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The k-Binomial Transforms and the Hankel Transform

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Abstract

We give a new proof of the invariance of the Hankel transform under the binomial transform of a sequence. Our method of proof leads to three variations of the binomial transform; we call these the k-binomial transforms. We give a simple means of constructing these transforms via a triangle of numbers. We show how the exponential generating function of a sequence changes after our transforms are applied, and we use this to prove that several sequences in the On-Line Encyclopedia of Integer Sequences are related via our transforms. In the process, we prove three conjectures in the OEIS. Addressing a question of Layman, we then show that the Hankel transform of a sequence is invariant under one of our transforms, and we show how the Hankel transform changes after the other two transforms are applied. Finally, we use these results to determine the Hankel transforms of several integer sequences.

1 Introduction

Given a sequence $A = \{a_0, a_1, \ldots\}$, define the *binomial transform B of a sequence A* to be the sequence $B(A) = \{b_n\}$, where b_n is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.$$

Define the *Hankel matrix of order n of A* to be the $(n+1) \times (n+1)$ upper left submatrix of

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let h_n denote the determinant of the Hankel matrix of order n. Then define the *Hankel transform H of A* to be the sequence $H(A) = \{h_0, h_1, h_2, \ldots,\}$. For example, the Hankel matrix of order 3 of the derangement numbers, $\{D_n\} = \{1, 0, 1, 2, 9, 44, 265, \ldots\}$, is

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 9 \\ 1 & 2 & 9 & 44 \\ 2 & 9 & 44 & 265 \end{bmatrix}.$$

The determinant of this matrix is 144, which is $(0!)^2(1!)^2(2!)^2(3!)^2$. Flajolet [4] and Radoux [9] have shown that the Hankel transform of the derangement numbers is $\{\prod_{i=0}^{n}(n!)^2\}$.

Although the determinants of Hankel matrices had been studied before, the term "Hankel transform" was introduced in 2001 by Layman [7]. Other recent papers involving Hankel determinants include those by Ehrenborg [3], Peart [8], and Woan and Peart [15].

Layman [7] proves that the Hankel transform is invariant under the binomial transform, in the sense that $H(B(A)) = H(A)$. He also proves that the Hankel transform is invariant under the invert transform (see the On-Line Encyclopedia of Integer Sequences [11] for the definition), and he asks if there are other transforms for which the Hankel transform is invariant. This article partly addresses Layman's question. We provide a new proof of the invariance of the Hankel transform under the binomial transform. Our method of proof generalizes to three variations of the binomial transform that we call the *k-binomial transform*, the *rising k-binomial transform*, and the *falling k-binomial transform*. Collectively, we refer to these as the *k-binomial transforms*. (There should be no confusion in context.) We give a simple means of constructing these transforms using a triangle of numbers, and we provide combinatorial interpretations of the transforms as well. We show how the exponential generating function of a sequence changes after applying our transforms, and we use these results to prove that several sequences in the On-Line Encyclopedia of Integer Sequences (OEIS) [11] are related by the transforms. In the process, we prove three conjectures listed in the OEIS concerning the binomial mean transform. Then, giving an answer to Layman's question, we show that the Hankel transform is invariant under the falling k-binomial transform. The Hankel transform is not invariant under the k-binomial transform and the rising k-binomial transform, but we give a formula showing how the Hankel transform changes under these two transforms. These results, together with our proofs of relationships between sequences in the OEIS, determine the Hankel transforms of several sequences in the OEIS.

(Unfortunately, this is some discrepancy in the literature in the definitions of the Hankel determinant and the binomial transform of an integer sequence. The definitions of the Hankel determinant in Layman [7] and in Ehrenborg [3] are slightly different from each other, and ours is slightly different from both of these. As many sequences of interest begin indexing with 0, we define the Hankel determinant and the Hankel transform so that the first elements in the sequences A and $H(A)$ are indexed by 0. Layman's definition begins indexing A and $H(A)$ by 1, whereas Ehrenborg's definition results in indexing A beginning with 0 and $H(A)$ beginning with 1. Our definition of the binomial transform is that used by Layman [7] and the OEIS [11]. Somewhat different definitions are given in Knuth [6] (p. 136) and by MathWorld [13].)

2 Invariance of the Hankel transform under the binomial transform

Layman [7] proves that $H(B(A)) = H(A)$ for any sequence A. He does this by showing that the Hankel matrix of order n of $B(A)$ can be obtained by multiplying the Hankel matrix of order n of A by certain upper and lower triangular matrices, each of which have determinant 1.

We present a new proof of this result. Our proof technique suggests generalizations of the binomial transform, which we discuss in subsequent sections. We require the following lemma.

Lemma 2.1. *Given a sequence $A = \{a_0, a_1, a_2, \ldots\}$, create a triangle of numbers T using the following rule:*

1. *The left diagonal of the triangle consists of the elements of A.*
2. *Any number off the left diagonal is the sum of the number to its left and the number diagonally above it to the left.*

Then the sequence on the right diagonal is the binomial transform of A.

For example, the binomial transform of the derangement numbers is the factorial numbers [7]. Figure 1 illustrates how the factorial numbers can be generated from the derangement numbers using the triangle described in Lemma 2.1.

Although they do not use the term *binomial transform*, Lemma 2.1 is essentially proven by Graham, Knuth, and Patashnik [5] (pp. 187–192). We present a different proof, one that allows us to prove similar results for the k-binomial transforms we discuss in subsequent sections.

1

0 1

1 1 2

2 3 4 6

9 11 14 18 24

44 53 64 78 96 120

265 309 362 426 504 600 720

Figure 1: Derangement triangle

Proof. Let t_n be the n^{th} element on the right diagonal of the triangle. By construction of the triangle, we can see from Figure 2 that the number of times element a_i contributes to the value of t_n is the number of paths from a_i to t_n. To move from a_i to t_n requires n path segments, i of which move directly to the right. Thus there are $\binom{n}{i}$ ways to choose which of the n ordered segments are the rightward-moving segments, and the down segments are completely determined by this choice. Therefore the contribution of a_i to t_n is $\binom{n}{i}a_i$, and the value of t_n, in terms of the elements on the left diagonal, is $\sum_{i=0}^{n}\binom{n}{i}a_i$. But this is the definition of the binomial transform, making the right diagonal of the triangle the binomial transform of A. $\square$

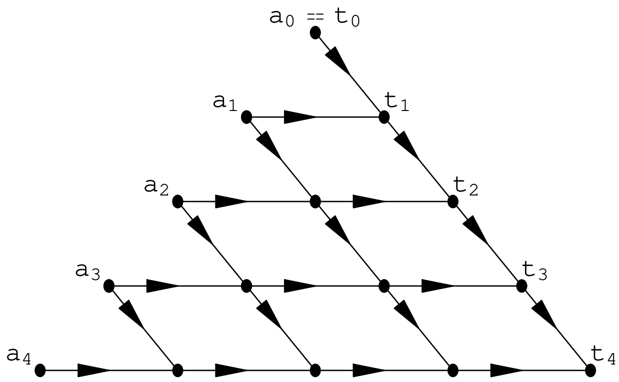

Figure 2: Directed graph underlying the binomial transform

The binomial transform has the following combinatorial interpretation: If a_n represents the number of arrangements of n labeled objects with some property P, then b_n represents the number of ways of dividing n labeled objects into two groups such that the first group has property P. In terms of the derangement and factorial numbers, then, as D_n is the number of permutations of n ordered objects in which no object remains in its original position, $n!$ is the number of ways that one can divide n labeled objects into two groups, order the objects in the first group, and then permute the first group objects so that none remains in its original position.

The numbers in the triangle described in Lemma 2.1, not just the right and left diagonals, can also have combinatorial interpretations. For instance, the triangle of numbers in Figure 1 is discussed as a combinatorial entity in its own right in another article by the first author [12]. The number in row i, position j, in the triangle is the number of permutations of i ordered objects such that every object after j does not remain in its original position.

We now give our proof of Layman's result.

Theorem 2.1. *(Layman) The Hankel transform is invariant under the binomial transform.*

Proof. We define a procedure for transforming the Hankel matrix of order n of a sequence A to the Hankel matrix of order n of $B(A)$ using only matrix row and column addition. While perhaps more complicated than Layman's proof, ours has the virtue of being easily modified to give proofs for the Hankel transforms of the k-binomial transforms that we discuss subsequently. The procedure is as follows:

1. Given a sequence $A = \{a_0, a_1, \ldots\}$, create the triangle of numbers described in Lemma 2.1, where $T_{i,j}$ is the $(i,j)^{th}$ entry in the triangle.

2. Let T_n be the following matrix consisting of numbers from the left diagonal of T:

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n,0} \\ T_{1,0} & T_{2,0} & T_{3,0} & \cdots & T_{n+1,0} \\ T_{2,0} & T_{3,0} & T_{4,0} & \cdots & T_{n+2,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n,0} & T_{n+1,0} & T_{n+2,0} & \cdots & T_{2n,0} \end{bmatrix}.$$

   Since $a_i = T_{i,0}$, T_n is the Hankel matrix of order n of A.

3. Then apply the following transformations to T_n, where rows and columns of the matrix are indexed beginning with 0.

   (a) Let i range from 1 to n. During stage i, for each row $j \geq i$, add row $j-1$ to row j and replace row j with the result.

   (b) Then let i again range from 1 to n. During stage i, for each column $j \geq i$, add column $j-1$ to column j and replace column j with the result.

Claim 1: After stage i in 3(a), row m of the matrix is of the following form:

$$\begin{bmatrix} T_{m,m} & T_{m+1,m} & \cdots & T_{n+m,m} \end{bmatrix}, \text{ if } m \leq i;$$

$$\begin{bmatrix} T_{m,i} & T_{m+1,i} & \cdots & T_{n+m,i} \end{bmatrix}, \text{ if } m > i.$$

The claim is clearly true initially, when $i = 0$. Now, assume the claim is true for all values of i from 0 to $k-1$. Then, in stage $i = k$, the only rows that change are rows $k, k+1, \ldots, n$. Row m, for $m \geq k$, is the sum of rows m and $m-1$ from the previous iteration:

$$\begin{bmatrix} T_{m,k-1} + T_{m-1,k-1} & T_{m+1,k-1} + T_{m,k-1} & \cdots & T_{n+m,k-1} + T_{n+m-1,k-1} \end{bmatrix}.$$

But, by the definition of T, $T_{i,j} + T_{i-1,j} = T_{i,j+1}$. Thus this row is equal to

$$\begin{bmatrix} T_{m,k} & T_{m+1,k} & \cdots & T_{n+m,k} \end{bmatrix},$$

proving the claim.

After the transformations in 3(a) are applied, then, we have the matrix

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n,0} \\ T_{1,1} & T_{2,1} & T_{3,1} & \cdots & T_{n+1,1} \\ T_{2,2} & T_{3,2} & T_{4,2} & \cdots & T_{n+2,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n,n} & T_{n+1,n} & T_{n+2,n} & \cdots & T_{2n,n} \end{bmatrix}.$$

Claim 2: After stage i in 3(b), column m of the matrix is of the following form:

$$\begin{bmatrix} T_{m,m} & T_{m+1,m+1} & \cdots & T_{n+m,n+m} \end{bmatrix}^T, \text{ if } m \leq i;$$

$$\begin{bmatrix} T_{m,i} & T_{m+1,i+1} & \cdots & T_{n+m,n+i} \end{bmatrix}^T, \text{ if } m > i.$$

The proof is almost the same as that for Claim 1. The claim is clearly true for $i = 0$. Assume the claim is true for all values of i from 0 to $k-1$. In stage $i = k$, the only columns that change are columns $k, k+1, \ldots, n$. Column m, for $m \geq k$, is the sum of columns m and $m-1$ from the previous iteration:

$$\begin{bmatrix} T_{m,k-1} + T_{m-1,k-1} & T_{m+1,k} + T_{m,k} & \cdots & T_{n+m,n+k-1} + T_{n+m-1,n+k-1} \end{bmatrix}^T.$$

Again, by the definition of T, $T_{i,j} + T_{i-1,j} = T_{i,j+1}$. Thus this column is equal to

$$\begin{bmatrix} T_{m,k} & T_{m+1,k+1} & \cdots & T_{n+m,n+k} \end{bmatrix}^T,$$

which proves the claim.

After applying the transformations in 3(b), then, we have the matrix

$$\begin{bmatrix} T_{0,0} & T_{1,1} & T_{2,2} & \cdots & T_{n,n} \\ T_{1,1} & T_{2,2} & T_{3,3} & \cdots & T_{n+1,n+1} \\ T_{2,2} & T_{3,3} & T_{4,4} & \cdots & T_{n+2,n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n,n} & T_{n+1,n+1} & T_{n+2,n+2} & \cdots & T_{2n,2n} \end{bmatrix}.$$

But this is the Hankel matrix of order n of $B(A)$, as $B(A)$ is the right diagonal of triangle T. Since the only matrix manipulations we used were adding a row to another row and adding a column to another column, and the determinant of a matrix is invariant under these operations [2] (p. 262), the determinant of the Hankel matrix of order n of A is equal to the determinant of the Hankel matrix of order n of $B(A)$. $\square$

The operations described in the proof of Theorem 2.1, when done in the order prescribed by the procedure, have a simple interpretation in terms of the triangle. Adding a row to another row shifts a left diagonal in the triangle one place to the right, and adding a column to another column shifts a right diagonal one place to the right. We can see this in the case of the Hankel matrix of order 2 of the derangement numbers by a comparison of Figure 1 and the sequence of matrices arising from the procedure described in the proof of Theorem 2.1.

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 1 & 2 & 9\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 3\\ 1 & 3 & 11\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 3\\ 2 & 4 & 14\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 2 & 6 & 18\end{bmatrix} \to \begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 6\\ 2 & 6 & 24\end{bmatrix}.$$

3 The k-binomial transforms

We now consider three variations of the binomial transform. All three transforms take two parameters: the input sequence A and a scalar k.

The *k-binomial transform W of a sequence A* is the sequence $W(A,k) = \{w_n\}$, where w_n is given by

$$w_n = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^n a_i = k^n \sum_{i=0}^{n} \binom{n}{i} a_i, & \text{if } k \neq 0 \text{ or } n \neq 0; \\ a_0, & \text{if } k = 0, n = 0. \end{cases}$$

The *rising k-binomial transform R of a sequence A* is the sequence $R(A,k) = \{r_n\}$, where r_n is given by

$$r_n = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^i a_i, & \text{if } k \neq 0; \\ a_0, & \text{if } k = 0. \end{cases}$$

The *falling k-binomial transform F of a sequence A* is the sequence $F(A,k) = \{f_n\}$, where f_n is given by

$$f_n = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i, & \text{if } k \neq 0; \\ a_n, & \text{if } k = 0. \end{cases}$$

The case $k = 0$ must be dealt with separately because 0^0 would occur in the formulas otherwise. Our definitions effectively take 0^0 to be 1. These turn out to be "good" definitions, in the sense that all the results discussed subsequently hold under our definitions for the $k = 0$ case. When $k = 0$, the k-binomial transform of A is the sequence $\{a_0, 0, 0, 0, \ldots\}$, the rising k-binomial transform of A is $\{a_0, a_0, a_0, \ldots\}$, and the falling k-binomial transform is the identity transform.

The k-binomial transform when $k = 1/2$ is of special interest; this is the *binomial mean transform*, defined in the OEIS [11], sequence A075271.

When k is a positive integer, these variations of the binomial transform all have combinatorial interpretations similar to that of the binomial transform, although, unlike the binomial transform, they have a two-dimensional component. If a_n represents the number of arrangements of n labeled objects with some property P, then w_n represents the number of ways of dividing n objects such that

- In one dimension, the n objects are divided into two groups so that the first group has property P.
- In a second dimension, the n objects are divided into k labeled groups.

The interpretation of the second dimension could be something as simple as a coloring of each object from a choice of k colors, independent of the division of the objects in the first dimension. For example, if the input sequence is the derangement numbers, w_n is the number of ways of dividing n labeled objects into two groups such that the objects in the first group are deranged and each of the n objects has been colored one of k colors, independently of the initial division into two groups. With this interpretation of w_n in mind, r_n represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and each object in the first group is further placed into one of k labeled groups (e.g., colored using one of k colors). Similarly, f_n represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and the objects in the second group are further placed into k labeled groups (e.g., colored using one of k colors).

Each of the k-binomial transforms can also be replicated via a triangle like that described in Lemma 2.1.

Theorem 3.1. *Given a sequence $A = \{a_0, a_1, a_2, \ldots\}$,*

1. *Create a triangle of numbers so that the left diagonal consists of the elements of A, and any number off the left diagonal is k times the sum of the numbers to its left and diagonally above it to the left. Then the right diagonal is the k-binomial transform $W(A, k)$ of A.*

2. *Create a triangle of numbers so that the left diagonal consists of the elements of A, and any number off the left diagonal is the sum of the number diagonally above it to the left and k times the number to its left. Then the right diagonal is the rising k-binomial transform $R(A, k)$ of A.*

3. *Create a triangle of numbers so that the left diagonal consists of the elements of A, and any number off the left diagonal is the sum of the number to its left and k times the number diagonally above it to the left. Then the right diagonal is the falling k-binomial transform $F(A, k)$ of A.*

For example, applying the procedure in Part 1 with $k = 2$ to the derangement numbers (sequence A000166) yields the triangle of numbers in Figure 3.

1
0 2
1 2 8
2 6 16 48
9 22 56 144 384
44 106 256 624 1536 3840
265 618 1448 3408 8064 19,200 46,080

Figure 3: Derangement triangle with 2-binomial transform

As we shall see, the sequence on the right diagonal is the double factorial numbers (sequence A000165), the 2-binomial transform of the derangement numbers.

We now prove Theorem 3.1.

Proof. Suppose $k \neq 0$.

(*Part 1.*) The proof is similar to that of Lemma 2.1. Let t_n be the n^{th} element on the right diagonal of the triangle. There are still n path segments from a_i to t_n; however, traversing a path segment now incurs a multiplicative factor of k rather than the understood factor of 1 incurred with the binomial transform. Thus each path from a_i to t_n contributes $k^n a_i$ to the value of t_n. With i moves to the right, there are still $\binom{n}{i}$ different paths from a_i to t_n; thus the total contribution from a_i to the value of t_n is $\binom{n}{i} k^n a_i$. Therefore, $t_n = \sum_{i=0}^{n} \binom{n}{i} k^n a_i$, the n^{th} element of $W(A, k)$.

(*Part 2.*) The proof is similar to that of Part 1. The difference is that the multiplicative factor of k only appears on moves to the right, not moves down. With i rightward-moving path segments, the contribution from a_i along one path to t_n is thus $k^i a_i$. The total contribution from a_i over all paths is therefore $\binom{n}{i} k^i a_i$, and we have $t_n = \sum_{i=0}^{n} \binom{n}{i} k^i a_i$, the n^{th} element of $R(A, k)$.

(*Part 3.*) The proof is identical to that of Part 2, except that the multiplicative factor of k occurs on down path segments rather than on right-moving segments. As there are $n - i$ down path segments from a_i to t_n, we have $t_n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i$, the n^{th} element of $F(A, k)$.

Now, suppose $k = 0$. For the triangle described in Part 1, every path from the left diagonal to t_n is a zero path, except for the empty path from a_0 to t_0. Thus $t_0 = a_0$, and $t_n = 0$ for $n \neq 0$. For the triangle in Part 2, the only nonzero path to t_n from the left diagonal is the one straight down from a_0; thus $t_n = a_0$ for all n. For Part 3, the only nonzero path to t_n from the left diagonal is the one straight right from a_n; therefore, $t_n = a_n$ for each n. ◻

There are also some nice relationships between our transforms, the binomial transform, and the inverse binomial transform. Given a sequence $A = \{a_0, a_1, \ldots\}$, the *inverse binomial transform of* A is defined to be the sequence $B^{-1}(A) = \{c_n\}$, where c_n is given by

$$c_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_i.$$

It is easy to show that $B^{-1}(B(A)) = B(B^{-1}(A)) = A$.

For $k \in \mathbb{Z}^+$, let $B^k(A)$ denote k successive applications of the binomial transform; i.e., $B^k(A) = B(B(\cdots B(A) \cdots))$, where B occurs k times. For $k \in \mathbb{Z}^-$, let $B^k(A)$ denote k successive applications of the inverse binomial transform. Finally, let $B^0(A)$ denote the identity transform. Then we have

Theorem 3.2. *If $k \in \mathbb{Z}$, $B^k(A) = F(A, k)$. In other words, k successive applications of the binomial transform (or the inverse binomial transform, if $k \in \mathbb{Z}^-$) is equivalent to the falling k-binomial transform.*

Proof. The theorem is clearly true when $k = 1$. Assume the theorem is true for values of k from 1 to m. Let b_n^k denote the n^{th} element in the sequence $B^k(A)$, and let f_n^k denote the

n^{th} element in the sequence $F(A,k)$. For $k = m+1$, we have

$$\begin{aligned}
b_n^{m+1} &= \sum_{i=0}^{n}\binom{n}{i}b_i^m = \sum_{i=0}^{n}\binom{n}{i}f_i^m = \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}m^{i-j}a_j = \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}m^{i-j}a_j \\
&= \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{j}\binom{n-j}{i-j}m^{i-j}a_j = \sum_{j=0}^{n}\binom{n}{j}a_j\sum_{i=j}^{n}\binom{n-j}{i-j}m^{i-j} = \sum_{j=0}^{n}\binom{n}{j}a_j\sum_{l=0}^{n-j}\binom{n-j}{l}m^l \\
&= \sum_{j=0}^{n}\binom{n}{j}a_j(m+1)^{n-j} = f_n^{m+1}.
\end{aligned}$$

The second equality holds by the induction hypothesis, the fifth by trinomial revision [10] (p. 104), and the second-to-last by the Binomial Theorem. This proves the relationship for positive values of k. The proof for $k \in \mathbb{Z}^-$ is an almost identical induction with $k = -1$ as the base case and moving down. When $k = 0$ we have the identity transform in both cases. □

We also have the following relationship between the W, R, and F transforms.

Theorem 3.3. $W(A,k) = F(R(A,k), k-1)$. *In other words, the* k*-binomial transform is equivalent to the composition of the falling* $(k-1)$*-binomial transform with the rising* k*-binomial transform.*

Proof. For $k = 1$, we have $W(A,1) = B(A)$, and $F(R(A,1),0) = F(B(A),0) = B(A)$.

For $k = 0$, $R(A,0) = \{a_0, a_0, a_0, \ldots\}$. Then the n^{th} term of $F(R(A,0),-1)$ is given by

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a_0 = a_0\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}.$$

But the summation on the right-hand side is 1 if $n = 0$ and 0 otherwise [10] (p. 104). Thus $F(R(A,0),-1) = \{a_0, 0, 0, \ldots\}$, which is the sequence $W(A,0)$.

Assuming $k \neq 0$ and $k \neq 1$, the n^{th} term of $F(R(A,k),k-1)$ is given by

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(k-1)^{n-i}\sum_{j=0}^{i}\binom{i}{j}k^j a_j = \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}(k-1)^{n-i}k^j a_j \\
&= \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{j}\binom{n-j}{i-j}(k-1)^{n-i}k^j a_j = \sum_{j=0}^{n}\binom{n}{j}k^j a_j\sum_{i=j}^{n}\binom{n-j}{i-j}(k-1)^{n-i} \\
&= \sum_{j=0}^{n}\binom{n}{j}k^j a_j\sum_{l=0}^{n-j}\binom{n-j}{l}(k-1)^{n-j-l} = \sum_{j=0}^{n}\binom{n}{j}k^j a_j k^{n-j} = \sum_{j=0}^{n}\binom{n}{j}k^n a_j.
\end{aligned}$$

The last term is the n^{th} term of $W(A,k)$. □

4 Integer sequences related by the k-binomial transforms

The k-binomial transforms relate many sequences listed in the On-Line Encyclopedia of Integer Sequences [11]. Several of these relationships are given in Layman [7]. He lists a number of tables of integer sequences related by repeated applications of the binomial transform and thus (via Theorem 3.2) by the falling k-binomial transform.

The sequences in Tables 1, 2, and 3 are related or appear to be related by the k-binomial transform, the rising k-binomial transform, and the falling k-binomial transform, respectively. (Table 3 is short because we do not duplicate Layman's lists [7].) The tables were mostly obtained by an investigation of several of the entries in the OEIS; undoubtedly there are more sequences related by these transforms. Under the "status" column, K ("known") means that we were able to find a reference for the transform relationship, P ("proved") means that we could not find a reference for the transform relationship but that it can be proved via the techniques in the following section, and C ("conjecture") means that we were not able to find a reference for the transform relationship and were not able to prove it. In addition, the expression (E) in front of a sequence means that the sequence has been shifted; its first term has been deleted.

Status	A	Name (if common)	k	$W(A,k)$	Name (if common)
P	A000142	Factorial Numbers	2	A082032	
P	A000142	Factorial Numbers	3	A097814	
P	A000142	Factorial Numbers	4	A097815	
P	A000142	Factorial Numbers	5	A097816	
P	A000166	Derangement Numbers	2	A000165	Double Factorial Numbers
P	A000166	Derangement Numbers	3	A032031	Triple Factorial Numbers
P	A000166	Derangement Numbers	4	A047053	Quadruple Factorial Numbers
P	A000166	Derangement Numbers	5	A052562	Quintuple Factorial Numbers
K	A000354		1/2	A000142	Factorial Numbers
P	(E)A000354		1/2	A007680	
P	A000354		2	A047053	Quadruple Factorial Numbers
K	A000364	Euler or Secant Numbers	1/2	A005799	
C	A000609		1/2	A002078	
P	A001006	Motzkin Numbers	2	A003645	
C	A001653		1/2	A007052	
P	A001907		1/2	A000165	Double Factorial Numbers
P	A002315		1/2	A007070	NSW Numbers
P	A002426	Central Trinomial Coefficients	2	A059304	
C	A007696		1/2	A002801	
K	A075271		1/2	A075272	

Table 1: k-binomial transforms

Status	A	Name (if common)	k	$R(A,k)$	Name (if common)
K	A000032	Lucas Numbers	2	A014448	Even Lucas Numbers, or L_{3n}
K	A000045	Fibonacci Numbers	2	A014445	Even Fibonacci Numbers, or F_{3n}
C	(E)A000108	Catalan Numbers	2	A059231	
P	A000142	Factorial Numbers	2	A010844	
P	A000142	Factorial Numbers	3	A010845	
P	A000142	Factorial Numbers	4	A056545	
P	A000142	Factorial Numbers	5	A056546	
P	A000984	Central Binomial Coefficients	2	A084771	

Table 2: Rising k-binomial transforms

Status	A	Name (if common)	k	$F(A,k)$	Name (if common)
P	A000354		2	A010844	

Table 3: Falling k-binomial transforms

A close examination of Table 1 indicates that a couple of new binomial transform relationships should exist as well; these are given in Table 4.

Status	A	Name (if common)	$B(A)$	Name (if common)
P	A000354		A000165	Double Factorial Numbers
P	A001907		A047053	Quadruple Factorial Numbers

Table 4: New binomial transforms

The relationships in the tables were found primarily by determining that the first several terms of a sequence correspond to a transform of another sequence. This, of course, does not constitute proof, and so the relationships that we were unable to prove and for which we could not find references remain conjectures. The relationships involving the Lucas numbers and the Fibonacci numbers are given in Benjamin and Quinn [1] (p. 135), and the other known relationships are mentioned in their respective entries in the OEIS [11]. The relationships indicated by a P in the status column we were able to prove via the generating function method we discuss in the next section. In particular, the entries in Table 1 involving the binomial mean transform $W(A, 1/2)$ and the input sequences (E)A000354, A001907, and A002315 are listed as conjectures in the OEIS, and we prove these conjectures in the next section.

Finally, we note that the Hankel transforms of many of these sequences, such as the derangement numbers, the factorial numbers, and the Catalan numbers, are known. Thus Tables 1, 2, 3, and 4, together with Theorems 6.1, 6.2, and 6.3 (proved in Section 6), contain several results and conjectures concerning the Hankel transforms of various integer sequences.

5 Exponential generating functions of the k-binomial transforms

The method we use to prove the relationships indicated with a P in Tables 1, 2, 3, and 4 is that of generating functions. (See Wilf's text [14] for an extensive discussion of generating functions and their uses.)

The *exponential generating function (egf) f of a sequence* $A = \{a_0, a_1, \ldots\}$ is the formal power series $f(x) = a_0 + a_1 x + a_2 x^2/2! + \cdots = \sum_{i=0}^{\infty} a_i x^i/i!$.

We have the following:

Theorem 5.1. *If $g(x)$ is the exponential generating function of a sequence $A = \{a_0, a_1, \ldots\}$, then:*

1. *The exponential generating function of the sequence $F(A, k)$ is $e^{kx} g(x)$.*
2. *The exponential generating function of the sequence $W(A, k)$ is $e^{kx} g(kx)$.*
3. *The exponential generating function of the sequence $R(A, k)$ is $e^{x} g(kx)$.*

Proof. For the special case $k = 0$, the theorem states that the egf of $F(A, k)$ is $g(x)$, which is the egf of A. The theorem gives the egf of $W(A, k)$ to be $g(0)$, which generates the sequence $\{a_0, 0, 0, 0, \ldots\}$. The theorem gives the egf of $R(A, k)$ to be $e^x g(0)$, which generates the sequence $\{a_0, a_0, a_0, \ldots\}$ [14] (p. 52). These are all consistent with the definitions of the k-binomial transforms when $k = 0$. Now, suppose $k \neq 0$.

(*Part 1.*) It is known [14] (p. 42) that if f is the egf of some sequence $\{a_n\}$ and h is the egf of some sequence $\{b_n\}$, then fh is the egf of the sequence

$$\left\{ \sum_{i=0}^{n} \binom{b}{i} a_i b_{n-i} \right\}. \tag{1}$$

(This is known as the *binomial convolution* of $\{a_n\}$ and $\{b_n\}$.) $F(A, k)$ is, again, the sequence

$$\left\{ \sum_{i=0}^{n} \binom{b}{i} k^{n-i} a_i \right\}.$$

With (1) in mind, then, $b_n = k^n$ for each n. The egf of the sequence $\{1, k, k^2, \ldots\}$ is e^{kx} [5] (p. 366). Thus the egf of $F(A, k)$ is $e^{kx} g(x)$.

(*Part 2.*) By definition, $W(A, k) = \{k^n b_n\}$, where $\{b_n\} = B(A)$. From the proof of Part 1, we know that the egf of $B(A)$ is $e^x g(x)$. By definition, it is also $\sum_{i=0}^{\infty} b_i x^i/i!$. The egf of $W(A, k)$ is thus $\sum_{i=0}^{\infty} k^i b_i x^i/i! = \sum_{i=0}^{\infty} b_i (kx)^i/i! = e^{kx} g(kx)$.

(*Part 3.*) By Theorem 3.3, $W(A, k) = F(R(A, k), k - 1)$. Thus the egf of the sequence on the left must equal that of the sequence on the right. Letting $G(x)$ denote the egf of $R(A, k)$, we have, by Parts 1 and 2, $e^{kx} g(kx) = e^{(k-1)x} G(x)$. Thus $G(x) = e^x g(kx)$. □

We now use Theorem 5.1 to prove three relationships listed in Table 1 that are given as conjectures in the OEIS. They all involve the binomial mean transform $W(A, 1/2)$. The proofs of the other relationships indicated with a P in Tables 1, 2, 3, and 4 are similar in flavor to these, and so we do not prove them explicitly here. In fact, most can be proved simply by applying Theorem 5.1 to a direct comparison of the egf's given in the OEIS [11].

Corollary 5.1. *The binomial mean transform of sequence (E)A000354 is sequence A007680.*

Proof. The egf of sequence A000354 is given by the OEIS as $e^{-x}/(1-2x)$. The egf of sequence (E)A000354 is thus $d/dx[e^{-x}/(1-2x)]$ [14] (p. 40), which is

$$\frac{e^{-x}(1+2x)}{(1-2x)^2}.$$

By Theorem 5.1, the egf of the binomial mean transform of (E)A000354 is therefore

$$e^{x/2}\left(\frac{e^{-x/2}(1+x)}{(1-x)^2}\right) = \frac{1+x}{(1-x)^2},$$

which is the egf of sequence A007680 as listed in the OEIS. □

Corollary 5.2. *The binomial mean transform of sequence A001907 is sequence A000165 (the double factorial numbers).*

Proof. The egf of sequence A001907 is given by the OEIS as $e^{-x}/(1-4x)$. Thus the egf of the binomial mean transform of A001907 is

$$e^{x/2}\left(\frac{e^{-x/2}}{1-2x}\right) = \frac{1}{1-2x},$$

which, according to the OEIS, is the egf of sequence A000165. □

Corollary 5.3. *The binomial mean transform of sequence A002315 is sequence A007070.*

Proof. The OEIS does not list egf's for these sequences. It does, however, list two-term recurrence relationships for the sequences, and we can use these relationships to set up differential equations for the egf's. Sequence A002315 is generated by the recursive relationship $a_n = 6a_{n-1} - a_{n-2}$, for $n \geq 2$, with initial values $a_0 = 1$, $a_1 = 7$. This means that the egf f of sequence A002315 satisfies the initial-value problem $f'' - 6f' + f = 0$, $f(0) = 1$, $f'(0) = 7$ [14] (p. 40). The characteristic equation for the differential equation is $r^2 - 6r + 1 = 0$, which has solution $r = 3 \pm 2\sqrt{2}$. Thus the solution to the differential equation, and therefore the egf of A002315, is $f(x) = c_1 e^{(3+2\sqrt{2})x} + c_2 e^{(3-2\sqrt{2})x}$. Using the initial conditions, the constants are found to be $c_1 = 1/2 + 1/\sqrt{2}$, $c_2 = 1/2 - 1/\sqrt{2}$. The recurrence relation for sequence A007070 is given by $a_n = 4a_{n-1} - 2a_{n-2}$, for $n \geq 2$, with $a_0 = 1$, $a_1 = 4$. Using the same method as before, its egf is found to be $c_1 e^{(2+\sqrt{2})x} + c_2 e^{(2-\sqrt{2})x}$, for the same values of c_1 and c_2 found previously. This is also the egf of the binomial mean transform of sequence A002315:

$$e^{x/2}\left[c_1 e^{(3/2+\sqrt{2})x} + c_2 e^{(3/2-\sqrt{2})x}\right] = c_1 e^{(2+\sqrt{2})x} + c_2 e^{(2-\sqrt{2})x}.$$

□

Theorem 5.1 can also be used to prove many additional relationships between sequences in the OEIS involving compositions of the k-binomial transforms. For example, let $D(a,b)$ denote the sequence of central coefficients of $(1+ax+bx^2)^n$, and let C denote the sequence of central binomial coefficients (A000984). Then we have

Corollary 5.4. $D(a,b) = F(R(C,\sqrt{b}), a-2\sqrt{b}-1)$.

Proof. The central binomial coefficients have $e^{2x}\mathrm{BesselI}(0,2x)$ as their exponential generating function. Thus the egf of $F(R(C,\sqrt{b}), a-2\sqrt{b}-1)$ is $e^{(a-2\sqrt{b}-1)x}e^{x}e^{2\sqrt{b}x}\mathrm{BesselI}(0,2\sqrt{b}x)$, which is $e^{ax}\mathrm{BesselI}(0,2\sqrt{b}x)$. This last expression, according to the comments under sequence A084770 in the OEIS, is the egf of the central coefficients of $(1+ax+bx^2)^n$. $\square$

Another example involves the Motzkin numbers (sequence A001006) and the super-Catalan numbers with the first element deleted (sequence (E)A001003). Denoting the sequence of Motzkin numbers by M and the sequence of super-Catalan numbers by S, we have

Corollary 5.5. $(E)S = F(R(M,\sqrt{2}), 2-\sqrt{2})$.

Proof. The Motzkin numbers have egf $e^{x}\mathrm{BesselI}(1,2x)/x$. The egf of $F(R(M,\sqrt{2}),2-\sqrt{2})$ is therefore $e^{(2-\sqrt{2})x}e^{x}e^{\sqrt{2}x}\mathrm{BesselI}(1,2\sqrt{2}x)/(\sqrt{2}x)$. Simplified, this is $e^{3x}\mathrm{BesselI}(1,2\sqrt{2}x)/(\sqrt{2}x)$, which is the egf of the super-Catalan numbers with the first element deleted. $\square$

6 Hankel transforms of the k-binomial transforms

At the end of his article [7], Layman asks if there are transforms besides the binomial and invert under which the Hankel transform is invariant. We now address this question for the three k-binomial transforms.

Theorem 6.1. *The Hankel transform is invariant under the falling k-binomial transform. In other words, for a given sequence $A = \{a_0, a_1, \ldots\}$, $H(F(A,k)) = H(A)$.*

For $k \in \mathbb{Z}$, this follows from Theorems 2.1 and 3.2. However, the theorem is true for $k \notin \mathbb{Z}$ as well. Our proof of this entails a slight modification of the proof of Theorem 2.1.

Proof. Create a triangle of numbers T by letting the left diagonal consist of A and every number off of the left diagonal be the sum of the number to its left and k times the number diagonally above it to the left. Then, via Theorem 3.1, the right diagonal consists of the sequence $F(A,k)$. Construct the matrix T_n as in the proof of Theorem 2.1. Change the transformation rules in part 3 of the procedure described in that proof so that one adds k times row (column) $j-1$ to row (column) j and replaces row (column) j with the result. By thus mimicking the rule for the creation of the numbers off the left diagonal of T, Claims 1 and 2 in the proof of Theorem 2.1 still hold. Therefore, the final matrix resulting from the transformation procedure is the Hankel matrix of order n of $F(A,k)$. Since the transformation rules only involve adding a multiple of a row to another row and adding a multiple of a column to another column, and the determinant is invariant under these operations, the determinant of the Hankel matrix of order n of $F(A,k)$ is equal to the determinant of the Hankel matrix of order n of A.

□

This proof technique works for determining how the Hankel transforms of A, $W(A,k)$, and $R(A,k)$ relate as well. However, the relationships are more complicated, as the Hankel transform is not invariant under $W(A,k)$ and $R(A,k)$. We begin with the transform for $R(A,k)$.

Theorem 6.2. *Given a sequence $A = \{a_0, a_1, \ldots\}$, let $H(A) = \{h_n\}$. Then $H(R(A,0)) = \{a_0, 0, 0, \ldots\}$. If $k \neq 0$, $H(R(A,k)) = \{k^{n(n+1)} h_n\}$.*

Proof. If $k = 0$, then H_n is the $(n+1) \times (n+1)$ matrix whose entries are all a_0. Thus $\det(H_0) = a_0$, and for $n > 0$, $\det(H_n) = 0$.

Now, assume $k \neq 0$. Create a triangle of numbers T by letting the left diagonal consist of A and letting each number off the left diagonal be the sum of the number diagonally above it to the left and k times the number to its left. By Theorem 3.1, the right diagonal of T is $R(A,k)$. Construct the matrix T_n as in the proofs of Theorems 2.1 and 6.1, but change the transformation rule so that one adds row (column) $j-1$ to k times row (column) j and replaces row (column) j with that result. Again, Claims 1 and 2 in the proof of Theorem 2.1 hold, and so the final matrix resulting from the transformation procedure is the Hankel matrix of order n of $R(A,k)$.

The transformation rule can be broken into two parts: First multiply a row (column) by k. Then replace that row (column) with the sum of itself and the previous row (column). Multiplying a row of a matrix by a factor of k changes the determinant by a factor of k, while replacing a row by the sum of itself and another row does not affect the determinant. The same is true for columns [2] (p. 262). In order to determine how the Hankel transform changes under the rising k-binomial transform, then, we need to determine the number of times a row or column is multiplied by a factor of k.

According to the transformation procedure, row i is multiplied by a factor of k in stages $1, 2, \ldots, i$. Thus row i is multiplied by a factor of k a total of i times. Therefore the total number of times a row is multiplied by a factor of k is $\sum_{i=0}^{n} i = n(n+1)/2$. Similarly, the number of times a column is multiplied by a factor of k is $n(n+1)/2$. Thus the determinant of the Hankel matrix of order n of $R(A,k)$ is $k^{n(n+1)}$ times that of the Hankel matrix of order n of A. □

Finally, we have

Theorem 6.3. *Given a sequence $A = \{a_0, a_1, \ldots\}$, let $H(A) = \{h_n\}$. Then $H(W(A,0)) = \{a_0, 0, 0, \ldots\}$. If $k \neq 0$, $H(W(A,k)) = \{k^{n(n+1)} h_n\}$.*

Proof. By Theorems 3.3 and 6.1, $H(W(A,k)) = H(F(R(A,k), k-1) = H(R(A,k))$. Alternatively, one can prove this result with a modification of the proofs of Theorems 6.1 and 6.2. The transformation rule would be adding row (column) $j-1$ to row (column) j and replacing row (column) j with k times that sum. This is equivalent to first multiplying row (column) j by a factor of k and then adding k times row (column) $j-1$ to that and replacing row (column) j with this sum. As we just noted, multiplying a row or column by a factor of

k changes the determinant by a factor of k, while adding a multiple of a row to another row or a multiple of a column to another column does not change the determinant. The analysis of the number of times a row or column is multiplied by a factor of k is exactly the same as that in the proof of Theorem 6.2. □

Theorems 6.1, 6.2, and 6.3, together the relationships given in Tables 1, 2, 3, and 4, allow us to determine the Hankel transforms of several integer sequences. Given that the factorial numbers (A000142) and the derangement numbers (A000166) have Hankel transform $\{\prod_{i=0}^{n}(i!)^2\}$ (sequence A055209), the Motzkin numbers (A001006) have Hankel transform $\{1, 1, 1, \ldots\}$ (sequence A000012), and the central binomial coefficients (A000984) and trinomial coefficients (A002426) have Hankel transform $\{2^n\}$ (sequence A000079) [7], we have the Hankel transforms of several integer sequences given in Table 5.

Hankel Transform	Some sequences with this Hankel transform
$\{2^{n(n+1)}\}$	A003645
$\{2^{n(n+2)}\}$	A059304, A084771
$\{2^{n(n+1)}\prod_{i=0}^{n}(i!)^2\}$	A000165, A000354, A010844, A082032
$\{3^{n(n+1)}\prod_{i=0}^{n}(i!)^2\}$	A010845, A032031, A097814
$\{4^{n(n+1)}\prod_{i=0}^{n}(i!)^2\}$	A001907, A047053, A056545, A097815
$\{5^{n(n+1)}\prod_{i=0}^{n}(i!)^2\}$	A052562, A056546, A097816

Table 5: Hankel transforms of some integer sequences

In addition, Theorems 6.1, 6.2, and 6.3 can be used to relate the Hankel transforms of sequences that are themselves related by compositions of the k-binomial transforms. For example, Theorems 6.1 and 6.2 applied to Corollary 5.4 tell us that the Hankel transform of the central coefficients of $(1 + ax + bx^2)^n$ is $\{2^n b^{n(n+1)/2}\}$. Applied to Corollary 5.5, they tell us that the Hankel transform of the super-Catalan numbers with the first element deleted is $\{2^{n(n+1)/2}\}$.

7 Conclusions

We have introduced three generalizations of the binomial transform: the k-binomial transform, the rising k-binomial transform, and the falling k-binomial transform. We have given a simple method for constructing these transforms, and we have given combinatorial interpretations of each of them. We have also shown how the generating function of a sequence changes after applying one of the transforms. This allows us to prove that several sequences in the On-Line Encyclopedia of Integer Sequences are related by one of these transforms, as well as prove three specific conjectures in the OEIS. In addition, we have shown how the Hankel transform of a sequence changes after applying one of the transforms. These results determine the Hankel transforms of several sequences listed in the OEIS.

We see several areas of further study. One involves continuing to answer Layman's question [7]. We have proved that the Hankel transform is invariant under the falling k-binomial transform, and he proves that the Hankel transform is invariant under the binomial and invert transforms. Are there other interesting transforms under which the Hankel transform is invariant?

Second, are there variations of our triangle method that would prove for other transforms T how $H(A)$ and $H(T(A))$ relate?

Third, we conjecture four relationships in Tables 1 and 2: Sequences A002078, A007052, and A002801 are the binomial mean transforms of A000609, A001653, and A007696, respectively, and sequence A059231 is the rising 2-binomial transform of (E)A000108. (The first three are also conjectured in the OEIS.) Can these conjectures be proved?

Finally, what sequences, other than those listed in Layman [7] and in Tables 1, 2, 3, and 4, are related by the k-binomial transforms?

References

[1] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs That Really Count*, MAA, Washington, D. C., 2003.

[2] Otto M. Bretscher, *Linear Algebra with Applications*, Pearson Prentice Hall, Upper Saddle River, NJ, third ed., 2005.

[3] Richard Ehrenborg, The Hankel determinant of exponential polynomials, *Amer. Math. Monthly*, **107** (2000), 557–560.

[4] Philippe Flajolet, On congruences and continued fractions for some classical combinatorial quantities, *Discrete Math.*, **41** (1982), 145–153.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, second ed., 1994.

[6] Donald E. Knuth, *The Art of Computer Programming, Vol. III: Sorting and Searching*, Addison-Wesley, Reading, MA, second ed., 1998.

[7] John W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.*, **4** (2001), Article 01.1.5.

[8] Paul Peart, Hankel determinants via Stieltjes matrices, *Congr. Numer.*, **144** (2000), 153–159.

[9] Christian Radoux, Déterminant de Hankel construit sur des polynômes liés aux nombres de dérangements, *European J. Combin.*, **12** (1991) 327–329.

[10] Kenneth H. Rosen, ed., *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Boca Raton, FL, 2000.

[11] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2005, published electronically at http://www.research.att.com/∼njas/sequences/.

[12] Michael Z. Spivey, Exams with deranged questions, in preparation.

[13] Eric W. Weisstein, Binomial transform, from MathWorld – A Wolfram Web Resource, http://mathworld.wolfram.com/BinomialTransform.html.

[14] Herbert S. Wilf, *Generatingfunctionology*, Academic Press, Boston, second ed., 1994.

[15] Win-Jin Woan and Paul Peart, Determinants of Hankel matrices of aerated sequences, *Congressus Numerantium*, **157** (2002), 103–111.

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