A Maximum Column Partition for Sparse Positive Definite Linear Systems Ordered by the Minimum Degree Ordering Algorithm

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Abstract In this paper we prove that for a matrix reordered by a minimum degree algorithm (MDA), the column partition of $L + L^T$ induced by the MDA in [3] is the best possible column partition (fewest number of diagonal and off diagonal dense blocks) where L is the Cholesky factor of the reordered matrix. As a by product we give a theorem which characterizes the local behavior of the MDA for general symmetric positive definite linear systems. The theorem is shown to be equivalent to Corollary 3.6 of [3].

1. Introduction

For a given symmetric matrix A, a column partition (CP) is a subdivision of A into square diagonal blocks with corresponding (block) columns below each square diagonal. Each (nonnull) off diagonal block in a (block) column will be defined by the largest number of consecutive nonzero rows. For example see **Figure 1.1.**

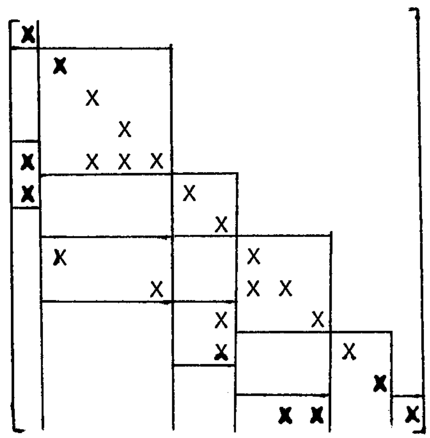

Figure 1.1 **A Column partition of** a matrix with off diagonal blocks.

We shall specify a CP of a matrix A in terms of a partition $\mathcal{P} = \{P_1,\ldots,P_p\}$ of the row numbers of A, where there are p diagonal blocks and if the i-th diagonal block consists of rows $r,r+1,\ldots,s$ then $P_i = \{r,r+1,\ldots,s\}$.

As shown in [3] the MDA induces a CP on the reordered symmetric positive definite matrix A (and on $L + L^T$ where L is the Cholesky factor of the reordered matrix) having the property that the diagonal and off diagonal blocks of $L + L^T$ are dense. For storage schemes in the linear solver which store the nonnull off diagonal blocks of L, the fact that the off diagonal blocks are dense is an asset in reducing storage. However each nonnull off diagonal block incurs a certain amount of storage overhead. Thus it is desirable for the number of off diagonal blocks to be as few as possible. In [3] a method was introduced which reordered equations within diagonal blocks (still a minimum degree ordering) in order to reduce the number of off diagonal blocks for a given partition induced by the MDA. By such a reordering it was often possible to coalesce several small off diagonal blocks in each (block) column into a few larger off diagonal blocks.

In this paper we consider the possibility of further reducing the number of dense diagonal and off diagonal blocks by a suitable choice of column partition for the reordered matrix (by MDA). We show that the column partition induced by the MDA in [3] is best possible. As a byproduct we give graph theoretic results which characterize the local behavior of the MDA for general symmetric positive definite linear systems. This characterization is shown to be equivalent to that of Corollary 3.6 in [3].

A summary of the remainder of the paper follows. In §2 we present some basic properties of a subclass of column partitions. We also review some graph theory notation and the elimination graph model for symmetric Gaussian elimination. In §3 we prove that for a matrix reordered by a MDA, the column partition of $L + L^T$ induced by the MDA in [3] is best possible (has fewest number of diagonal and off diagonal dense blocks) where L is the Cholesky factor of the reordered matrix. Section 4 contains our concluding remarks. The appendix contains proofs of the equivalence of Theorem 3.2 and Corollary 3.6 of [3].

2. Preliminary Results and Notation

In this section we derive some properties for a class of CP's. We then review some graph theoretic notation and the graph model of the Gaussian elimination process for symmetric positive definite linear systems.

A dense column partition (DCP) of a symmetric positive definite matrix is a CP in which each diagonal and off diagonal block is dense (full).

A maximal dense column partition (MDCP) of a symmetric positive definite matrix is a DCP which is not a refinement of another DCP.

Clearly the set of all DCP's of a symmetric positive definite matrix is partially ordered set (poset) under the relation (≤) of refinement ($P_1 \le P_2 \Leftrightarrow P_1$ is a refinement of P_2). In the following theorem we show that in fact the poset is a lattice (and thus any MDCP is the maximum DCP of the set of all DCP's for a given symmetric positive definite matrix).

THEOREM 2.1. The set of all DCP's of a symmetric positive definite matrix is a lattice under the relation of refinement ($P_1 \le P_2 \Leftrightarrow P_1$ is a refinement of P_2).

Proof. Consider any MDCP $\mathcal{P}_{MD}$ of the positive definite matrix. We show that it is a maximum element of the poset. We show that every DCP is a refinement of $\mathcal{P}_{MD}$. Suppose not and let $\mathcal{P}'$ be a DCP which contains a block P_i' not contained in any block of $\mathcal{P}_{MD}$. Then there must exist two adjacent blocks P_r and P_{r+1} in $\mathcal{P}_{MD}$ sharing diagonal elements with P_i'. Thus the 2×2 diagonal block consisting of one diagonal element from P_r and P_{r+1} must be dense and contain dense off diagonal blocks. But this implies that $P_r \cup P_{r+1}$ is a dense diagonal block containing dense off diagonal blocks. This contradicts $\mathcal{P}_{MD}$ being maximal.

Thus, since the DCP consisting of 1x1 diagonal blocks is clearly the minimum partition, and since the poset is finite, the poset is a lattice. □

Following [3] we shall, in the remainder of this section, review some graph theory notation and the graph model of the Gaussian elimination process for symmetric positive definite linear systems.

An undirected graph G is an ordered pair of sets $G = (X,E)$ where X is a finite set of nodes and E is a set of unordered pairs of distinct nodes of X called edges.

In the following definitions the graph $G = (X,E)$ will be assumed.

For $Y \subset X$ the adjacent set of Y, denoted by $\mathrm{Adj}_G(Y)$ is $\mathrm{Adj}_G(Y) = \{x | x \in X \backslash Y \text{ and } y \in Y \text{ and } \{x,y\} \in E\}$. When Y consists of the single node y, we write $\mathrm{Adj}_G(y)$ rather than $\mathrm{Adj}_G(\{y\})$.

The degree of a node x, denoted $\deg_G(x)$, is the number $\deg_G(x) = |\mathrm{Adj}_G(x)|$, where $|Y|$ denotes the cardinality of the finite set Y. We will denote by $\delta(G)$ the minimum degree of all nodes in the graph G.

The incidence set of Y, $Y \subset X$, denoted by $\mathrm{Inc}_G(Y)$, is defined by $\mathrm{Inc}_G(Y) = \{\{x,y\} | y \in Y \text{ and } x \in \mathrm{Adj}_G(Y)\}$.

The deficiency, $\mathrm{Def}_G(Y)$, of a set $Y \subset X$ is the set of all pairs of distinct nodes in $\mathrm{Adj}_G(Y)$ which are not themselves adjacent. Thus

$$\mathrm{Def}_G(Y) = \{\{x,y\} | x,y \in \mathrm{Adj}_G(Y),\ x \neq y,\ \{x,y\} \notin E\}.$$

A clique of a graph $G = (X,E)$ is a set $C \subset X$ such that $x,y \in C \Rightarrow \{x,y\} \in E$, [1].

A path from x to y of length k is an ordered set of distinct nodes $(v_1, v_2, \ldots, v_{k+1})$ where $x = v_1$, $y = v_{k+1}$, and $v_i \in \mathrm{Adj}_G(v_{i+1})$, $1 \le i \le k$.

For a graph $G = (X,E)$ with $|X| = N$, an ordering (labelling) of G is a bijection $\alpha:\{1,2,\ldots,N\} \to X$. We denote the labelled graph $G^\alpha = (X^\alpha, E^\alpha)$ and the labelled node $x_i = \alpha(i)$.

We now establish a relationship between matrices and graphs where the graph is used to represent the zero-nonzero structure of the matrix. Let A be an N by N symmetric matrix. The labelled, undirected graph associated with A is denoted by $G^{\alpha_A} = (X^{\alpha_A}, E^{\alpha_A})$ where X^{α_A} is labelled as the rows of A and $\{x_i, x_j\} \in E^{\alpha_A}$ if and only if $A_{ij} \neq 0$, $i \neq j$ ($\alpha_A(i) = x_i$). The unlabelled graph corresponding to A is simply G^{α_A} with its labels removed. For any N by N permutation matrix $P \neq I$, the unlabelled graphs of A and PAP^T are identical but the associated labellings differ.

As in [4] and [5] we examine the graph theoretic elimination model of the Gaussian elimination process for symmetric positive definite matrices. The triangular factorization of an N by N symmetric positive definite matrix A into LL^T can be described by the following outer-product formulation [6]:

$$A = A_0 = H_0 = \begin{bmatrix} d_1 & V_1^T \\ V_1 & \bar{H}_1 \end{bmatrix} = \begin{bmatrix} \sqrt{d_1} & 0 \\ \frac{V_1}{\sqrt{d_1}} & I_{N-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_1 \end{bmatrix} \begin{bmatrix} \sqrt{d_1} & V_1^T/\sqrt{d_1} \\ 0 & I_{N-1} \end{bmatrix}$$

$$= L_1 A_1 L_1^T \quad \text{where} \quad H_1 = \bar{H}_1 - V_1 V_1^T / d_1 .$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & d_2 & V_2^T \\ 0 & V_2 & \bar{H}_2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & \sqrt{d_2} & \\ 0 & \frac{V_2}{\sqrt{d_2}} & I_{N-2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & H_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & \sqrt{d_2} & V_2^T/\sqrt{d_2} \\ & & I_{N-2} \end{bmatrix}$$

$$= L_2 A_2 L_2^T \quad \text{where} \quad H_2 = \bar{H}_2 - V_2 V_2^T / d_2 .$$

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$A_{N-1} = L_N I_N L_N^T$.

It is easy to verify that $A = LL^T$ where

$$L = (\sum_{i=1}^{N} L_i) - (N-1)I.$$

The Y-elimination graph, denoted by G_Y, of a set $Y \subset X$ is $G_Y = \{X\backslash Y, E\backslash \bigcup_{y\in Y} \mathrm{Inc}_G(y) \cup \mathrm{Def}_G(Y)\}$. When $Y = \{y\}$ we write G_y and refer to the graph as the y-elimination graph.

The sequence of elimination graphs $G_1, G_2, \ldots, G_{N-1}$ for a matrix A is then defined by $G_0 = G^{\alpha_A}$, and for $i = 1,2,\ldots,N-1$, $G_i = (G_{i-1})_{x_i}$. The elimination graph G_i, $0 \le i < N$ is simply the graph associated with the matrix $H_i : G_i = G^{\alpha_{H_i}}$, as in Rose [5].

Following [2] the reachable set of a node $x \in X\backslash Y$ through a set $Y \subset X$ is

$$\mathrm{Reach}(x,Y) = \{y \in X\backslash Y \mid \exists \text{ path } (x,y) \text{ or } (x,v_1,\ldots,v_k,y) \ni v_i \in Y, 1 \le i \le k\}.$$

The following lemma will be useful later and establishes the relationship between Y-elimination graph G_Y and the sequence of elimination graphs G_i, $1 \le i \le N-1$.

LEMMA 2.2. For any graph $G = (X,E)$,

$$G_Y = G_i \quad \text{where} \quad Y = \{x_1,\ldots,x_i\} \subset X.$$

PROOF. Clearly the elimination of a node x_j from elimination graph G_{j-1} results in $\mathrm{Adj}_{G_{j-1}}(x_j)$ forming a clique in G_j. Thus

$$\mathrm{Adj}_{G_{i-1}}(x_i) = \mathrm{Reach}_G(x_i, \{x_1,\ldots,x_{i-1}\})$$

$$= \mathrm{Adj}_G(\{x_1,\ldots,x_i\}).$$

The result then follows immediately. □

A CP $\mathcal{P} = \{P_1,P_2,\ldots,P_p\}$ of a matrix A induces a partition $\mathcal{P}' = \{P_1',P_2',\ldots,P_p'\}$ of the labelled undirected graph $G^{\alpha_A} = (X^{\alpha_A},E^{\alpha_A})$ corresponding to A in the sense that

$$i \in P_i \Leftrightarrow x_i \in P_i' \quad \text{where} \quad x_i = \alpha_A(i) \in X^{\alpha_A}$$

is the labelled node corresponding to row i of A. Thus a CP of a matrix can be thought of as a partition of the nodes of the corresponding labelled undirected graph provided that for $a \in P_i'$, $b \in P_j'$, $i < j \Rightarrow \alpha_A^{-1}(a) < \alpha_A^{-1}(b)$.

3. A Maximum DCP for $L + L^T$ of a Matrix Ordered by a MDA

We now define the partition $P_{MD} = \{P_1, P_2, \ldots, P_p\}$ of X^{PAP^T} induced by the following ordering algorithm where PAP^T is the reordered matrix and p is the number of times the for-loop is executed.

(3.1) $N_0 = 0$

<u>for</u> i := 1 <u>step</u> 1 <u>while</u> $N_{i-1} < N$ <u>do</u>

1. Find $y \in X_{N_{i-1}} \ni \deg_{G_{N_{i-1}}}(y) = \delta(G_{N_{i-1}})$
2. Set $P_i = \{x \in X_{N_{i-1}} | Adj_{G_{N_{i-1}}}(x) \cup \{x\} = Adj_{G_{N_{i-1}}}(y) \cup \{y\}\}$
3. Set $N_i = |\bigcup_{j=1}^{i} P_j|$
4. Form $G_{N_i} = (G_{N_{i-1}})_{P_i}$

Whenever ordering algorithm (3.1) chooses a minimum degree node y in an elimination graph $G_{N_{i-1}} = (X_{N_{i-1}}, E_{N_{i-1}})$, $1 \le i \le p$, there is a set $P_i \subset X_{N_{i-1}}$ containing node y and having the property that

$$x,z \in P_i \Rightarrow Adj_{G_{N_{i-1}}}(x) \cup \{x\} = Adj_{G_{N_{i-1}}}(z) \cup \{z\}$$

(possibly $P_i = \{y\}$). The following theorem shows that once ordering algorithm (3.1) has choosen such a minimum degree node y to be ordered next, a MDA would order the remaining nodes of P_i next. Hence the ordering algorithm (3.1) is a MDA.

THEOREM 3.2. Let C_i be a maximal clique in $G_i = (X_i, E_i)$, $0 \le i < N$ having the property that $x,y \in C_i \Rightarrow Adj_{G_i}(x) \cup \{x\} = Adj_{G_i}(y) \cup \{y\}$. If the MDA chooses a node of C_i at the i+1 step, then every ordering $\{x_j\}_{j=i+1}^{N}$ of X_i generated by the MDA satisfies $\{x_{i+1}, x_{i+2}, \ldots, x_{i+|C_i|}\} = C_i$.

Proof. Let x_{i+1} be any node of C_i. If the MDA chooses x_{i+1}, then $\deg_{G_i}(x_{i+1}) = \delta(G_i)$. Let $G_{i+1} = (G_i)_{x_{i+1}}$. Thus $\deg_{G_i}(x_{i+1}) - 1 \le \delta(G_{i+1})$. If $C_i \backslash \{x_{i+1}\} = \phi$, then there is nothing to prove. Suppose $C_i \backslash \{x_{i+1}\} \ne \phi$. Let x_{i+2} be any node of $C_{i+1} = C_i \backslash \{x_{i+1}\}$. Using the properties of C_i in G_i, $\{x_{i+1}, x_{i+2}\} \in E_i$ and thus $\deg_{G_{i+1}}(x_{i+2}) = \deg_{G_i}(x_{i+2}) - 1$. Thus, since $x_{i+1}, x_{i+2} \in C_i$, $\deg_{G_{i+1}}(x_{i+2}) = \deg_{G_i}(x_{i+1}) - 1$ and as a result we have $\deg_{G_{i+1}}(x_{i+2}) \le \delta(G_{i+1})$. Note also that $\delta(G_i) - 1 = \delta(G_{i+1})$.

We now show that for $x \in X_i \backslash C_i$, $\deg_{G_{i+1}}(x) > \delta(G_{i+1})$. Consider $x \in Adj_{G_i}(C_i)$. Now $C_i \cup \{x\}$ is a clique in G_i and since C_i is maximal, $\forall y \in C_i$, $Adj_{G_i}(x) \cup \{x\} \ne Adj_{G_i}(y) \cup \{y\} = Adj_{G_i}(C_i) \cup C_i$. Thus either $\{Adj_{G_i}(x) \cup \{x\}\} \backslash \{Adj_{G_i}(C_i) \cup C_i\} \ne \phi$ or $\{Adj_{G_i}(C_i) \cup C_i\} \backslash \{Adj_{G_i}(x) \cup \{x\}\} \ne \phi$. If the latter holds then since $\deg_{G_i}(x_{i+1}) = \delta(G_i)$ and $x_{i+1} \in C_i$, $|Adj_{G_i}(C_i) \cup C_i| \le |Adj_{G_i}(x) \cup \{x\}|$ and so $\{Adj_{G_i}(x) \cup \{x\}\} \backslash \{Adj_{G_i}(C_i) \cup C_i\} \ne \phi$. Thus $\{\{Adj_{G_i}(x) \cup \{x\}\} \backslash \{x_{i+1}\}\} \backslash \{\{Adj_{G_i}(C_i) \cup C_i\}\ \{x_{i+1}\}\} \ne \phi$ and since $\{Adj_{G_i}(x) \cup \{x\}\} \backslash \{x_{i+1}\} \subset Adj_{G_{i+1}}(x) \cup \{x\}$ we have $\{Adj_{G_{i+1}}(x) \cup \{x\}\} \backslash \{\{Adj_{G_i}(C_i) \cup C_i\} \backslash \{x_{i+1}\}\} \ne \phi$. Now the elimination of node x_{i+1} from G_i results in $Adj_{G_i}(x_{i+1}) = \{Adj_{G_i}(C_i) \cup C_i\} \backslash \{x_{i+1}\}$ being a clique in G_{i+1}. Hence since $x \in Adj_{G_i}(C_i)$ we have $\{Adj_{G_i}(C_i) \cup C_i\} \backslash \{x_{i+1}\} \subset Adj_{G_{i+1}}(x) \cup \{x\}$. Combining results gives $\{Adj_{G_i}(C_i) \cup C_i\} \backslash \{x_{i+1}\} \subsetneq Adj_{G_{i+1}}(x) \cup \{x\}$. Thus $\deg_{G_{i+1}}(x) > |Adj_{G_i}(C_i) \cup C_i \backslash \{x_{i+1}\}| - 1 = \delta(G_i) - 1 = \delta(G_{i+1})$. Consider $x \in X_i \backslash \{Adj_{G_i}(C_i) \cup C_i\}$. Then $\forall y \in C_i$ it follows that $\{x,y\} \notin E_i$. Hence $\deg_{G_{i+1}}(x) = \deg_{G_i}(x) \ge \delta(G_i) > \delta(G_i) - 1 = \delta(G_{i+1})$.

Now clearly C_{i+1} is a clique in G_{i+1} and has the property that $x,y \in C_{i+1} \Rightarrow Adj_{G_{i+1}}(x) \cup \{x\} = Adj_{G_{i+1}}(y) \cup \{y\}$. It remains to show that C_{i+1} is a maximal clique having this property in G_{i+1}. Suppose C_{i+1} is not maximal and that there exists a clique $D \subset X_{i+1}$ properly containing C_{i+1} and satisfying $x,y \in D \Rightarrow Adj_{G_{i+1}}(x) \cup \{x\} = Adj_{G_{i+1}}(y) \cup \{y\}$. Consider $y \in D \backslash C_{i+1}$, $x \in C_{i+1} \subset D \Rightarrow Adj_{G_{i+1}}(y) \cup \{y\} = Adj_{G_{i+1}}(x) \cup \{x\}$ $= Adj_{G_{i+1}}(C_{i+1}) \cup C_{i+1}$. But then $y \in C_{i+1}$ since if $y \in Adj_{G_{i+1}}(C_{i+1})$ **then $y \in Adj_{G_i}(C_i)$ and since $deg_{G_i}(x_{i+1}) = \delta(G_i)$, then**

$Adj_{G_i}(y) \backslash \{C_i \cup Adj_{G_i}(C_i)\} \neq \phi$ and thus $Adj_{G_{i+1}}(y) \backslash \{C_{i+1} \cup Adj_{G_{i+1}}(C_{i+1})\} \neq \phi$, contradiction. But $y \in C_{i+1}$ is a contradiction and hence C_{i+1} is maximal.

The above process can now be repeated by setting $i \leftarrow i+1$ until $C_i \backslash \{x_{i+1}\} = \phi$. □

COROLLARY 3.3. Under the conditions of Theorem 3.2, C_j is a maximal clique in $G_j = (X_j, E_j)$ having the property that

$$x,y \in C_j \Rightarrow Adj_{G_j}(x) \cup \{x\} = Adj_{G_j}(y) \cup \{y\}$$

$$\text{for } j = i,\ldots,i+|C_i|-1.$$

Hence the ordering algorithm (3.1) is a MDA. The importance of MDA (3.1) is that for each minimum degree search of an elimination graph G_j a set P_i of nodes can be ordered. Also the data structures can be adjusted directly to reflect the resulting elimination graph $(G_j)_{P_i}$.

Theorem 3.2 also characterizes the local behavior of the MDA for an undirected graph. For completeness we show in the appendix the equivalence of Theorem 3.2 with that of Corollary 3.6 of [3].

We now complete this section with a proof that the partition P_{MD} produced by MDA (3.1) is the maximum DCP of the lattice of DCP's of $L + L^T$, **where $PAP^T = LL^T$ is the matrix ordered by the MDA (3.1).**

THEOREM 3.4. For any symmetric positive definite matrix A ordered by a MDA, the partition P_{MD} is a DCP of $L + L^T$ where L is the Cholesky factor of the reordered matrix.

Proof. Follows directly from the definition of P_i in MDA (3.1). □

THEOREM 3.5. For any symmetric positive definite matrix A ordered by a MDA, the partition P_{MD} is a MDCP of $L + L^T$ where L is the Cholesky factor of the reordered matrix.

Proof. Suppose P_{MD} is not maximal. Then there exists a DCP $P' = \{P_1', \ldots, P_{p'}'\}$ such that $P_{MD} = \{P_1, P_2, \ldots, P_p\}$ is a refinement of P', since by Theorem 2.1 the set of all DCP's of $L + L^T$ form a finite lattice under refinement. But then there exists $P_i' \in P'$ such that $P_i' = P_s \cup P_{s+1} \cup \ldots \cup P_{s+t}$ for some i, s and $t \geq 1$. But this implies that P_i' is a clique in elimination graph G_j, $j = |\bigcup_{k=1}^{i-1} P_k'| = |\bigcup_{k=1}^{s-1} P_k|$. But P_s is a maximal clique in G_j having the property that

$x, y \in P_s \Rightarrow Adj_{G_j}(x) \cup \{x\} = Adj_{G_j}(y) \cup \{y\}$,

$\Rightarrow \forall y \in P_{s+1} \cup \ldots \cup P_{s+t},\ x \in P_s \Rightarrow Adj_{G_j}(x) \cup \{x\} \neq Adj_{G_j}(y) \cup \{y\}$,

$\Rightarrow$ diagonal or off diagonal block corresponding to P_i' is not dense. This contradicts P' being a DCP. □

THEOREM 3.6. For any symmetric positive definite matrix A ordered by a MDA, the partition P_{MD} is the maximum DCP of $L + L^T$ where L is the Cholesky factor of the reordered matrix.

Proof. Follows from Theorem 3.5 and the fact that (Theorem 2.1) the **set of DCP's of $L + L^T$ form a finite lattice under refinement.** □

4. Conclusions

We have shown that the class of all DCP's for a symmetric positive definite matrix forms a finite lattice under the relation of refinement. We then showed that the DCP produced by the MDA (3.1) is the maximum element of the lattice of DCP's of $L + L^T$ where L is the Cholesky factor of the matrix reordered by the MDA. Finally, as a byproduct we have derived a theorem which characterizes the local behavior of the MDA for general symmetric positive definite linear systems. This theorem is shown to be equivalent to Corollary 3.6 of [3].

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Appendix

In this section we derive several theorems which are equivalent to Theorem 3.2. In particular we show the equivalence of Theorem 3.2 and Corollary 3.6 of [3].

We require the following lemma:

LEMMA 1. Let P be a maximal clique in $G = (X,E)$ having the property that $x,y \in P \Rightarrow Adj_G(x) \cup \{x\} = Adj_G(y) \cup \{y\}$. Let $Q = \{x \in X | x \in \bigcap_{k=1}^{r} C_k$ and $Adj_G(x) \subset \bigcup_{k=1}^{r} C_k\}$ where $C_1, C_2, \ldots, C_r$ are distinct maximal cliques in G. Then if $P \cap Q \neq \phi$ then $P = Q$.

Proof. Let $x \in P$ and $y \in P \cap Q$. Then $y \in Q \Rightarrow y \in \bigcap_{k=1}^{r} C_k$ and $Adj_G(y) \subset \bigcup_{k=1}^{r} C_k \Rightarrow Adj_G(y) \cup \{y\} = \bigcup_{k=1}^{r} C_k$. Now $x,y \in P \Rightarrow Adj_G(x) \cup \{x\} = Adj_G(y) \cup \{y\} = \bigcup_{k=1}^{r} C_k \Rightarrow x \in \bigcup_{k=1}^{r} C_k$ and $Adj_G(x) = \bigcup_{k=1}^{r} C_k \backslash \{x\}$. Thus $x \in \bigcap_{k=1}^{r} C_k$ since C_k, $k = 1,\ldots,r$ are maximal. Hence $x \in Q \Rightarrow P \subset Q$.

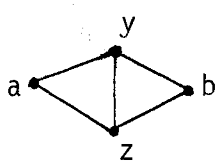
Let $x \in Q$ and $y \in P \cap Q$. Then $x,y \in Q \Rightarrow x,y \in \bigcap_{k=1}^{r} C_k$ and $Adj_G(x) \subset \bigcup_{k=1}^{r} C_k$ and $Adj_G(y) \subset \bigcup_{k=1}^{r} C_k$. Hence $Adj_G(x) \cup \{x\} = \bigcup_{k=1}^{r} C_k = Adj_G(y) \cup \{y\}$. But since $y \in P$, therefore $x \in P \Rightarrow Q \subset P$. □

Thus the following is an equivalent statement of Theorem 3.2 and is identical to Corollary 3.6 of [3].

THEOREM 2. Let $C_i = \{x | x \in \bigcap_{k=1}^{r} C_k$ and $Adj_G(x) \subset \bigcup_{k=1}^{r} C_k\}$ where $C_1, C_2, \ldots, C_r$ are distinct maximal cliques in G. If the MDA chooses a node of C_i at the $i+1$ step, then every ordering $\{x_j\}_{j=i+1}^{N}$ of X_i generated by the MDA satisfies

$$\{x_{i+1}, x_{i+2}, \ldots, x_{i+|C_i|}\} = C_i.$$

Note that maximal cliques, C_i, $1 \le i \le r$, are necessary as shown by the following example.



$C_1 = \{y,a\}$, $C_2 = \{y,b\}$, $C_3 = \{y,z\}$, $P = \{y,z\}$, $Q = \{y\}$

$\therefore \; P \cap Q \neq \phi$ but $P \neq Q$!

In a similar manner the following equivalent statement of Theorem 3.2 can be proved.

THEOREM 3. Let $C_i = \{x | Adj_G(x) \cup \{x\} = \bigcup_{k=1}^{r} C_k\}$ where $C_1, C_2, \ldots, C_r$ are cliques in G. If the MDA chooses a node of C_i at the $i+1$ step, then every ordering $\{x_j\}_{j=i+1}^{N}$ of X_i generated by the MDA satisfies

$$\{x_{i+1}, x_{i+2}, \ldots, x_{i+|C_i|}\} = C_i.$$