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On the Divisibility of Generalized Central Trinomial Coefficients

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Abstract

We present several methods of computing sequences of generalized central trinomial coefficients. We generalize the Schur and Holt congruences for Legendre polynomials in order to prove divisibility properties of these sequences and a conjecture of Deutsch and Sagan.

1 Introduction

For integers a, b, c, we call the coefficient of x^n in the expression

$$(a + bx + cx^2)^n \tag{1}$$

the generalized central trinomial coefficient, T_n. The sequences $\{T_n\}_{n\geq 0}$ have combinatorial interpretations and, as shown in Table 1 at the end of this paper, appear frequently in Sloane's On-Line Encyclopedia of Integer Sequences [6]. By using the binomial theorem twice and the identity $\binom{n}{k}\binom{n-k}{n-2k} = \binom{2k}{k}\binom{n}{2k}$, we obtain

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k}\binom{n}{2k} b^{n-2k}(ac)^k. \tag{2}$$

Assuming b is fixed, note that for any a and c having a fixed nonzero product, the sequences will be the same. Hence, we assume $a = 1$. Because $b = 0$ is a degenerate case and the absolute value of T_n is the same for b and $-b$, we can assume $b > 0$ for divisibility purposes.

As we will see below, an important quantity that determines many properties of the sequence is the the discriminant $d = b^2 - 4ac$. There are an infinite number of pairs (b, c)

that yield the same discriminant. Also, the discriminant is always of the form $4k$ or $4k+1$ for some integer k.

We begin by considering several ways of evaluating these sequences. We will see that the representation as values of Legendre polynomials is crucial in studying the divisibility of the sequences by primes.

2 Ordinary Generating Function

Using the solution to an exercise in [4, page 575], we compute the sequence from the ordinary generating function

$$\frac{1}{\sqrt{1-2bx+dx^2}} = \sum_{k=0}^{\infty} T_k x^k. \tag{3}$$

Euler [3] found this generating function for the case $a = b = c = 1$.

Observe that when c is a perfect square, the quadratic in the radical factors into the product of two linear terms $1-(b+2\sqrt{c})x$ and $1-(b-2\sqrt{c})x$ having integer coefficients. This factorization allows us to represent equation (3) as the convolution of two sequences of the form

$$\frac{1}{\sqrt{1-fx}} = \sum_{k=0}^{\infty} \frac{1}{4^k}\binom{2k}{k} f^k$$

for constant f. For $f=1$, see sequence A001790. Forming the convolution, we obtain the interesting

$$T_n = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k} f^k g^{n-k},$$

where $f = b+2\sqrt{c}$ and $g = b-2\sqrt{c}$ are not required to be integers.

3 Recursion Equation

The exercise in [4] states that the recurrence equation for the sequence is

$$T_0 = 1, \quad T_1 = b, \quad T_n = \frac{(2n-1)bT_{n-1} - (n-1)dT_{n-2}}{n}, \tag{4}$$

which provides an efficient formula for computing T_n. Euler [3] found this recursion equation for the case $a = b = c = 1$. Observe that the recursion actually needs only T_0 to start; T_1 can be computed. The recursion also allows us to show that

$$T_{n+1} \equiv bT_n \pmod{n}.$$

4 Exponential Generating Function

Using the infinite series expansion of the modified Bessel function of the first kind,

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(\frac{1}{4}x^2)^k}{(k!)^2}$$

and the infinite series expansion of the exponential function, it is not hard to show that the sequence can be computed from the exponential generating function

$$e^{bx} I_0(2x\sqrt{c}) = \sum_{k=0}^{\infty} \frac{T_k x^k}{k!}.$$

5 Legendre Polynomials

When the discriminant d is nonzero, the sequence can be defined in terms of the Legendre polynomials

$$T_n = d^{n/2}\, P_n\left(\frac{b}{\sqrt{d}}\right), \tag{5}$$

which is easy to derive from the generating function for Legendre polynomials

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Note that in equation (5), after multiplying each term of the polynomial $P_n(b/\sqrt{d})$ by $d^{n/2}$, there will be no occurrences of $\sqrt{d}$ because the degree of the polynomial is n and its terms have either all even or all odd exponents if n is even or odd, respectively.

The cases of $d = 1$ and $d = 0$ have particularly simple forms. When $d = 1$, which occurs when $b = 2k + 1$ and $c = k(k + 1)$ for integer $k \geq 0$, we obtain $T_n = P_n(b)$. When $d = 0$, equation (1) can be written as $(1 + bx/2)^{2n}$ and the sequence is

$$T_n = \left(\frac{b}{2}\right)^n \binom{2n}{n}. \tag{6}$$

6 Schur and Holt Congruences

The remainder of this paper contains many polynomial congruences. A statement such as

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \equiv 0 \pmod{p}$$

in the indeterminant x means that $a_k \equiv 0 \pmod{p}$ for all k. The coefficients a_k are allowed to be fractions if the denominator is not divisible by p.

Papers by Carlitz [1] and Wahab [7] have congruences for Legendre polynomials modulo an odd prime p. In particular, Carlitz cites the congruence of Schur: if n is written in base p as

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \quad (0 \leq n_i < p), \tag{7}$$

then

$$P_n(x) \equiv P_{n_0}(x)\ (P_{n_1}(x))^p\ (P_{n_2}(x))^{p^2} \dots (P_{n_r}(x))^{p^r} \pmod p. \tag{8}$$

This beautiful congruence, which is reminiscent of Lucas' theorem for binomial coefficients, is proved by Wahab [7, Theorem 6.1]. The congruence implies that the divisibility of $P_n(x)$ by p is completely determined by the divisibility of $P_k(x)$ by p for $0 \le k < p$. We may use Fermat's little theorem to write congruence (8) as

$$P_n(x) \equiv P_{n_0}(x)\ P_{n_1}(x^p)\ P_{n_2}(x^{p^2}) \dots P_{n_r}(x^{p^r}) \pmod p.$$

If we replace x by an integer $s > 0$, then Schur's congruence still holds and we use Fermat's little theorem to obtain a congruence without exponents:

$$P_n(s) \equiv P_{n_0}(s)\ P_{n_1}(s)\ P_{n_2}(s) \dots P_{n_r}(s) \pmod p. \tag{9}$$

Another result mentioned by Carlitz and Wahab is due to Holt [5]:

$$P_k(x) \equiv P_{p-k-1}(x) \pmod p \tag{10}$$

for $k < p$. A surprising implication of this congruence is that for $(p-1)/2 < k < p$, the polynomial $P_k(x) \pmod p$ has degree less than k. It also means, in conjunction with the Schur congruence (8), that the divisibility of $P_n(x)$ by p is completely determined by the divisibility of $P_k(x)$ by p for $0 \le k \le (p-1)/2$.

7 Scaled Legendre Polynomials

We would like to use the Schur and Holt congruences to prove divisibility properties of the sequence T_n using equation (5). However, the factor of $d^{n/2}$ in that equation makes using the congruences dubious. To remedy this problem, we define a scaled Legendre polynomial having a parameter d:

$$Q_n(x,d) = d^{n/2}\ P_n\left(\frac{x}{\sqrt{d}}\right) = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2 \rfloor}(-1)^k \binom{n}{k}\binom{2n-2k}{n-2k} d^k x^{n-2k}, \tag{11}$$

which, except for the factor of d^k in the summation, is the same as the usual Legendre polynomial definition. The first few polynomials are $Q_0(x,d) = 1, Q_1(x,d) = x$, and $Q_2(x,d) = \frac{1}{2}(3x^2 - d)$. This formulation also allows the case of $d = 0$. Note that if we replace b by the indeterminant x in recursion equation (4), the scaled Legendre polynomials are generated. Equation (5) becomes

$$T_n = Q_n(b,d).$$

In the appendix we prove the Schur and Holt congruences for these scaled Legendre polynomials. Using the same reasoning as in the previous section, congruence (9) becomes

$$Q_n(s,d) \equiv Q_{n_0}(s,d)\ Q_{n_1}(s,d)\ Q_{n_2}(s,d) \dots Q_{n_r}(s,d) \pmod p, \tag{12}$$

which can be used with $s = b$ to obtain the important congruence

$$T_n \equiv T_{n_0}\, T_{n_1}\, T_{n_2}\, \ldots\, T_{n_r} \pmod{p}. \tag{13}$$

Let $u = (p-1)/2 - k$ for $k < p$. Then, assuming $p \nmid d$, Holt's congruence (10) becomes

$$d^u\, Q_k(x,d) \equiv Q_{p-k-1}(x,d) \pmod{p}$$

for scaled Legendre polynomials, which leads to

$$d^u\, T_k \equiv T_{p-k-1} \pmod{p}. \tag{14}$$

8 Divisibility Properties

The divisibility properties of these sequences are quite interesting. However, as shown in equation (6), the case $d = 0$ concerns essentially just the central binomial coefficients, whose divisibility is handled by Kummer's theorem [2, Theorem 5.12].

In the following theorems, we assume that $\{T_n\}_{n\ge 0}$ is a sequence of generalized central trinomial coefficients with $b > 0$ and discriminant $d \neq 0$. For specific integer $n > 0$ and prime p, let $(n)_p = (n_i)$ denote the base p representation of n, as in equation (7). We begin with the case $p = 2$, which cannot be handled with Legendre polynomials due to the powers of 2 in their denominators.

Theorem 8.1. *If $p = 2$, then $T_n \equiv b \pmod{p}$.*

Proof. In the sum in equation (2), the $k = 0$ term is b^n; the other terms are $0 \pmod 2$ because $\binom{2k}{k}$ is even for $k > 0$. The theorem follows from $b^n \equiv b \pmod 2$ for $n > 0$. □

When p is an odd prime there are four cases to consider: whether or not $p \mid b$ and whether or not $p \mid d$. We do not treat the case of $p \nmid bd$ except in the general case treated by Theorem 8.5. The following three theorems cover the three remaining cases. The easiest case is p dividing both b and d, which we treat first.

Theorem 8.2. *If p is an odd prime with $p \mid b$ and $p \mid d$, then $p \mid T_n$.*

Proof. Because $d = b^2 - 4c$, $p \mid b$ and $p \mid d$ imply $p \mid c$. Then, using equation (2), it is easy to see that p divides every term of the sum. Hence, $p \mid T_n$. □

Theorem 8.3. *Let p be an odd prime with $p \mid b$ and $p \nmid d$. Then the following are true:*

(1) if n_i is odd for some $0 \le i \le r$, then $p \mid T_n$,

(2) if n_i is even and $n_i \le (p-1)/2$ for all $0 \le i \le r$, then $p \nmid T_n$,

(3) if $p \mid T_n$, then either n_i is odd or $n_i > (p-1)/2$ for some $0 \le i \le r$.

Proof. Part 1: Using equation (2), b is a factor in every term of the sum when n is odd. Hence, because $p \mid b$, we obtain $p \mid T_n$ for odd n. By congruence (13), this divisibility extends to all n such that n_i is odd for some $0 \le i \le r$.

Part 2: If n_i is even for all i, then n is even. For even n, the last term of equation (2) is $\binom{2n}{n} c^{n/2}$; all the other terms are divisible by p because those terms have a factor of b, which p

divides. Note that $d = b^2 - 4c$, $p \mid b$, and $p \nmid d$ imply $p \nmid c$. Because we assume $n_i \le (p-1)/2$ for all i, there are no carries when n is added to n in base p. Hence, by Kummer's theorem, $p \nmid \binom{2n}{n}$. Thus, p does not divide $\binom{2n}{n} c^{n/2}$ and we obtain $p \nmid T_n$.

Part 3: this is just the contrapositive of part 2. ∎

Theorem 8.4. *Let p be an odd prime with $p \nmid b$ and $p \mid d$, then $p \mid T_n$ if and only if $(p+1)/2 \le n_i \le p-1$ for some $0 \le i \le r$.*

Proof. From congruence (13), we see that $p \mid T_n$ iff $p \mid T_{n_i}$ for some $0 \le i \le r$. Hence, we assume that $n < p$. Recall that $T_n = Q_n(b, d)$. We see from equation (11), that all the terms of $Q_n(b, d)$ except the first are divisible by p because $p \mid d$. The coefficient of the first term is $\binom{2n}{n} b^n / 2^n$. Assuming $(p+1)/2 \le n \le p-1$, the sum $n+n$ has a carry in base p. Therefore, Kummer's theorem implies that p divides $\binom{2n}{n}$. We conclude that $p \mid T_n$. For the converse, we prove the contrapositive. Assume $0 \le n \le (p-1)/2$. The coefficient of the first term of equation (11) is $\binom{2n}{n} b^n / 2^n$, which is not divisible by p because $2n$ and b are not divisible by p. Hence, $p \nmid T_n$. ∎

The next theorem generalizes Deutsch and Sagan [2, Corollary 4.8], which they prove for the case $a = b = c = 1$.

Theorem 8.5. *Let p be a prime. Then $p \mid T_n$ if and only if $p \mid T_{n_i}$ for some $0 \le i \le r$.*

Proof. For $p = 2$, n_i is either 0 or 1, and at least one n_i is 1 because $n > 0$. Using Theorem 8.1, we conclude $p \mid T_n$ if and only if $p \mid T_1$. For odd prime p, the theorem follows from congruence (13). ∎

For the case $a = b = c = 1$, Deutsch and Sagan [2, Conjecture 5.8] and David W. Wilson conjecture that, for $n < p$, p divides T_n if and only if p divides T_{n-p-1}. The next theorem proves and generalizes the conjecture.

Theorem 8.6. *Let p be an odd prime with $p \nmid d$ and $n < p$. Then $p \mid T_n$ if and only if $p \mid T_{p-n-1}$.*

Proof. Due to symmetry, we may assume $n < (p-1)/2$. Assuming $p \mid T_n$, congruence (14) yields $0 \equiv T_{p-n-1} \pmod{p}$, which means $p \mid T_{p-n-1}$. Conversely, assuming $p \mid T_{p-n-1}$, congruence (14) yields $d^u T_n \equiv 0 \pmod{p}$, which means $p \mid T_n$ if $p \nmid d$. ∎

The following three theorems compute the values of T_p, T_{p-1}, and $T_{p-2} \pmod{p}$. We see that if $p \nmid bd$, then p does not divide T_p, T_{p-1}, and T_{p-2}. Theorem 8.8 is a rare instance in which Legendre polynomials and the Legendre symbol appear together.

Theorem 8.7. *If p is a prime, then $T_p \equiv b \pmod{p}$*

Proof. The theorem follows from equation (2), the fact that $\binom{p}{2k} \equiv 0 \pmod{p}$ for all indices $k > 0$, and Fermat's little theorem. ∎

Theorem 8.8. *For odd prime p,*

$$T_{p-1} \equiv d^{(p-1)/2}\, P_{p-1}\left(\frac{b}{\sqrt{d}}\right) \equiv \left(\frac{d}{p}\right) \pmod{p},$$

where $(\frac{d}{p})$ is the Legendre symbol.

Proof. The first congruence is equation (5). By congruence (14) and the fact that $T_0 = 1$, we obtain $T_{p-1} \equiv d^{(p-1)/2}\, T_0 \equiv d^{(p-1)/2} \pmod{p}$. By Euler's criterion, $(\frac{d}{p}) \equiv d^{(p-1)/2} \pmod{p}$. Hence, $T_{p-1} \equiv (\frac{d}{p}) \pmod{p}$. □

Theorem 8.9. *For odd prime p,*

$$T_{p-2} \equiv bd^{(p-3)/2} \pmod{p}.$$

Proof. If $p = 3$, then $T_1 = b$. If $p \geq 5$ and $p \mid d$, then clearly $T_{p-2} \equiv 0 \pmod{p}$ by Theorems 8.2 and 8.4. If $p \nmid d$, then the theorem follows from $T_1 = b$ and congruence (14). □

9 Appendix

Here we prove the generalization of the Schur congruence (8) and Holt congruence (10) for the scaled Legendre polynomials $Q(x, d)$ defined by (11). In several cases, the parameter d forces the proofs to be somewhat different than the ones by Holt [5] and Wahab [7]. In all cases, p is an odd prime and all congruences are modulo p. We use Q_n as shorthand for the polynomial $Q_n(x, d)$. As mentioned above, Q_n satisfies the recursion equation

$$nQ_n = (2n-1)xQ_{n-1} - (n-1)dQ_{n-2} \tag{15}$$

starting with $Q_0 = 1$. Note that because all the denominators in Q_n are powers of 2, which are relatively prime to p, we replace equation (15) by the congruence

$$nQ_n \equiv (2n-1)xQ_{n-1} - (n-1)dQ_{n-2}. \tag{16}$$

We begin by proving four lemmas that are stated, but not proved in Wahab [7, Lemma 6.2]. The first proof is almost identical to Holt's proof.

Lemma 9.1. *If $0 \leq r < p$, then $Q_{kp+r} \equiv Q_r Q_{kp}$.*

Proof. The case $r = 0$ is obvious, so assume $0 < r < p$. Letting $n = kp+r$ in congruence (16), we obtain

$$rQ_{kp+r} \equiv (2r-1)xQ_{kp+r-1} - (r-1)dQ_{kp+r-2}.$$

Observe that with $r = 1$ we obtain

$$Q_{kp+1} \equiv xQ_{kp} \equiv Q_1 Q_{kp}$$

because $Q_1 = x$. Similarly, with $r = 2$ we have

$$2Q_{kp+2} \equiv 3xQ_{kp+1} - dQ_{kp} \equiv (3x^2 - d)Q_{kp} \equiv 2Q_2 Q_{kp},$$

which allows us to conclude that $Q_{kp+2} \equiv Q_2 Q_{kp}$. Assuming

$$Q_{kp+r-1} \equiv Q_{r-1} Q_{kp} \quad \text{and} \quad Q_{kp+r-2} \equiv Q_{r-2} Q_{kp},$$

the general case is

$$rQ_{kp+r} \equiv (2r-1)Q_{kp+r-1} - (r-1)dQ_{kp+r-2} \equiv [(2r-1)Q_{r-1} - (r-1)dQ_{r-2}]Q_{kp}.$$

However, the bracketed expression is merely the right-hand-side of (16) with $n = r$. Hence,

$$rQ_{kp+r} \equiv rQ_r Q_{kp}.$$

Because $0 < r < p$, r is relatively prime to p, and we conclude $Q_{kp+r} \equiv Q_r Q_{kp}$. □

The next lemma shows the effect of the parameter d in the scaled Legendre polynomials. The parameter d also forces the proof to be slightly different than Holt's proof.

Lemma 9.2. *If $1 \leq r < p$ and $p \nmid d$, then $d^{r-1} Q_{kp-r} \equiv Q_{r-1} Q_{kp-1}$.*

Proof. The case $r = 1$ is obvious. Letting $n = kp - r + 1$ in congruence (16), we obtain

$$(-r+1)Q_{kp-r+1} \equiv (-2r+1)xQ_{kp-r} + rdQ_{kp-r-1},$$

which is rearranged to yield

$$rdQ_{kp-r-1} \equiv (2r-1)xQ_{kp-r} - (r-1)Q_{kp-r+1}. \tag{17}$$

With $r = 1$ we obtain, using $Q_1 = x$,

$$dQ_{kp-2} \equiv xQ_{kp-1} \equiv Q_1 Q_{kp-1}.$$

Similarly, with $r = 2$ we have

$$2dQ_{kp-3} \equiv 3xQ_{kp-2} - Q_{kp-1}.$$

Multiplying by d, and using the result with $r = 1$, gives us

$$2d^2 Q_{kp-3} \equiv 3x^2 Q_{kp-1} - dQ_{kp-1} \equiv (3x^2 - d)Q_{kp-1} \equiv 2Q_2 Q_{kp-1},$$

and we conclude that $d^2 Q_{kp-3} \equiv Q_2 Q_{kp-1}$. For the case $r \geq 3$, we assume

$$d^{r-2} Q_{kp-r+1} \equiv Q_{r-2} Q_{kp-1} \quad \text{and} \quad d^{r-3} Q_{kp-r+2} \equiv Q_{r-3} Q_{kp-1}, \tag{18}$$

and substitute $r-1$ for r in the general case (17), to obtain

$$(r-1)dQ_{kp-r} \equiv (2r-3)xQ_{kp-r+1} - (r-2)Q_{kp-r+2}.$$

Multiplying by d^{r-2}, and using the assumptions (18), gives us

$$d^{r-1}(r-1)Q_{kp-r} \equiv [(2r-3)xQ_{r-2} - (r-2)dQ_{r-3}]Q_{kp-1}.$$

However, the bracketed expression is merely the right-hand-side of (16) for $n = r-1$. Hence,

$$d^{r-1}(r-1)Q_{kp-r} \equiv (r-1)Q_{r-1}Q_{kp-1}$$

For $r \neq 1$, $r-1$ is relatively prime to p, and we conclude $d^{r-1} Q_{kp-r} \equiv Q_{r-1} Q_{kp-1}$. □

Although the statement of the next lemma is the same as Holt's, the proof is different.

Lemma 9.3. *If $0 \le r < p$, then $Q_{p+r} \equiv x^p Q_r$.*

Proof. By Lemma 9.1, we have $Q_{p+r} \equiv Q_r Q_p$. From the definition of Q_p in equation (11), it is obvious that p divides every term of the sum except the first. The remaining term is $\binom{2p}{p} x^p/2^p$. Hence, $Q_p \equiv Ax^p$ for some constant A that is the solution to the congruence $2^p A \equiv \binom{2p}{p}$. By Fermat's little theorem, $2^p \equiv 2$. Also, $\binom{2p}{p} \equiv \binom{2}{1} = 2$. We conclude that $A \equiv 1$, proving the lemma. $\square$

The following lemma is a generalization of Holt's congruence mentioned earlier in this paper. Its statement and proof are different than Holt's version.

Lemma 9.4. *If $0 \le r < p$ and $p \nmid d$, then $Q_{p-r-1} \equiv d^u Q_r$, where $u = (p-1)/2 - r$.*

Proof. Letting $k = 1$ and substituting $r + 1$ for r in Lemma 9.2, we have

$$d^r Q_{p-r-1} \equiv Q_r Q_{p-1}. \tag{19}$$

From the definition of Q_{p-1} in equation (11), we use Kummer's theorem to show that p divides every term of the sum except the last. The remaining term is the constant

$$\frac{(-1)^m}{2^{p-1}} \binom{p-1}{m} d^m,$$

where $m = (p-1)/2$. But $\binom{p-1}{m} \equiv (-1)^m$ for any m, making the constant $d^m/2^{p-1}$. Calling this constant A and using Fermat's little theorem as in the previous lemma, we obtain $A \equiv d^{(p-1)/2}$. Thus, equation (19) becomes

$$d^r Q_{p-r-1} \equiv d^{(p-1)/2} Q_r.$$

Dividing by d^r proves the lemma. $\square$

The following lemma is proved by Wahab for the usual Legendre polynomials.

Lemma 9.5. *$Q_{kp} \equiv (Q_k)^p$.*

Proof. By the definition of scaled Legendre polynomials (11),

$$2^{kp} Q_{kp}(x,d) = \sum_{j=0}^{\lfloor kp/2 \rfloor} (-1)^j \binom{kp}{j} \binom{2kp-2j}{kp-2j} d^j x^{kp-2j}.$$

However, $\binom{kp}{j} \equiv 0$ for $j \ne ip$ and $\binom{kp}{ip} \equiv \binom{k}{i}$. Consequently,

$$2^{kp} Q_{kp}(x,d) \equiv \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{ip} \binom{k}{i} \binom{2k-2i}{k-2i} d^{ip} x^{kp-2ip}.$$

Using Fermat's little theorem several times, we obtain

$$2^{kp} Q_{kp}(x,d) \equiv 2^k Q_k(x^p, d) \equiv 2^{kp} (Q_k(x,d))^p.$$

Dividing by 2^{kp} proves the lemma. $\square$

The following theorem proves Schur's congruence for scaled Legendre polynomials. The proof is identical to Wahab's for the usual Legendre polynomials.

Theorem 9.1. *If* $n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r$, *with* $0 \le n_i < p$, *then*

$$Q_n \equiv Q_{n_0}\ (Q_{n_1})^p\ (Q_{n_2})^{p^2} \ldots (Q_{n_r})^{p^r}.$$

Proof. Induction on n is used. The statement is true for $0 \le n < p$. By Lemma 9.1,

$$Q_n \equiv Q_{n_0}\ Q_{n_1 p + n_2 p^2 + \cdots + n_r p^r}.$$

By Lemma 9.5,

$$Q_n \equiv Q_{n_0}\ (Q_{n_1 + n_2 p + \cdots + n_r p^{r-1}})^p.$$

By the inductive assumption,

$$Q_n \equiv Q_{n_0}\ [Q_{n_1}\ (Q_{n_2})^p \cdots (Q_{n_r})^{p^{r-1}}]^p.$$

Hence,

$$Q_n \equiv Q_{n_0}\ (Q_{n_1})^p\ (Q_{n_2})^{p^2} \cdots (Q_{n_r})^{p^r}.$$

□

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a	b	c	d	A-number	Comments
1	1	1	-3	A002426	central trinomial coefficients
1	1	2	-7	A084601	
1	1	3	-11	A084603	
1	1	4	-15	A084605	
1	1	5	-19	A098264	
1	1	6	-23	A098265	
1	1	8	-31	A098329	
1	1	12	-47	A098439	
1	1	15	-59	A098440	
1	1	16	-63	A098441	
1	1	24	-95	A098442	
1	1	-1	5	A098331	
1	1	-2	9	A098332	
1	1	-3	13	A098333	
1	1	-4	17	A098334	
1	2	1	0	A000984	binomial$(2n, n)$
1	2	2	-4	A006139	
1	2	3	-8	A084609	
1	2	4	-12	A098453	
1	2	5	-16	A084770	
1	2	-1	8	A098335	
1	2	-2	12	A098336	
1	2	-3	16	A012000	$4^n P_n(1/2)$
1	2	-4	20	A098337	
1	3	1	5	A026375	
1	3	2	1	A001850	$P_n(3)$, central Delannoy numbers
1	3	5	-11	A098444	
1	3	-1	13	A098338	
1	3	-2	17	A098339	
1	3	-3	21	A098340	
1	3	-4	25	A098341	
1	4	1	12	A081671	
1	4	2	8	A080609	
1	4	3	4	A069835	$2^n P_n(2)$
1	4	4	0	A059304	2^n binomial$(2n, n)$
1	4	5	-4	A098443	
1	4	6	-8	A106258	
1	5	1	21	A098409	
1	5	4	9	A084771	
1	5	6	1	A006442	$P_n(5)$
1	6	1	32	A098410	
1	6	9	0	A098658	3^n binomial$(2n, n)$
1	7	1	45	A104454	
1	7	12	1	A084768	$P_n(7)$
1	8	16	0	A098430	4^n binomial$(2n, n)$
1	9	20	1	A084769	$P_n(9)$

Table 1: Generalized Central Trinomial Coefficients

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