CONTINUOUS DATA TYPES*

by

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1. Introduction

Data types play a central role in programming and it is therefore important to find ways of giving semantic characterisations of data types. Some authors have suggested that data types are (many-sorted) algebras (ADJ[1], Guttag) and ADJ[1] have shown that data types may be characterised as a quotient algebra which is initial in the class of algebras satisfying a set of equations. This algebra is found by factoring a "term algebra" T_Σ by an appropriate congruence q and is denoted T_Σ/q.

A particular class of data types which is of additonal interest is the class of data types whose operators are continuous and whose set of objects is a complete partial order or complete lattice (Scott, ADJ[2]). These data types arise when considering any types with infinite objects. Circular lists, for example, can be treated as infinite objects of a continuous type (Reynolds). It has been shown (ADJ[2]) that the class of all such data types (hereafter called continuous data types) has an initial algebra, denoted CT_Σ, which is (intuitively) the algebra of finite and infinite terms. It is natural to ask whether the elegant characterisation of data types in terms of quotients given by ADJ[1] extends simply to continuous data types. In this paper we show that the quotient CT_Σ/q (where q is obtained from a set of equations in the usual way (ADJ[1])) is sometimes, but not always, initial in the class of continuous algebras satisfying the equations. Firstly, we show that in general the quotient CT_Σ/q does not admit a partial order which is consistent with the partial order on CT_Σ. Thus, even though CT_Σ/q is a Σ-algebra, it is not a member of the class of continuous Σ-algebras and hence cannot be initial in this class. We then define a function nf called a normaliser which is a continuous function that selects a normal form from each class in a congruence q. In order for such a function to exist, the congruence will have to have a property of continuity, namely that the congruence respects limits. (That is, if two directed sets are pairwise congruent, their least upper bounds must be congruent.) It is shown that, given any set of equations, there exists a unique least continuous congruence containing these equations. CT_Σ/q is then "made" into a partial order by defining a partial order relation on CT_Σ/q in terms of the relationship between normal forms. It is then possible to establish the main result of the paper, namely that if q is a continuous congruence generated by a set of equations and a normaliser exists, then CT_Σ/q is initial in the class of continuous Σ-algebras satisfying the equations. Hence continuous data types can be characterised as initial quotients of CT_Σ (just as data types were characterised as quotients of T_Σ) by giving a set of equations for the type (called the specification equations) and by finding a normaliser function for the type.

Finally, it is sometimes easier to find an algebra of normal or canonical terms for a data type than to find directly the normaliser function. We show that if such a normal algebra exists, then a normaliser function exists, and thus CT_Σ/q is initial in the appropriate class.

2. Relation to Other Work

Several authors have studied quotient algebras in some form (ADJ[1,3], Courcelle [1], Lehmann and Hennessy). ADJ[1] is concerned with the class of all Σ-algebras (rather than of continuous Σ-algebras), and it is the main results of ADJ[1] that have been generalised here, using the notion of normal forms. In Courcelle [1], Courcelle and Nivat investigate quotients of Σ-algebras taken from congruences that have been defined in terms of pre-orders (rather than simply the least congruence generated by a set of equations), but they do not examine the initiality of CT_Σ/q. Hennessy has shown, independently of the present work, that the completion of T_Σ/q is initial in the class of Σ-algebras satisfying q where q is the congruence obtained using Courcelle and Nivat's construction on pre-orders and the class of algebras of interest is expressed in terms of a set of <u>in</u>equations rather than with equations. As a consequence of the main theorem of this paper (theorem 12) the initial algebra of Hennessy will be isomorphic to CT_Σ/q when normal forms exist. (Note that a set of equations

$$\{t_1 = t_1', \ t_2 = t_2', \dots, t_n = t_n'\}$$

may be regarded as the set of inequations

$$\{t_1 \le t_1', \ t_1' \le t_1, \ t_2 \le t_2', \ t_2' \le t_2, \dots, t_n \le t_n', \ t_n' \le t_n\}.)$$

Lehmann has also investigated independently the initial algebra in a continuous equational class using a categorical framework, and has shown that the completion of T_Σ/q is initial in this class. This result is essentially the same as that of Hennessy. ADJ[3] have investigated quotients in so-called rational algebraic theories.

The results in this paper were strongly motivated by the consideration of types where either it would be desirable for normal forms to exist or it was clear that they did exist. (See for example Levy [1,2].) Normal forms are also important for expressing simply the "value" of a computation or when considering the problem of decidability of two expressions. Huet has investigated the existence of normal forms in a non-continuous framework, and Berry and Courcelle (in Berry) have investigated classes of interpretations where normal forms (called canonical terms by them) exist. In additon Courcelle (in Courcelle [2]) has studied conditions under which (what are essentially our) normal forms exist for an equationally specified continuous class of algebras. It would be worthwhile to investigate whether the conditions of Courcelle are also sufficient to guarantee the existence of a normal form function in our sense and hence guarantee the initiality of CT_{Σ}/q in the appropriate class.

The present paper thus provides a simple extension of ADJ[1] avoiding the more complex constructions of Lehammn, Hennessy and ADJ[3] in the useful case where normal forms exist. In practice the biggest advantage of this approach is that the congruence q considered is just the "usual" <u>least</u> congruence containing a set of equations, or possibly the least continuous congruence containing a set of equations. Further, the algebra CT_{Σ}/q is just the quotient of CT_{Σ} by q in the usual algebraic sense rather than being a more complex completion. This minimality of q is an extremely useful fact that can be used for proving various properties of continuous data types, a property in general absent from congruences derived from completions of pre-orders. (See ADJ[1] and Levy [1,2] for some uses of minimality of congruences in proofs.) Thus the main thrust of this paper differs from the other papers cited in that the concern is not so much "Does an initial algebra exist in a continuous equational class?" but "Is CT_Σ/q initial in this class?"

3. Mathematical Preliminaries

A data type is viewed here as a many-sorted algebra. (For a discussion of algebras, see Cohn or Gratzer.) This view was put forward previously by ADJ [1], Guttag and also Levy [1]. The notation and results in the section are adopted from ADJ [1,2]. We also assume familiarity with the definitions and results of ADJ [1,2].

Definition 1 Let S be a set whose elements are called sorts. An S-sorted operator domain Σ is a family of sets $\Sigma_{w,s}$ of symbols, for $s \in S$ and $w \in S^*$ where S^* is the free monoid on S. $\Sigma_{w,s}$ is the set of operator symbols of type $\langle w,s\rangle$, arity w and sort s.

A Σ-algebra consists of a family $\langle A_s\rangle_{s \in S}$ of sets called the carrier of A, and for each $\langle w,s\rangle \in S^* \times S$ and each $\sigma \in \Sigma_{w,s}$ a function

$$\sigma_A\colon\ A_{s_1} \times A_{s_2} \times \dots \times A_{s_n} \to A_s$$

(where $w = s_1 s_2 \dots s_n$) called the operation of A named by σ. (If $w = s_1 s_2 \dots s_n$, then let A^w denote $A_{s_1} \times A_{s_2} \times \dots \times A_{s_n}$.)

□

We use $\langle x_s\rangle_{s \in S}$ to denote a family of objects x_s indexed by s, such that there is exactly one object x_s for each $s \in S$. The subscript $s \in S$ will be omitted when the index set S can be determined from the context. For $\sigma \in \Sigma_{\lambda,s}$ where λ is the empty string, $\sigma_A \in A_s$ (also written $\sigma_A\colon \to A_s$). These operators are called constants of A of sort s. If $s \in S$, we usually denote the set A_s by s. If S has only one element then we get the standard definition of a (one-sorted) Σ-algebra. In this case let Σ be a family of sets $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \dots$ such that for each $\sigma \in \Sigma_n$ there is a function

$$\sigma_A: \; A_1 \times A_2 \times \ldots \times A_n \to A.$$

From this point on, all definitions and results in the paper will be for the one-sorted case, although they could be generalised to many-sorted algebras.

Definition 2 If A and A' are both Σ-algebras, then a Σ-homomorphism is a function

$$h: \; A \to A'$$

such that if $\sigma \in \Sigma_n$ and $\langle a_1,\ldots,a_n\rangle \in A^n$ then $h(\sigma_A(a_1,\ldots,a_n)) = \sigma_{A'}(h(a_1),\ldots,h(a_n))$.

Definition 3 A Σ-algebra A in a class $\underline{\underline{C}}$ of Σ-algebras is said to be initial in $\underline{\underline{C}}$ iff for every B in $\underline{\underline{C}}$ there exists a unique homomorphism h: A $\to$ B.

Theorem 1 The class of all Σ-algebras has an initial algebra called T_Σ. It also has an algebra $T_\Sigma(X)$, called the free algebra on X in the class, such that for any function f: X $\to$ A, where A is a Σ-algebra, there is a unique homomorphism $\bar{f}$: $T_\Sigma(X) \to A$ extending f. □

Intuitively T_Σ is the algebra of finite terms, and $T_\Sigma(X)$ is the algebra of finite terms with variables.

Definition 4 A Σ-equation is a pair $e = \langle L,R\rangle$ where $L,R \in T_\Sigma(X)$.

A Σ-algebra A satisfies e if

$$\bar{\theta}(L) = \bar{\theta}(R)$$

for all assignments θ: X $\to$ A. If ε is a set of Σ-equations, then A satisfies ε iff A satisfies each $e \in \varepsilon$. □

Thus a set of equations ε can be viewed as a set of axioms whose free variables are implicitly universally quantified. The class of Σ-algebras which satisfy ε is denoted $\underline{\underline{Alg}}_{\Sigma,\varepsilon}$.

<u>Theorem 2</u> $\underline{\underline{Alg}}_{\Sigma,\varepsilon}$ has an initial algebra called $T_{\Sigma,\varepsilon}$. □

The structure of $T_{\Sigma,\varepsilon}$ can be characterised as an algebraic quotient of T_Σ where intuitively two elements of T_Σ are equivalent if and only if one can be derived from the other by using the equations. That is $T_{\Sigma,\varepsilon}$ groups together all equivalent terms.

An important concept in the theory of abstract data types is the idea of quotients mentioned above. A quotient partitions the carrier of an algebra, and when this quotient is over T_Σ, it can be interpreted as a way of equating syntactic terms over the alphabet of the type. The importance of such "equations" is that they provide a means for expressing the difficult concept of abstraction. Furthermore quotients are defined in terms of equivalence relations which are congruences; intuitively, terms that have been equated must behave in the same way with respect to the operators of the type (referential transparency).

<u>Definition 5</u> A Σ-congruence $\equiv$ on a Σ-algebra A is an equivalence relation on A such that if $\sigma \in \Sigma_n$ and for $1 \leq i \leq n$ if $a_i, a_i' \in A$ and $a_i \equiv a_i'$ then

$$\sigma_A(a_1,\ldots,a_n) \equiv \sigma_A(a_1',\ldots,a_n').$$

If A is a Σ-algebra and $\equiv$ is a Σ-congruence on A, let $A/\equiv$ be the set of $\equiv$-equivalence classes of A. For $a \in A$ let $[a]$ denote the $\equiv$-class containing a. It is possible to make $A/\equiv$ into a Σ-algebra by defining the operations $\sigma_{A/\equiv}$ as follows.

(i) If $\sigma \in \Sigma_0$, then $\sigma_{A/\equiv} = [\sigma_A]$

(ii) If $\sigma \in \Sigma_n$ and $[a_i] \in A/\equiv$ for $1 \leq i \leq n$,
then $\sigma_{A/\equiv}([a_1],\ldots,[a_n]) = [\sigma_A(a_1,\ldots,a_n)]$

Then it can be shown that $A/\equiv$ is a Σ-algebra called the quotient of A by $\equiv$. (The property of $\equiv$ being a congruence ensures that $\sigma_{A/\equiv}$ is well defined.) □

Let K(A) be the class of congruences on the Σ-algebra A. It is well known that K(A) is a complete lattice.

A set of Σ-equations $\varepsilon = \{\langle t,t'\rangle | t,t' \in T_\Sigma(X)\}$ generates a binary relation $R \subseteq A \times A$. This relation is the set of all pairs $\{\langle\bar{\theta}(t), \bar{\theta}(t')\rangle | \theta$ is an assignment$\}$.

Theorem 3 If A is a Σ-algebra and R is a relation of A, then there exists a least Σ-congruence relation on A containing R; it is called the congruence relation generated by R on A. (The ordering on Σ-congruences is the subset ordering.) □

Theorem 4 If ε is a set of Σ-equations generating a congruence q on T_Σ, then T_Σ/q is initial in $\underline{\underline{Alg}}_{\Sigma,\varepsilon}$. □

The importance of the above theorems is that any set of Σ-equations (axioms) "automatically" defines an algebra which can be regarded as the sybmolic model of the object being defined. This model can be used to answer such questions as "Do the axioms characterise some particular model of the type?" and "Is a given implementation of the type correct?".

Definition 6 A partially ordered set (poset) $(P, \leq)$ is a set P together with a binary relation $\leq$ which is reflexive, transtitive and antisymmetric. □

All posets are here assumed to have a minimum element denoted $\perp$ ("bottom" or undefined) such that $\perp \leq p$ for any $p \in P$.

Definition 7 A subset S of P is said to be directed iff every finite subset of S has an upper bound in S. A function $f\colon P \to P'$ is said to be monotonic iff for all $p_1 \leq p_2$ in P, $f(p_1) \leq f(p_2)$ in P'. Such a function is said to be continuous if it preserves all least upper bounds of directed sets that exist in P. That is, f is continuous iff

$$f(\bigsqcup_{i\in I} p_i) = \bigsqcup_{i\in I} f(p_i)$$

where $\langle p_i \rangle_{i\in I}$ is a directed set in P and $\bigsqcup_{i\in I} p_i$ denotes the least upper bound of $\langle p_i \rangle_{i\in I}$ if it exists. A poset P is complete iff all directed sets have least upper bounds in P. □

Definition 8 A Σ-algebra is continuous iff its carrier is strict (has a minimum element ⊥), is complete, and if its operations are continuous. A data type is said to be continuous if it is continuous as an algebra. A function f: A → B is strict if $f(\bot_A) = \bot_B$. □

The following important result is proved in ADJ [2].

Theorem 5 The class of continuous Σ-algebras with strict continuous Σ-homomorphisms, called $\mathbf{CAlg}_\Sigma$, has an initial algebra called CT_Σ. □

As before with T_Σ, we let $CT_\Sigma(X_n)$ denote the free Σ-algebra in $\mathbf{CAlg}_\Sigma$ generated by X_n. An element $x_i \in X_n$ is called a variable.

Definition 9 The class of all continuous Σ-algebras that satisfy ε together with continuous Σ-homomorphisms between them is denoted $\mathbf{CAlg}_{\Sigma,\varepsilon}$. □

We now investigate whether $\mathbf{CAlg}_{\Sigma,\varepsilon}$ has an initial algebra which can be expressed as a quotient of CT_Σ.

4. Normal Forms and Initiality

Consider the following two equations:

$$x + x = x \tag{1}$$

$$x + (y+z) = (x+y) + z. \tag{2}$$

Suppose also that Σ is a signature containing $\perp$ and at least one other constant. Let $t_1, t_2, t_3 \in CT_\Sigma$ where

$$t_1 \leq t_2 \leq t_3 \tag{3}$$

are three distinct terms. Clearly such an algebra can be found. Let q be the least Σ-congruence on CT_Σ generated by (1) and (2). Now suppose there exists a partial order relation $\sqsubseteq$ on CT_Σ/q which is consistent with $\leq$ the partial order relation on CT_Σ. (That is $t_1 \leq t_2 \Rightarrow [t_1] \sqsubseteq [t_2]$). Then clearly

$$[t_1+t_3] = [(t_1+t_1)+t_3] \text{ by } (1)$$
$$\sqsubseteq [(t_1+t_2)+t_3] \text{ by } (3).$$

Also

$$[(t_1+t_2)+t_3] \sqsubseteq [(t_1+t_3)+t_3] \text{ by } (3)$$
$$= [t_1+(t_3+t_3)] \text{ by } (2)$$
$$= [t_1+t_3] \text{ by } (1).$$

Hence $[t_1+t_3] = [(t_1+t_2)+t_3]$ since $\sqsubseteq$ is a partial order on CT_Σ/q. But clearly, by examination of (1) and (2) this cannot be true, and thus there exists no partial order relation on CT_Σ/q consistent with $\leq$.

Thus when taking the quotient of CT_Σ by some congruence of q, it will not always be the case that an appropriate partial order relation on CT_Σ/q exists, and hence clearly CT_Σ/q will not be initial in $\underline{\underline{CAlg}}_{\Sigma,\varepsilon}$.

For practical reasons, it is often useful to try to characterise a class of values which are equivalent in some equivalence relation by a single representative of the class. We call such a representative a <u>normal form</u> of the class. This practical consideration in fact leads to a sufficient condition for guaranteeing the initiality of CT_Σ/q.

<u>Definition 10</u> Suppose there exists a function

$$nf: CT_\Sigma \to CT_\Sigma$$

such that for t, t_1, $t_2 \in CT_\Sigma$, and any congruence q,

1. $[t_1] = [t_2] \Rightarrow nf(t_1) = nf(t_2)$;
2. $[nf(t)] = [t]$;
3. nf is continuous (in the usual ordering on CT_Σ).

$[t] = \theta(t)$ where θ is the natural homomorphism induced by the congruence q. nf is called a <u>normaliser function</u> (or <u>normaliser</u>) for CT_Σ/q. □

<u>Lemma 5</u> (i) $nf(nf(t)) = nf(t)$. That is, nf is idempotent.

(ii) $nf(t_1) = nf(t_2) \Rightarrow [t_1] = [t_2]$.

That is, two normal forms are equal only if the corresponding terms are equivalent.

<u>Proof</u>: (i) From property 2 we get $[nf(t)] = [t]$

so $nf(nf(t)) = nf(t)$ from property 1.

(ii) Suppose $nf(t_1) = nf(t_2)$. Then

$[t_1] = [nf(t_1)] = [nf(t_2)] = [t_2]$. □

Until now, given a set of equations ε, we have considered the least congruence q generated by these equations. However consider two directed sets $\langle t_i \rangle$, $\langle t_i' \rangle$ in CT_Σ such that

$$t = \sqcup_i t_i \text{ and } t' = \sqcup_i t'_i$$

and such that for each $i \in I, (t_i, t'_i) \in q$. In general it would not be true that $(t,t') \in q$, but this condition is necessary for nf to exist. This is because if nf exists, then

$$\begin{aligned} nf(\sqcup_i t_i) &= \sqcup_i nf(t_i) && \text{continuity of nf} \\ &= \sqcup_i nf(t'_i) && \text{since } (t_i,t'_i) \in q \text{ and by property 1 of nf} \\ &= nf(\sqcup_i t'_i) && \text{again by continuity.} \end{aligned}$$

Hence $(t,t') \in q$ by lemma 5. This motivates a concept called continuous congruence which is defined below. We show that a unique continuous congruence exists for an arbitrary set of equations, and that continuous congruences have desirable properties.

Definition 11 A Σ-congruence q on a continuous Σ-algebra A is said to be continuous if whenever there exist two directed sets $\langle t_i \rangle_{i \in I}$ and $\langle t'_i \rangle_{i \in I}$ in A such that for all $i \in I, (t_i, t'_i) \in q$, then $(\sqcup_i t_i, \sqcup_i t'_i) \in q$. □

We now generalise theorem 3 to continuous Σ-algebras.

Theorem 6 If A is a continuous Σ-algebra and R is a relation on A, then there exists a least continuous Σ-congruence on A containing R, called the continuous congruence relation on A generated by R.

Proof: Let $K(R)$ be the class of all continuous Σ-congruence relations on A that contain R. $K(R) \neq \phi$ since

$$U = \langle u_s = A_s \times A_s \mid s \in S \rangle$$

is in $K(R)$, and is continuous since $(a_1,a_2) \in U$ for any $a_1,a_2 \in A$. Let $\equiv_R = \cap K(R)$. It is shown in ADJ [1] that $\equiv_R$ is a Σ-congruence relation. We show that $\equiv_R$ is continuous. Suppose that $\langle a_i \rangle_{i \in I}$ and $\langle a'_i \rangle_{i \in I}$ are directed sets in A with

$$a = \bigsqcup_i a_i, \quad a' = \bigsqcup_i a_i'.$$

Also, suppose that

$$\langle a_i, a_i' \rangle \in \equiv_R \qquad \text{for each } i \in I.$$

Hence

$$\langle a_i, a_i' \rangle \in K \qquad \text{for each } i \in I \text{ and}$$
$$\text{each } K \in \mathcal{K}(R) \text{ by definition of } \equiv_R.$$

But each $K \in \mathcal{K}(R)$ is continuous, so

$$\langle a, a' \rangle \in K \qquad \text{for each } K \in \mathcal{K}(R),$$

thus $\langle a, a' \rangle \in \equiv_R$ as required. □

Let K'(A) be the class of continuous Σ-congruences on A.

<u>Lemma 7</u> Let A be a continuous Σ-algebra. Then K'(A) is a complete lattice.

<u>Proof:</u> Clearly, U as defined in the proof above is the greatest continuous Σ-congruence on A. The least is clearly the collection of identity relations on A. The intersection of two continuous Σ-congruences is clearly continuous. If q_1 and $q_2 \in K'(A)$ and we let $R = q_1 \cup q_2$, then we can use the proof above to show that q_1 and q_2 have a least upper bound in K'(A). □

<u>Definition 12</u> The <u>kernel</u> of a Σ-homomorphism h: A → B is the relation $Ker(h) = \{\langle a, a' \rangle \mid a, a' \in A \text{ and } h(a) = h(a')\}$. □

It is well known that the kernel of a homomorphism is a congruence.

<u>Lemma 8</u> If A and B are continuous Σ-algebras and h: A → B is a continuous Σ-homomorphism, then Ker(h) is a continuous Σ-congruence.

<u>Proof:</u> We know Ker(h) is a Σ-congruence and we must prove continuity. Let $\langle a_i \rangle_{i \in I}$ and $\langle a_i' \rangle_{i \in I}$ be directed sets in A such that $(a_i, a_i') \in Ker(h)$ for

all $i \in I$. Now

$$\begin{aligned} h(\sqcup_i a_i) &= \sqcup_i h(a_i) && \text{since h is continuous} \\ &= \sqcup_i h(a_i') && \text{since } h(a_i) = h(a_i') \text{ for all } i \in I \\ &= h(\sqcup a_i') && \text{since h is continuous.} \end{aligned}$$

Thus $(\sqcup_i a_i, \sqcup_i b_i) \in Ker(h)$. □

We now define a partial order on CT_Σ/q, where q is a continuous congruence, in the case that a normaliser nf exists for q.

Definition 13 Suppose $[t_1], [t_2] \in CT_\Sigma/q$. Then define a partial order relation $\sqsubseteq$ on CT_Σ/q by

$$[t_1] \sqsubseteq [t_2] \text{ iff } nf(t_1) \le nf(t_2)$$

where $\le$ is the partial order relation on CT_Σ. □

(Note that nf need not be unique, and so the order relation $\sqsubseteq$ also depends on nf. See the corollary of theorem 12 for clarification.)

Lemma 9 $\sqsubseteq$ is a partial order on CT_Σ/q.

Proof: Obvious, since $\le$ is a partial order on CT_Σ. □

Lemma 10 Let $\langle t_i \rangle_{i \in I}$ be a set directed in CT_Σ, and q a continuous congruence, and suppose a normaliser nf exists. Then $\langle [t_i] \rangle$ is directed in CT_Σ/q and it has a least upper bound denoted $\sqcup_i [t_i]$ such that $\sqcup_i [t_i] = [\sqcup_i t_i]$.

Proof: Let $t = \sqcup_i t_i$, which exists since CT_Σ is complete. By monotonicity of nf and definition of $\le$, $\langle [t_i] \rangle$ is directed. Now for all $i \in I$, $t_i \le t$ since t is the least upper bound of $\langle t_i \rangle_{i \in I}$.

Hence $\forall i \; nf(t_i) \le nf(t)$

so $\forall i \; [t_i] \le [t]$ by definition of $\le$.

So $[t]$ is an upper bound of $<[t_i]>_{i\in I}$. Now suppose that for some t', for all $i\in I$, $[t_i] \le [t']$. Then for all $i\in I$, $nf(t_i) \le nf(t')$. But since nf is continuous and $<t_i>_{i\in I}$ is directed, $<nf(t_i)>_{i\in I}$ is directed and

$$\sqcup_i nf(t_i) \le nf(t') \qquad \text{by continuity of nf}$$

so

$$nf(\sqcup_i t_i) \le nf(t') \qquad \text{by definition of } \le.$$

and

$$[\sqcup_i t_i] \le [t'].$$

Hence $[\sqcup_i t_i]$ is the least upper bound of $<[t_i]>_{i\in I}$. That is

$$\sqcup_i [t_i] = [\sqcup_i t_i].$$

□

<u>Lemma 11</u> If B is a continuous Σ-algebra satisfying ε, q is the continuous congruence generated by ε, $t_1,t_2 \in CT_\Sigma$ and $(t_1,t_2) \in q$ and

$$h_B: \; CT_\Sigma \to B$$

is the unique homomorphism guaranteed to exist by the initiality of CT_Σ, then $h_B(t_1) = h_B(t_2)$.

<u>Proof</u>: Let $Ker(h_B)$, the kernel of h_B, be defined as before. We know $Ker(h_B)$ is a continuous congruence and, moreover, $\varepsilon(B) \subseteq Ker(h_B)$ where $\varepsilon(B)$ is the relation on B generated by the set of equations ε. This is because for each assignment θ: $X \to B$ and each $<L,R> \in \varepsilon$, $\bar\theta(L) = \bar\theta(R)$. Now $\bar\theta = h_B$ by uniqueness of h_B and hence $h_B(L) = h_B(R)$. But q is the least continuous congruence satisfying ε (containing $\varepsilon(B)$) and so $q \subseteq Ker(h_B)$. Thus $(t_1,t_2) \in Ker(h_B)$ and hence $h_B(t_1) = h_B(t_2)$ as required. □

Much of the power of considering abstract data types as many-sorted algebras centres around the property of isomorphism. Different implementations of the same data type can be considered members of a class of isomorphic algebras. In order to characterise this class precisely, the concept of initial algebra is used. The initial algebra in a class of algebras contains in some sense the least amount of information needed to specify a member of the class. Thus we would like to say that a particular abstract data type is the initial algebra in a class of algebras satisfying the specifications. Initiality ensures that the operators do no more than required by the specification. ADJ [1] shows that T_Σ/q is initial in the class of algebras satisfying the equations which generate the congruence q. It is natural to ask whether or not CT_Σ/q is initial, and we show that if the normaliser nf exists, then indeed CT_Σ/q is initial (where q now is the least continuous Σ-congruence generated by the equations).

Theorem 12 If a normaliser nf exists for q, then CT_Σ/q is initial in $\underline{\underline{CAlg}}_{\Sigma,\varepsilon}$.

Proof: We must find a unique

$$h_B\colon\ CT_\Sigma/q \to B$$

for any $B \in \underline{\underline{CAlg}}_{\Sigma,\varepsilon}$. By Theorem 5, $h_1\colon CT_\Sigma \to B$ exists, and is unique. Now define

$$h_B([t]) = h_1(nf(t)).$$

(i) h_B is a Σ-homomorphism. We must show that

$$h_B([\sigma(t_1,\ldots,t_n)]) = \sigma(h_B([t_1]),\ldots,h_B([t_n]))$$

for any $\sigma \in \Sigma_n$.

$h_B([\sigma(t_1,\ldots,t_n)]) = h_1(nf(\sigma(t_1,\ldots,t_n)))$

by definition of h_B

$= h_1(\sigma(t_1,\ldots,t_n))$

by lemma 11 and property 2 of nf

$= \sigma(h_1(t_1),\ldots,h_1(t_n))$

since h_1 is a homomorphism

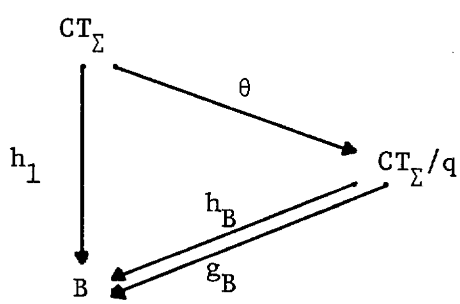
$= \sigma(h_1(nf(t_1)),\ldots,h_1(nf(t_n)))$

again by lemma 11 and property 2 of nf

$= \sigma(h_B([t_1]),\ldots,h_B([t_n]))$

by definition of h_B.

(ii) h_B is unique. Suppose there is a g_B: $CT_\Sigma/q \to B$ such that g_B is a homomorphism. Now consider the following diagram:



where θ is the natural homomorphism induced by a. If g_B exists, then we must have $\theta \circ h_B = \theta \circ g_B = h_1$ since $\theta \circ h_B$ and $\theta \circ g_B$ are both homomorphisms into B. But θ is onto, hence $h_B = g_B$.

(iii) CT_Σ/q is complete. Let $<[t_i]>_{i\in I}$ be directed in CT_Σ/q. Then by definition of ≤, $<nf(t_i)>_{i\in I}$ is directed in CT_Σ. Applying lemma 10 and since $[t] = [nf(t)]$ for any $t \in CT_\Sigma$, we get

$$\bigsqcup_i [t_i] = \bigsqcup_i [nf(t_i)] = [\bigsqcup_i nf(t_i)]$$
$$= [t]$$

where $t = \bigsqcup_i nf(t_i)$ exists since CT_Σ is complete.

(iv) CT_Σ/q is continuous. By (iii) CT_Σ/q is complete. $[\perp]$ is the minimum element of CT_Σ/q since nf is continuous and hence $nf(\perp) = \perp$. Now we must show that for each $\sigma \in \Sigma_n$, and each $1 \le j \le n$, $\sigma([t_1],\ldots,\bigsqcup_i [t_j^i],\ldots,[t_n]) = \bigsqcup_i \sigma([t_1],\ldots,[t_j^i],\ldots,[t_n])$.

$\sigma([t_1],\ldots,\bigsqcup_i [t_j^i],\ldots,[t_n])$

$= \sigma([t_1],\ldots,[\bigsqcup_i t_j^i],\ldots,[t_n])$
by lemma 10

$= [\bigsqcup_i (\sigma(t_1,\ldots,t_j^i,\ldots,t_n))]$
by definition and continuity of σ.

$= \bigsqcup_i [\sigma(t_1,\ldots,t_j^i,\ldots,t_n)]$
by lemma 10

$= \bigsqcup_i (\sigma([t_1],\ldots,[t_j^i],\ldots,[t_n]))$
by definition of σ.

(v) h_B is continuous. We must show that if $[t]$ is the least upper bound of a directed set $<[t_i]>_{i \in I}$ in CT_Σ/q then

$$h_B([t]) = \bigsqcup_i h_B([t_i]).$$

By lemma 10 and part (iii) above we know that if $<[t_i]>_{i \in I}$ is directed, and if $t = \bigsqcup_i nf(t_i)$, then

$$\sqcup_i[t_i] = \sqcup_i[nf(t_i)] = [t] = [\sqcup_i t_i] = [\sqcup_i nf(t_i)].$$

Now $h_B([t]) = h_1(nf(t))$

by definition of h_B

$= h_1(nf(\sqcup_i nf(t_i)))$

by definition of t

$= h_1(\sqcup_i nf(t_i))$

since nf is continuous and idempotent

$= \sqcup_i h_1(nf(t_i))$

since h_1 is continuous

$= \sqcup_i h_B([t_i])$

by definition of h_B as required.

<u>Corollary</u>

The particular normaliser chosen will not affect the ordering on CT_Σ/q, because any two initial algebras must be isomorphic and have the same structure. □

In practice, an algebra of normal forms is useful for establishing properties about an abstract data type and we make the following definition. This will be a generalisation of the concept of canonical term algebra in ADJ [1].

<u>Definition 14</u> A continuous Σ-algebra L_Σ is called a <u>normal term algebra</u> for q if

(i) The carrier of L_Σ is a subset of the carrier of CT_Σ,

and (ii) $L_\Sigma \cong CT_\Sigma/q$. □

If a normaliser exists, then it is possible to construct a normal term algebra.

<u>Theorem 13</u> Let

$$nf: CT_\Sigma \to CT_\Sigma$$

be a normaliser, and define a Σ-algebra L_Σ as follows

(i) The carrier is L,

$L = \{nf(t) \mid t \in CT_\Sigma\}$;

(ii) For each $\sigma \in \Sigma$

$\sigma_L(nf(t_1),\ldots,nf(t_n)) = nf(\sigma(t_1,\ldots,t_n))$.

Then L_Σ is a normal term algebra.

<u>Proof</u>: Let $g: L_\Sigma \to CT_\Sigma/q$ be defined as the restriction of the natural homomorphism Θ to L_Σ.

(i) g is a homomorphism, since θ is.

(ii) g is surjective since if $[t] \in CT_\Sigma/q$, then $[t] = [nf(t)]$ since nf is a normaliser and so $g(nf(t)) = [t]$ by definition of g.

(iii) g is injective, since if $g(nf(t_1)) = g(nf(t_2))$ then $[nf(t_1)] = [nf(t_2)]$ by definition of g, and so $nf(t_1) = nf(t_2)$ since nf is a normaliser. □

We now demonstrate the converse of this theorem.

<u>Theorem 14</u> If a normal term algebra L_Σ exists, then there is a normaliser function for q.

<u>Proof</u>: By initiality of CT_Σ, $h: CT_\Sigma \to L_\Sigma$ exists. Let $nf(t) = h(t)$. We must show that

(i) $[t_1] = [t_2] \Rightarrow nf(t_1) = nf(t_2)$.

This follows by lemma 11 since L_Σ satisfies ε.

(ii) $[nf(t)] = [t]$.

That is, we must show that $[h(t)] = [t]$.

Since $L_\Sigma \subseteq CT_\Sigma$, $h \circ h = h$. Also, since $L_\Sigma \cong CT_\Sigma/q$, $Ker(h) = q$. Now since $h \circ h(t) = h(t)$, $(h(t),t) \in Ker(h)$

so $(h(t),t) \in q$

and hence $[h(t)] = [t]$ as required.

(iii) Continuity of nf is immediate since h is continuous.

□

5. Conclusions

Continuous data types arise naturally in many settings when studying the semantics of programs and data. Data types are elegantly characterised by universal algebras, where one of the most powerful tools used in the study of universal algebra is the construction of quotient algebras. It would have been useful to be able to use this technique in the more restricted domain of continuous algebras. This, however, turned out to be impossible as the quotient of a continuous algebra by an arbitrary congruence may not yield a continuous algebra (as the quotient set may not admit a partial order or, even if it does, the partial order may not be complete.)

Our purpose in this report was to characterise continuous data types by finding conditions under which the quotient of the initial continuous algebra by some congruence would yield a continuous quotient algebra. We were particularly interested in the case where the congruence was generated by a set of equations (axioms). Two simply stated conditions suffice for this purpose. Firstly, the congruence has to have a special property called continuity. A continuous congruence is one which relates (puts in the same congruence class) upper bounds of directed sets if the elements of the directed sets are pairwise related by the congruence. This seems to be a highly desirable and natural property of congruences. For example, we showed that the kernel of a continuous homomorphism between continuous algebras is a continuous congruence.

Secondly, the congruence has to be such that from each congruence class a unique representative can be chosen by using a (continuous) map called a normaliser. The normal form (image under the normaliser) of an expression is a generalisation of the canonical form of an expression introduced in the study of abstract data types (see ADJ [1]). (In contrast to canonical forms, however, normalisers do not always exist.) Since our motivation for studying quotients was to generalise the work on abstract data types to the setting of continuous algebras, this was a natural place to start. Moreover, continuous data types which did not have normal forms would be impossible to represent (even aside from problems of representing infinite objects by finite ones).

As indicated above, the motivation for this work was the need to generalise the work on abstract data types to the continuous setting in order to handle "naturally infinite" objects (such as lists represented by equations of the form x = cons(a,x). The "list" represented by this equation is clearly cons(a,cons(a,cons(a,......)...))).) Such objects do not exist in the usual algebras of finite objects studied in the theory of abstract data types. Moreover, it turns out that concepts such as sharing and circularity in the definition of structures are best handled by resorting to continuous data types. These ideas are developed further in Levy [1] and Levy [2].

Other applications of these ideas would be in the study of control structures as operations in an abstract data type and in providing a basis for unifying proof rules for data and control structures (to make their application easier).

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