\ldots,P(n)\rangle$, and the subsequence M_1, then for $i < j$, $i \overset{G}{-} j$ if and only if:

(a) $i, j \in P_1-M_1$ and $P_1^{-1}(i) > P_1^{-1}(j)$

or

(b) $i, j \in M_1$ and $P_1^{-1}(i) > P_1^{-1}(j)$

or

(c) $i \in M_1$ and $j \in P_1-M_1$ and $P_1^{-1}(i) < P_1^{-1}(j)$

or

$j \in M$ and $i \in P_1-M_1$ and $P_1^{-1}(i) < P_1^{-1}(j)$

<u>Proof</u>: (a) Since $i < j$, i' precedes j' on C_1 moving from $1'$ anticlockwise. Suppose $i \overset{G}{-} j$, then on C_2 j must precede i anticlockwise therefore $P_1^{-1}(j) < P_1^{-1}(i)$. Conversely, if $i \overset{G}{\not -} j$, i precedes j on C_2 and $P_1^{-1}(i) < P_1^{-1}(j)$.

Cases (b) and (c) can be proved using similar arguments. □

For an illustration see Figure 6(a).

We can get a representing permutation P_i for G using an arbitrary chord $\bar{i} \neq \bar{1}$ as a reference chord by simply relabelling the endpoints of the chords on C_1 and C_2 . The chord $\bar{i}$ must have its endpoints relabelled $1'$ on C_1 and 1 on C_2 , the endpoints of $\bar{i} - 1$ are relabelled n' and n , and in general the chord $\bar{k}$ will now have endpoints $(k - i + 1)' \bmod n$ and $(k - i + 1) \bmod n$ for $k \neq i - 1$. (See Figure 6(b)). We then obtain P_i as we obtained P_1 previously.

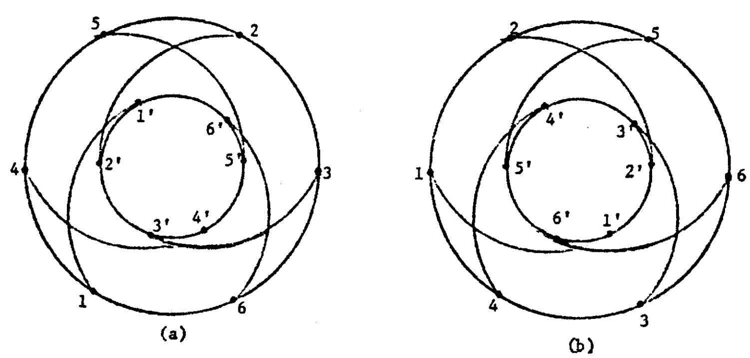


Figure 6. $P_1 = \langle 1,6,3,2,5,4 \rangle$ and

$P_4 = \langle 1,4,3,6,5,2 \rangle$

Given a diagram for a CPG G we can find a maximum independent set in G using the permutations $P_1,\ldots,P_n$ as follows. We observe that an independent set in G which contains vertex i has all its elements in $G - \Gamma(i)$. Therefore it corresponds to an increasing subsequence in $P_i - M_i$ by Lemma 6 , where an increasing subsequence in a permutation P is a sequence of elements $x_1 < x_2 < \ldots < x_k$ with $P^{-1}(x_1) < P^{-1}(x_2) < \ldots < P^{-1}(x_k)$.

Let I_i be a maximum independent set which includes vertex i . Then the elements of I_i correspond to a longest increasing subsequence in $P_i - M_i$. A longest increasing subsequence can be found in $O(n \log_2 n)$ steps using an algorithm by Fredman [2]. The maximum independent set in G is $\max_i\{|I_i| \,/\, i = 1,2,\ldots,n\}$ and can be found in $O(n^2 \log_2 n)$ steps since constructing each P_i requires only $O(n)$ steps. This compares favourably with Golumbic's algorithm [4] for finding a maximum independent set in a comparability graph which requires $O(|V| \,.\, |E|)$ steps.

From a computational point of view it is useful to be able to find $P_1,\ldots,P_n$ for a CPG without explicitly constructing the diagram. This can be done by using the directed graph $D_{\Gamma(v)}$ used in the recognition algorithm. We recall that $D_{\Gamma(v)}$ is a permutation graph which represents $C(\Gamma(v))$ where v is an isolated vertex in $D_{\Gamma(v)}$. In order to construct a diagram for G , we only 'switch' chords in $C(\Gamma(v))$. This operation clearly does not change the positions of the endpoints of the chords on C_1 or C_2 . Therefore we can construct a permutation P_1 which represents G , by labelling the chord $\bar{v}$ by $\bar{1}$ and finding $\langle P(2),\ldots,P(n)\rangle$ on $\{2,\ldots,n\}$ which represents $D_{\Gamma(v)} - v$ using the algorithm of [6] .

Then let $P_1 = \langle 1, P(2), \ldots, P(n) \rangle$ where M_1 is the sequence of end-points of chords intersecting $\bar{v}$ (i.e. vertices in $\Gamma(v)$) . To get P_i $i \neq 1$ from P_1 we proceed as follows:

(a) Find the elements in P_1 which represent endpoints of chords which intersect $\bar{i}$. This requires $n - 1$ comparisons using the rules of Lemma 6. Call those elements M_i' .

(b) We relabel the elements of P_1 such that i becomes 1 , $i + 1$ becomes 2 etc. and $i - 1$ becomes n, as we did previously to obtain P_i when the diagram was given. This can be done by changing P_1 to the permutation P_1' defined as follows: For $k = 1,2,\ldots,n$

$$P_1'(k) = (P_1(k) - i + 1) \bmod n \quad \text{if} \quad P_1(k) \neq i - 1$$

$$P_1'(k) = n \quad \text{if} \quad P_1(k) = i - 1$$

In P_i the set of marked elements M_i consists of exactly the relabelled elements of M_i' .

(c) Read cyclically P_1' from position $P_1^{-1}(i)$ (which in P_1' contains the number 1) to obtain P_i. This is equivalent to reading the permutation P_1' from C_2 starting at the endpoint 1 (previously in P_1 this endpoint was labelled i), i.e. for $v = 1,\ldots,n$ $P_i(k) = P_1'(\ell)$ where $\ell = (P_1'^{-1}(1) + k - 1) \bmod n$. For example consider the cycle of length 6 (See Figure 6).

$$P_1 = (1, 6, 3, 2, 5, 4)$$

$$M_1 = (2, 4) \quad .$$

We construct P_4 as follows:

Step (a) $M'_4 = (1, 3)$

Step (b) $P'_1 = (4, 3, 6, 5, 2, 1)$

$M_4 = (4, 6)$ and finally

Step (c) $P_4 = (1, 4, 3, 6, 5, 2)$.

In this case by symmetry $P_2 = P_4 = P_6$ and $P_1 = P_3 = P_5$. The maximum independent set is of size 3 since $|I_i| = 3$ for $i = 1,\ldots,6$, e.g. $I_4 = \langle 1, 3, 5 \rangle$. □

Summary

A new class of graphs called circular permutation graphs was introduced. It was shown that this class generalizes the class of permutation graphs which were studied in [6], and it is embedded in the class of comparability graphs . The characterization uses properties of permutation graphs and comparability graphs and is of the same complexity as the known recognition algorithms for these types of graphs. Also it was shown that CPG can be represented as permutations with marked elements, this representation was used in an efficient algorithm for determining a maximum independent set.

References

[1] Even, S., Pnueli, A., Lempel, A. : Permutation graphs and transitive graphs. J. ACM 19, 400-410 (1972).

[2] Fredman, M. L. : On Computing the length of longest increasing subsequences. Discrete Math. 11, (1975), 29-35.

[3] Gilmore, P. C., Hoffman, A. J.: A characterization of comparability graphs and of interval graphs, Can. J. Math 16, 539-548 (1964).

[4] Golumbic, M.C. : The Complexity of Comparability Graph : Recognition and Coloring, Computing 18, 199-208, (1977).

[5] Golumbic, M.C. : Comparability graphs and a new matroid, J. Comb. Th., Series B 22, 68-90 (1977).

[6] Pnueli, A, Lempel, A., Even, S.: Transitive orientation of graphs and identification of permutation graphs, Can. J. Math. 23, 160-175 (1971).