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Sums Involving the Inverses of Binomial Coefficients

Jin-Hua Yang
Department of Applied Mathematics
Dalian University of Technology
Dalian, 116024
China
and
Zhoukou Normal University
Zhoukou, 466001
China

Feng-Zhen Zhao
Department of Applied Mathematics
Dalian University of Technology
Dalian, 116024
China
fengzhenzhao@yahoo.com.cn

Abstract

In this paper, we compute certain sums involving the inverses of binomial coefficients. We derive the recurrence formulas for certain infinite sums related to the inverses of binomial coefficients.

1 Introduction

As usual, the binomial coefficient $\binom{n}{m}$ is defined by

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!}, & \text{if } n \geq m; \\ 0, & \text{if } n < m; \end{cases}$$

where n and m are nonnegative integers.

There are many identities involving binomial coefficients. However, computations related to the inverses of binomial coefficients are difficult. For some results involving the inverses of binomial coefficients, see [5, 4, 1, 6, 8, 9, 7, 10]. In order to compute sums involving the inverses of binomial coefficients, using integrals is an effective approach. This idea is based on Euler's well-known Beta function defined by (see [8])

$$B(n,m)=\int_0^1 t^{n-1}(1-t)^{m-1}dt$$

for all positive integers n and m. Since $B(n,m)=\dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}=\dfrac{(n-1)!(m-1)!}{(n+m-1)!}$, the inverse binomial coefficient $\binom{n}{m}^{-1}$ satisfies the identity

$$\binom{n}{m}^{-1} = (n+1)\int_0^1 t^m(1-t)^{n-m}\,dt. \tag{1}$$

with this method, a series of identities related to the inverses of binomial coefficients is obtained (see [8, 9, 7, 10]). In this paper, we also use Eq. (1) to evaluate the sums

$$\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n(n+k)\binom{2n}{n}},\qquad \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n^2(n+k)\binom{2n}{n}},$$

$$\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n(n+k)\binom{2n+k}{n}},\quad\text{and}\quad \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n(n+k)\binom{2n+2k}{n+k}},$$

where $|\varepsilon|=1$, and k is an arbitrary positive integer with $k>1$. For convenience, we put

$$\begin{aligned}
S_1(k) &= \sum_{n=1}^{\infty}\frac{1}{n(n+k)\binom{2n}{n}},\quad S_2(k)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+k)\binom{2n}{n}},\\
T_1(k) &= \sum_{n=1}^{\infty}\frac{1}{n^2(n+k)\binom{2n}{n}},\quad T_2(k)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n+k)\binom{2n}{n}},\\
Q_1(k) &= \sum_{n=1}^{\infty}\frac{1}{n(n+k)\binom{2n+k}{n}},\quad Q_2(k)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+k)\binom{2n+k}{n}},\\
R_1(k) &= \sum_{n=1}^{\infty}\frac{1}{n(n+k)\binom{2n+2k}{n+k}},\quad R_2(k)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+k)\binom{2n+2k}{n+k}}.
\end{aligned}$$

In the next section, we evaluate the sums above. In the third section, we define $W_k=\sum_{n=1}^{\infty}\dfrac{1}{n^k\binom{2n}{n}}$ and $X_r=\dfrac{1}{r}\sum_{k=1}^{r}W_k$; our aim is to compute $\lim_{r\to+\infty}X_r$.

2 Some Results For $S_i(k)$ And $T_i(k)$ $(1 \leq i \leq 2)$

Theorem 2.1. *Let k be a positive integer with $k \geq 2$. Then*

$$\begin{aligned}
S_1(k) &= \frac{1-2k}{k}\int_0^1 \frac{\ln[1-t(1-t)]+\sum_{i=1}^{k} t^i(1-t)^i/i}{t^k(1-t)^k}dt+\frac{1}{k}\left(2-\frac{\sqrt{3}\pi}{3}\right), &(2)\\
S_2(k) &= (-1)^k\frac{(1-2k)}{k}\int_0^1 \frac{\ln[1+t(1-t)]+\sum_{i=1}^{k}(-1)^i t^i(1-t)^i/i}{t^k(1-t)^k}dt\\
&\quad -\frac{2}{k}\left(\sqrt{5}\ln\frac{\sqrt{5}+1}{2}-1\right), &(3)\\
T_1(k) &= \frac{\pi^2}{18k}-\frac{S_1(k)}{k}, &(4)\\
T_2(k) &= -\frac{2}{k}\left(\ln\frac{\sqrt{5}-1}{2}\right)^2-\frac{S_2(k)}{k}. &(5)
\end{aligned}$$

Proof. It follows from Eq. (1) that

$$\begin{aligned}
S_1(k) &= \frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 t^n(1-t)^n dt+\left(2-\frac{1}{k}\right)\sum_{n=1}^{\infty}\frac{\int_0^1 t^n(1-t)^n dt}{n+k}\\
&= \frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 t^n(1-t)^n dt+\left(2-\frac{1}{k}\right)\sum_{n=k+1}^{\infty}\frac{\int_0^1 t^{n-k}(1-t)^{n-k} dt}{n},
\end{aligned}$$

$$S_2(k) = \frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^n\int_0^1 t^n(1-t)^n dt}{n}+\left(2-\frac{1}{k}\right)\sum_{n=k+1}^{\infty}\frac{(-1)^{n-k}\int_0^1 t^{n-k}(1-t)^{n-k} dt}{n}.$$

It is well known that

$$\sum_{n=1}^{\infty}\frac{u^n}{n}=-\ln(1-u),\quad \text{for}\quad u\in[-1,1). \tag{6}$$

On the other hand,

$$\begin{aligned}
&\int_0^1 \ln(1-t+t^2)dt=-2+\frac{\sqrt{3}\pi}{3}, &(7)\\
&\int_0^1 \ln(1+t-t^2)dt=2\left(\sqrt{5}\ln\frac{\sqrt{5}+1}{2}-1\right). &(8)
\end{aligned}$$

From Eqs. (6-8) we get Eqs. (2) and (3).

Now we give the proofs of Eqs. (4-5).
One can verify that

$$T_1(k) = \frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} - \frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n(n+k)\binom{2n}{n}} = \frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} - \frac{1}{k}S_1(k),$$

$$T_2(k) = \frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\binom{2n}{n}} - \frac{1}{k}S_2(k).$$

It follows from [2, 10] that

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\binom{2n}{n}} = -2\left(\ln\frac{\sqrt{5}-1}{2}\right)^2.$$

Hence Eqs. (4-5) are valid. □

Next, we extend $S_i(k)$ and $T_i(k)(i=1,2)$ to the following forms:

$$\begin{aligned} S_1(k,m) &= \sum_{n=1}^{\infty}\frac{1}{n(n+k)\binom{2mn}{mn}}, \quad S_2(k,m) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+k)\binom{2mn}{mn}}, \\ T_1(k,m) &= \sum_{n=1}^{\infty}\frac{1}{n^2(n+k)\binom{2mn}{mn}}, \quad \text{and} \quad T_2(k,m) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n+k)\binom{2mn}{mn}}. \end{aligned}$$

we give the corresponding results as a corollary:

Corollary 2.1. *Let k and m be positive integers. Then*

$$\begin{aligned} S_1(k,m) &= \left(\frac{1}{k}-2m\right)\int_0^1 \frac{\ln[1-t^m(1-t)^m] + \sum_{i=1}^{k} t^{mi}(1-t)^{mi}/i}{t^{mk}(1-t)^{mk}}dt \\ &\quad -\frac{1}{k}\int_0^1 \ln[1-t^m(1-t)^m]dt, \end{aligned} \tag{9}$$

$$\begin{aligned} S_2(k,m) &= (-1)^k\left(\frac{1}{k}-2m\right)\int_0^1 \frac{\ln[1+t^m(1-t)^m] + \sum_{i=1}^{k} (-1)^i t^{mi}(1-t)^{mi}/i}{t^{mk}(1-t)^{mk}}dt \\ &\quad -\frac{1}{k}\int_0^1 \ln[1+t^m(1-t)^m]dt, \end{aligned} \tag{10}$$

$$T_1(k,m) = -\frac{m}{2k}\int_0^1 \frac{\ln[1-t^m(1-t)^m]dt}{t(1-t)} - \frac{1}{k}S_1(k,m), \tag{11}$$

$$T_2(k,m) = -\frac{m}{2k}\int_0^1 \frac{\ln[1+t^m(1-t)^m]dt}{t(1-t)} - \frac{1}{k}S_2(k,m). \tag{12}$$

Proof. We only give the proofs of Eqs. (11-12), and leave the proofs of Eqs. (9-10) to the reader. We can immediately obtain that

$$T_1(k,m)=\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2mn}{mn}}-\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n(n+k)\binom{2mn}{mn}}=\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2mn}{mn}}-\frac{1}{k}S_1(k,m),$$

$$T_2(k,m)=\frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\binom{2mn}{mn}}-\frac{1}{k}S_2(k,m).$$

Owing to the conclusions (see [10]):

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2mn}{mn}}=-\frac{m}{2}\int_0^1\frac{\ln[1-t^m(1-t)^m]dt}{t(1-t)}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\binom{2mn}{mn}}=-\frac{m}{2}\int_0^1\frac{\ln[1+t^m(1-t)^m]dt}{t(1-t)},$$

we can show that Eqs. (11-12) hold. □

It is evident that Eqs. (9-12) are the generalizations of Eqs. (2-5), respectively. Now we give the recurrence relation for $S_i(k)$ and $T_i(k)$.

Theorem 2.2. *Let k be a positive integer with $k\geq 2$. Then*

$$S_1(k+1) = \frac{2(2k+1)}{k+1}S_1(k)+\frac{1}{(k+1)^2}-\frac{\sqrt{3}\pi}{3(k+1)}, \tag{13}$$

$$S_2(k+1) = -\frac{2(2k+1)}{k+1}S_2(k)+\frac{1}{(k+1)^2}-\frac{2\sqrt{5}}{k+1}\ln\frac{\sqrt{5}+1}{2}, \tag{14}$$

$$T_1(k+1) = -\frac{(3k+1)\pi^2}{18(k+1)^2}+\frac{2k(2k+1)}{(k+1)^2}T_1(k)-\frac{1}{(k+1)^3}+\frac{\sqrt{3}\pi}{3(k+1)^2}, \tag{15}$$

$$\begin{aligned}T_2(k+1) &= -\frac{2(5k+3)}{(k+1)^2}\left(\ln\frac{\sqrt{5}-1}{2}\right)^2-\frac{2k(2k+1)}{(k+1)^2}T_2(k)-\frac{1}{(k+1)^3}\\&\quad+\frac{2\sqrt{5}}{(k+1)^2}\ln\frac{\sqrt{5}+1}{2}.\end{aligned} \tag{16}$$

Proof. For $0<a\leq 1$, we consider the integrals:

$$I_k(a)=\left(\frac{1}{k}-2\right)\int_0^1\frac{\ln[1-at(1-t)]+\sum_{i=1}^{k}a^it^i(1-t)^i/i}{t^k(1-t)^k}dt$$

$$J_k(a) = (-1)^k\left(\frac{1}{k}-2\right)\int_0^1\frac{\ln[1+at(1-t)]+\sum_{i=1}^{k}(-1)^ia^it^i(1-t)^i/i}{t^k(1-t)^k}dt,$$

where $I_k(0)=0$ and $J_k(0)=0$. It is clear that

$$S_1(k)=I_k(1)+\frac{1}{k}\Big(2-\frac{\sqrt{3}\pi}{3}\Big) \quad\text{and}\quad S_2(k)=J_k(1)-\frac{2}{k}\Big(\sqrt{5}\ln\frac{\sqrt{5}+1}{2}-1\Big).$$

When $0<a\le 1$, we have

$$\begin{aligned}
I_k'(a) &= \Big(\frac{1}{k}-2\Big)a^{k-1}\Big(1-\int_0^1\frac{dt}{1-at+at^2}\Big).\\
&= \Big(\frac{1}{k}-2\Big)\Big(a^{k-1}-4a^{k-2}\sqrt{\frac{a}{4-a}}\arctan\sqrt{\frac{a}{4-a}}\Big),\\
J_k'(a) &= \Big(\frac{1}{k}-2\Big)\int_0^1\frac{a^k t(1-t)dt}{1+at(1-t)}\\
&= \Big(\frac{1}{k}-2\Big)\Big(a^{k-1}-\frac{2a^{k-2}\sqrt{a}}{\sqrt{a+4}}\ln\frac{\sqrt{\frac{a+4}{a}}+1}{\sqrt{\frac{a+4}{a}}-1}\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
I_k(a) &= -\frac{(2k-1)a^k}{k^2}+\frac{4(2k-1)}{k}\int a^{k-2}\sqrt{\frac{a}{4-a}}\arctan\sqrt{\frac{a}{4-a}}da.\\
J_k(a) &= -\frac{(2k-1)a^k}{k^2}+\frac{2(2k-1)}{k}\int a^{k-2}\sqrt{\frac{a}{a+4}}\ln\frac{\sqrt{\frac{a+4}{a}}+1}{\sqrt{\frac{a+4}{a}}-1}da.
\end{aligned}$$

Let $u=\sqrt{\frac{a}{4-a}}$ and $v=\sqrt{\frac{a+4}{a}}$. Then we get

$$I_k(a) = -\frac{(2k-1)a^k}{k^2}+\frac{(2k-1)2^{2k+1}}{k}\int\frac{u^{2k-2}\arctan u}{(1+u^2)^k}du, \tag{17}$$

$$J_k(a) = -\frac{(2k-1)a^k}{k^2}-\frac{(2k-1)4^k}{k}\int\frac{1}{(v^2-1)^k}\ln\frac{v+1}{v-1}dv. \tag{18}$$

It is well known that

$$\begin{aligned}
\int\frac{u^{2k}\arctan u}{(1+u^2)^{k+1}}du &= \frac{2k-1}{2k}\int\frac{u^{2k-2}\arctan u}{(1+u^2)^k}du-\frac{u^{2k-1}\arctan u}{2k(1+u^2)^k}\\
&\quad+\frac{1}{2k}\int\frac{u^{2k-1}}{(1+u^2)^{k+1}}du,\\
\int\frac{1}{(v^2-1)^{k+1}}\ln\frac{v+1}{v-1}dv &= -\frac{2k-1}{2k}\int\frac{1}{(v^2-1)^k}\ln\frac{v+1}{v-1}dv+\frac{1}{2k^2(v^2-1)^k}\\
&\quad-\frac{v}{2k(v^2-1)^k}\ln\frac{v+1}{v-1}.
\end{aligned}$$

In the meantime, we note that

$$\int \frac{u^{2k-1}}{(1+u^2)^{k+1}}du = \frac{2}{4^{k+1}}\int a^{k-1}da \quad \text{and} \quad I_k(0) = J_k(0) = 0.$$

Therefore $I_k(a)$ and $J_k(a)$ satisfy that

$$\frac{k+1}{2k+1}I_{k+1}(a) = 2I_k(a) - \frac{a^{k+1}}{k+1} + \frac{4a^k}{k} - \frac{4\sqrt{4-a}\,a^{k-1/2}}{k}\arctan\sqrt{\frac{a}{4-a}}, \tag{19}$$

$$\frac{k+1}{2k+1}J_{k+1}(a) = -2J_k(a) - \frac{a^{k+1}}{k+1} - \frac{4a^k}{k} + \frac{2a^{k-1/2}\sqrt{a+4}}{k}\ln\frac{\sqrt{a+4}+\sqrt{a}}{\sqrt{a+4}-\sqrt{a}}. \tag{20}$$

From Eqs. (19-20) we can derive Eqs. (13-14). According to Eqs. (4-5) we can obtain Eqs. (15-16). □

We note that the recurrences given in Eqs. (13-14) are similar to [3, Eq. (28)].

Theorem 2.3. *Let k be a positive integer with $k \geq 2$. Then*

$$Q_1(k) = \left(\frac{1}{k}-1\right)\int_0^1 \frac{\ln[1-t(1-t)] + \sum_{i=1}^{k} t^i(1-t)^i/i}{t^k}dt - \left(1+\frac{1}{k}\right)\int_0^1 (1-t)^k \ln[1-t(1-t)]dt, \tag{21}$$

$$Q_2(k) = (-1)^k\left(\frac{1}{k}-1\right)\int_0^1 \frac{\ln[1+t(1-t)] + \sum_{i=1}^{k} (-1)^i t^i(1-t)^i/i}{t^k}dt - \left(1+\frac{1}{k}\right)\int_0^1 (1-t)^k \ln[1+t(1-t)]dt, \tag{22}$$

$$R_1(k) = \frac{1}{k}\int_0^1 \left\{\ln[1-t(1-t)] + \sum_{i=1}^{k}\frac{t^i(1-t)^i}{i}\right\}dt - \left(2+\frac{1}{k}\right)\int_0^1 t^k(1-t)^k \ln[1-t(1-t)]dt, \tag{23}$$

$$R_2(k) = \frac{(-1)^k}{k}\int_0^1 \left\{\ln[1+t(1-t)] + \sum_{i=1}^{k}\frac{(-1)^i t^i(1-t)^i}{i}\right\}dt - \left(2+\frac{1}{k}\right)\int_0^1 t^k(1-t)^k \ln[1+t(1-t)]dt. \tag{24}$$

Proof. We only give the proof of Eq. (21). The proofs of Eqs. (22-24) follow the same pattern and are omitted here. It follows from Eq. (1) and Eq. (6) that

$$\begin{aligned}
Q_1(k) &= \sum_{n=1}^{\infty} \frac{2n+k+1}{n(n+k)} \int_0^1 t^n (1-t)^{n+k} dt \\
&= \left(1 - \frac{1}{k}\right) \sum_{n=k+1}^{\infty} \int_0^1 \frac{t^{n-k}(1-t)^n}{n} dt + \left(1 + \frac{1}{k}\right) \sum_{n=1}^{\infty} \int_0^1 \frac{t^n (1-t)^{n+k}}{n} dt. \\
&= \left(\frac{1}{k} - 1\right) \int_0^1 \frac{\ln[1-t(1-t)] + \sum_{i=1}^{k} t^i (1-t)^i / i}{t^k} dt \\
&\quad -\left(1 + \frac{1}{k}\right) \int_0^1 (1-t)^k \ln[1-t(1-t)] dt.
\end{aligned}$$

Hence Eq. (21) holds. □

By computing integrals of Eqs. (17-18) and Eqs. (21-24), we can establish a series of identities involving inverses of binomial coefficients. In the final part of this section, we evaluate special cases of $S_i(k)$, $T_i(k)$, and $R_i(k)(1 \leq i \leq 2)$ according to the particular choice of k. For example, when $k = 2$ in Eqs. (17-18), we have

$$\begin{aligned}
I_2(a) &= -\frac{3a^2}{4} + 48 \int \frac{u^2 \arctan u du}{(1+u^2)^2} \\
&= -\frac{3a^2}{4} - \frac{24u \arctan u}{1+u^2} + 12(\arctan u)^2 - \frac{12}{1+u^2} + c_2 \\
&= -\frac{3a^2}{4} - 6\sqrt{a(4-a)} \arctan \sqrt{\frac{a}{4-a}} + 12\left(\arctan \sqrt{\frac{a}{4-a}}\right)^2 - 12 + 3a + c_2. \\
J_2(a) &= -\frac{3a^2}{4} - 24 \int \frac{1}{(v^2-1)^2} \ln \frac{v+1}{v-1} dv \\
&= -\frac{3a^2}{4} + \frac{12v}{v^2-1} \ln \frac{v+1}{v-1} - \frac{12}{v^2-1} - 3\ln^2\left(\frac{v+1}{v-1}\right) + c_2' \\
&= -\frac{3a^2}{4} + 3\sqrt{a(a+4)} \ln \frac{\sqrt{a+4} + \sqrt{a}}{\sqrt{a+4} - \sqrt{a}} - 3a - 3\ln^2\left(\frac{\sqrt{a+4} + \sqrt{a}}{\sqrt{a+4} - \sqrt{a}}\right) + c_2'.
\end{aligned}$$

Since $I_2(0) = 0$ and $J_2(0) = 0$, then $c_2 = 12$, $c_2' = 0$,

$$I_2(1) = \frac{9}{4} - \sqrt{3}\pi + \frac{\pi^2}{3}, \quad J_2(1) = -\frac{15}{4} + 6\sqrt{5} \ln \frac{\sqrt{5}+1}{2} - 12\ln^2\left(\frac{\sqrt{5}+1}{2}\right).$$

Hence, we get

$$\begin{aligned} S_1(2) &= \frac{13}{4} - \frac{7\sqrt{3}\pi}{6} + \frac{\pi^2}{3}, \quad T_1(2) = \frac{7\sqrt{3}\pi}{12} - \frac{13}{8} - \frac{5\pi^2}{36}, \\ S_2(2) &= -\frac{11}{4} + 5\sqrt{5}\ln\frac{\sqrt{5}+1}{2} - 12\ln^2\left(\frac{\sqrt{5}+1}{2}\right), \\ T_2(2) &= \frac{11}{8} - \frac{5\sqrt{5}}{2}\ln\frac{\sqrt{5}+1}{2} + 5\ln^2\left(\frac{\sqrt{5}+1}{2}\right). \end{aligned}$$

By means of $S_i(2)$, $T_i(2)$, and Eqs. (13-16), we can compute other values of $S_i(k)$ and $T_i(k)(1 \leq i \leq 2, k > 2)$.

If $k = 2$ in Eqs. (23-24), we can obtain

$$\begin{aligned} R_1(2) &= \frac{1}{2}\int_0^1 \left[\ln(1-t+t^2) + t(1-t) + \frac{t^2(1-t)^2}{2}\right]dt \\ &\quad -\frac{5}{2}\int_0^1 t^2(1-t)^2\ln(1-t+t^2)dt \\ &= \frac{17}{36} - \frac{\sqrt{3}\pi}{12}, \\ R_2(2) &= \frac{1}{2}\int_0^1 \left[\ln(1+t-t^2) - t(1-t) + \frac{t^2(1-t)^2}{2}\right]dt \\ &\quad -\frac{5}{2}\int_0^1 t^2(1-t)^2\ln(1+t-t^2)dt \\ &= \frac{1}{36} + \frac{5\sqrt{5}}{6}\ln\frac{\sqrt{5}-1}{2}. \end{aligned}$$

3 The Value of $\lim\limits_{r\to+\infty} X_r$

We know that $W_1 = \frac{\sqrt{3}\pi}{9}$, $W_2 = \frac{\pi^2}{18}$, and $W_4 = \frac{17\pi^4}{3240}$ (see [2]). However, we do not know how to evaluate W_k in closed form when $k \geq 5$. In this section, we are interested in the average X_r of W_k. We compute $\lim\limits_{r\to+\infty} X_r$ by Eq. (1).

Theorem 3.1. *Let r be a positive integer with $r > 4$. Then* $\lim\limits_{r\to+\infty} X_r = \frac{1}{2}$.

Proof. One can verify that

$$X_r = \frac{1}{2} + \frac{1}{r}\sum_{n=2}^{\infty} \frac{1 - \frac{1}{n^r}}{(n-1)\binom{2n}{n}}.$$

Since $1-\dfrac{1}{n^{r+1}}<1$, the series $\displaystyle\sum_{n=2}^{\infty}\frac{1-\dfrac{1}{n^r}}{(n-1)\binom{2n}{n}}$ satisfies that

$$\sum_{n=2}^{\infty}\frac{1-\frac{1}{n^r}}{(n-1)\binom{2n}{n}} \leq \sum_{n=2}^{\infty}\frac{1}{(n-1)\binom{2n}{n}}.$$

It follows from Eq. (1) that

$$\sum_{n=2}^{\infty}\frac{1}{(n-1)\binom{2n}{n}} = \sum_{n=2}^{\infty}\frac{(2n+1)\int_0^1 t^n(1-t)^n dt}{n-1}.$$

Then

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{(n-1)\binom{2n}{n}} &= \sum_{n=1}^{\infty}\frac{(2n+3)\int_0^1 t^{n+1}(1-t)^{n+1} dt}{n} \\ &= 2\int_0^1\frac{t^2(1-t)^2 dt}{[1-t(1-t)]^2} - 3\int_0^1 t(1-t)\ln[1-t(1-t)]dt.\end{aligned}$$

Hence $\lim_{r\to+\infty} X_r = \dfrac{1}{2}$. □

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