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Evaluations of Some Variant Euler Sums

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Abstract

In this note we present some elementary methods for the summation of certain Euler sums with terms involving $1 + 1/3 + 1/5 + \cdots + 1/(2k-1)$.

1 Introduction

In the last decade, based on extensive experimentation with computer algebraic systems, a large class of Euler sums have been explicitly evaluated in terms of the Riemann zeta function $\zeta(k)$. For example, let

$$H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2} H_k &= 2\,\zeta(3), \\ \sum_{k=1}^{\infty} \frac{1}{2^k\,k^2} H_k &= \zeta(3) - \frac{\pi^2}{12} \ln 2, \\ \sum_{k=1}^{\infty} \frac{1}{k^2} H_k^2 &= \frac{17}{4}\,\zeta(4), \end{aligned}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} H_k = \frac{5}{8}\,\zeta(3).$$

More details can be found in [1, 2, 3, 4] In particular, Borwein and Bradley [3] collected 32 beautiful proofs of the first sum above.

Motivated by the above results, in this note, replacing H_k by

$$h_k = H_{2k} - \frac{1}{2} H_k = 1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}, \tag{1}$$

we study the following variant Euler sums

$$\sum_{k=1}^{\infty} a_k \, h_k$$

where the a_k are relatively simple function of k.

2 The Main Results

We begin to derive some series involving h_k. Since

$$-\ln(1-x) = \int_0^x \frac{dt}{1-t} = \sum_{k=1}^{\infty} \frac{x^k}{k},$$

replacing x by $-x$ gives

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}.$$

Averaging these two series gives us

$$\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \, x^{2k-1}. \tag{2}$$

In term of the Cauchy product and partial fractions, we have

$$\begin{aligned}
\frac{1}{4} \ln^2\left(\frac{1+x}{1-x}\right) &= \sum_{k=1}^{\infty} \left(\frac{1}{(2k-1)\cdot 1} + \frac{1}{(2k-3)\cdot 3} + \cdots + \frac{1}{1\cdot(2k-1)}\right) x^{2k} \\
&= \sum_{k=1}^{\infty} \frac{1}{2k} \left[\left(\frac{1}{2k-1} + \frac{1}{1}\right) + \left(\frac{1}{2k-3} + \frac{1}{3}\right) + \cdots + \left(\frac{1}{1} + \frac{1}{2k-1}\right)\right] x^{2k} \\
&= \sum_{k=1}^{\infty} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right) \frac{x^{2k}}{k}.
\end{aligned}$$

Noting that h_k is given by (1), we have

$$\sum_{k=1}^{\infty} \frac{h_k}{k} \, x^{2k} = \frac{1}{4} \ln^2\left(\frac{1+x}{1-x}\right). \tag{3}$$

This enables us to evaluate a wide variety of interesting series via specialization, differentiation and integration.

First, setting $x = 1/2$, we find

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}\,k} = \frac{1}{4}\ln^2 3. \tag{4}$$

For $x = \sqrt{2}/2$,

$$\sum_{k=1}^{\infty} \frac{h_k}{2^k\,k} = \frac{1}{4}\ln^2(3+2\sqrt{2}). \tag{5}$$

Putting $x = (\sqrt{5}-1)/2 = \phi$, the golden ratio, we get

$$\sum_{k=1}^{\infty} \frac{h_k}{k}\,\phi^{2k} = \frac{1}{4}\ln^2(2+\sqrt{5}). \tag{6}$$

Furthermore, for any $\alpha \geq 2$, putting $x = (\sqrt{5}+1)/2\alpha$ and $x = (\sqrt{5}-1)/2\alpha$ in (3) respectively, we get

$$\sum_{k=1}^{\infty} \frac{h_k}{\alpha^{2k}\,k} \left(\frac{\sqrt{5}+1}{2}\right)^{2k} = \frac{1}{4}\ln^2\left(\frac{(2\alpha+1)+\sqrt{5}}{(2\alpha-1)-\sqrt{5}}\right) \tag{7}$$

and

$$\sum_{k=1}^{\infty} \frac{h_k}{\alpha^{2k}\,k} \left(\frac{\sqrt{5}-1}{2}\right)^{2k} = \frac{1}{4}\ln^2\left(\frac{(2\alpha-1)+\sqrt{5}}{(2\alpha+1)-\sqrt{5}}\right). \tag{8}$$

Recalling the Fibonacci numbers which are defined by

$$F_1 = 1, F_2 = 1, F_k = F_{k-1} + F_{k-2} \text{ for } k \geq 2$$

and Binet's formula

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right),$$

combining (7) and (8), we find

$$\sum_{k=1}^{\infty} \frac{h_k}{\alpha^{2k}\,k} F_{2k} = \frac{\sqrt{5}}{20}\ln\left(\frac{\alpha^2+\alpha-1}{\alpha^2-\alpha-1}\right)\ln\left(\frac{\alpha^2+\alpha\sqrt{5}+1}{\alpha^2-\alpha\sqrt{5}+1}\right). \tag{9}$$

In particular, for $\alpha = 2$

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}\,k} F_{2k} = \frac{\sqrt{5}}{4}\ln 5\,\ln(9+4\sqrt{5}). \tag{10}$$

Another step along this path is to change variables. Setting $x = \cos\theta$ in (3) leads to

$$\sum_{k=1}^{\infty} \frac{h_k}{k} \cos^{2k}\theta = \ln^2\left(\cot(x/2)\right). \tag{11}$$

Integrating both sides from 0 to π, and using

$$\int_0^{\pi} \cos^{2k}\theta \, d\theta = \frac{\pi}{2^{2k}} \binom{2k}{k}$$

and

$$\int_0^{\pi} \ln^2\left(\cot(x/2)\right) \, d\theta = \frac{\pi^3}{4},$$

we find

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}\, k} \binom{2k}{k} = \frac{\pi^2}{4}. \tag{12}$$

This adds another interesting series to Lehmer's list [6].

Next, for $0 < x < 1$, differentiating (3), then multiplying both sides by x, we obtain

$$\sum_{k=1}^{\infty} h_k x^{2k} = \frac{x}{2(1-x^2)} \ln\left(\frac{1+x}{1-x}\right). \tag{13}$$

Setting $x = 1/2$, we get

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}} = \frac{1}{3} \ln 3. \tag{14}$$

For $x = \sqrt{2}/2$,

$$\sum_{k=1}^{\infty} \frac{h_k}{2^k} = \frac{\sqrt{2}}{2} \ln(3 + 2\sqrt{2}). \tag{15}$$

Similar to (10), we have

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}} F_{2k} = \frac{\sqrt{5}}{50} (10 \ln(5 + 2\sqrt{5}) + 3\sqrt{5} \ln 5 - 5 \ln 5). \tag{16}$$

Finally, for $0 < x \leq 1$, dividing both sides of (3) by x and integrating from 0 to x, we obtain

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} = \frac{1}{2} \int_0^x \frac{1}{t} \ln^2\left(\frac{1+t}{1-t}\right) dt. \tag{17}$$

Using the substitution $u = (1-x)/(1+x)$ and integration by parts, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} &= \int_{(1-x)/(1+x)}^{1} \frac{\ln^2 u}{1-u^2} \, du \\ &= \frac{1}{2} \ln x \ln^2\left(\frac{1-x}{1+x}\right) + \int_{(1-x)/(1+x)}^{1} \frac{\ln u}{u} \ln\left(\frac{1-u}{1+u}\right) du. \end{aligned}$$

In view of (2), we have

$$\int_{(1-x)/(1+x)}^{1} \frac{\ln u}{u} \ln\left(\frac{1-u}{1+u}\right) du \quad = \quad -2\sum_{k=0}^{\infty} \frac{1}{2k+1} \int_{(1-x)/(1+x)}^{1} u^{2k} \ln u \, du.$$

Since

$$\int u^{2k} \ln u \, du = \frac{1}{2k+1} u^{2k+1} \ln u - \frac{1}{(2k+1)^2} u^{2k+1} + C,$$

we find

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} = \frac{1}{2} \ln x \ln^2\left(\frac{1-x}{1+x}\right) + 2\ln\left(\frac{1-x}{1+x}\right) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{1-x}{1+x}\right)^{2k+1}$$

$$+2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} - 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left(\frac{1-x}{1+x}\right)^{2k+1}. \tag{18}$$

In terms of the polylogarithm function [5]

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^n}{k^n},$$

and noting that

$$\sum_{k=0}^{\infty} \frac{x^n}{(2k+1)^n} = \frac{1}{2}\left(Li_n(x) - Li_n(-x)\right)$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \sum_{k=0}^{\infty} \frac{1}{k^3} - \sum_{k=0}^{\infty} \frac{1}{(2k)^3} = \frac{7}{8}\zeta(3),$$

we finally obtain

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} = \frac{7}{4}\zeta(3) + \frac{1}{2} \ln x \ln^2\left(\frac{1-x}{1+x}\right)$$

$$+\ln\left(\frac{1-x}{1+x}\right)\left(Li_2\left(\frac{1-x}{1+x}\right) - Li_2\left(\frac{x-1}{1+x}\right)\right) - \left(Li_3\left(\frac{1-x}{1+x}\right) - Li_3\left(\frac{x-1}{1+x}\right)\right).$$

Setting $x = 1$, we get

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} = \frac{7}{4}\zeta(3). \tag{19}$$

For $x = 1/3$,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k}{3^{2k} k^2} &= \frac{7}{8}\zeta(3) - \frac{1}{2} \ln 3 \ln^3 2 \\ &+ \frac{1}{3} \ln^3 2 + \ln 2 \, Li_2(-1/2) + Li_3(-1/2), \end{aligned}$$

where we have used

$$
\begin{aligned}
Li_2(1/2) &= \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2; \\
Li_3(1/2) &= \frac{7}{8}\zeta(3) + \frac{1}{6}\ln^3 2 - \frac{\pi^2}{12}\ln 2.
\end{aligned}
$$

Moreover, noting that

$$
h_k = \sum_{i=1}^{k} \int_0^1 x^{2(i-1)}\, dt = \int_0^1 (\sum_{i=1}^{k} x^{2(i-1)})\, dt = \int_0^1 \frac{1-x^{2k}}{1-x^2}\, dx
$$

and rewriting (8) as

$$
\sum_{k=1}^{\infty} \frac{h_k}{k^2}(1-x^{2k}) = \frac{1}{2}\int_x^1 \frac{1}{t}\ln^2\left(\frac{1+t}{1-t}\right) dt,
$$

we have

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{h_k^2}{k^2} &= \sum_{k=1}^{\infty} \frac{h_k}{k^2}\int_0^1 \frac{1-x^{2k}}{1-x^2}\, dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{1-x^2}\int_x^1 \frac{1}{t}\ln^2\left(\frac{1+t}{1-t}\right) dt\right) dx.
\end{aligned}
$$

Exchanging the order of the integration, we get

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{h_k^2}{k^2} &= \frac{1}{2}\int_0^1 \left(\frac{1}{t}\ln^2\left(\frac{1+t}{1-t}\right)\int_0^t \frac{1}{1-x^2}\, dx\right) dt. \\
&= \frac{1}{4}\int_0^1 \frac{1}{t}\ln^3\left(\frac{1+t}{1-t}\right) dt.
\end{aligned}
$$

Using the substitution $x = (1-t)/(1+t)$ and the well-known fact that

$$
\int_0^1 x^k \ln^3 x\, dx = -\frac{6}{(k+1)^3},
$$

we find

$$
\sum_{k=1}^{\infty} \frac{h_k^2}{k^2} = -\frac{1}{2}\int_0^1 \frac{\ln^3 x}{1-x^2}\, dx
$$

$$
= -\frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 x^{2k}\ln^3 x\, dx = 3\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{45}{16}\zeta(4) \tag{20}
$$

Another path out of (3) is to bring in complex variables. Since

$$
\frac{1}{i}\tan^{-1}(iz) = \tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)
$$

Replacing x by ix in (3), we obtain

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k} x^{2k} = (\tan^{-1} x)^2. \tag{21}$$

This series may be evaluated at values such as $x = 2 - \sqrt{3}, \sqrt{3}/3, 1$ explicitly:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k (2-\sqrt{3})^{2k}}{k} = \frac{\pi^2}{144} = \frac{3}{72}\,\zeta(2), \tag{22}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{3^k\, k} = \frac{\pi^2}{36} = \frac{1}{6}\,\zeta(2), \tag{23}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k} = \frac{\pi^2}{16} = \frac{3}{8}\,\zeta(2). \tag{24}$$

Similarly, applying differentiation and integration to (21), we deduce the corresponding formulas

$$\sum_{k=1}^{\infty} (-1)^{k-1} h_k\, x^{2k} = \frac{x}{1+x^2} \tan^{-1} x, \tag{25}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k^2} x^{2k} = 2 \int_0^x \frac{(\tan^{-1} t)^2}{t}\, dt. \tag{26}$$

In particular, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{3^k} = \frac{\sqrt{3}}{24}\,\pi, \tag{27}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k^2} = G\,\pi - \frac{7}{4}\,\zeta(3), \tag{28}$$

where G is the Catalan's constant which is defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Finally, following the excellent suggestion of an anonymous referee, recalling that

$$h_k = H_{2k} - \frac{1}{2}\, H_k, \tag{29}$$

we find from (19)

$$\sum_{k=1}^{\infty} \frac{1}{k^2}\, H_{2k} = \sum_{k=1}^{\infty} \frac{1}{k^2}\, h_k + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}\, H_k = \frac{11}{4}\,\zeta(3). \tag{30}$$

Furthermore, in terms of the multiple series [7]

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)}=2\zeta(3),\quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{i+j}}{ij(i+j)}=\frac{1}{4}\,\zeta(3),$$

the difference gives

$$\sum_{i,j>1,i+j=\text{odd}}\frac{1}{ij(i+j)}=\frac{7}{8}\,\zeta(3).$$

Setting $i+j=2k+1$ and using partail fractions, we have

$$\begin{aligned}\sum_{i,j>1,i+j=\text{odd}}\frac{1}{ij(i+j)} &= \sum_{k=1}^{\infty}\sum_{j=1}^{2k}\frac{1}{j(2k+1-j)(2k+1)}\\ &= \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\sum_{j=1}^{2k}\left(\frac{1}{j}+\frac{1}{2k+1-j}\right)\\ &= \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\,2H_{2k}.\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\,H_{2k}=\frac{7}{16}\,\zeta(3). \tag{31}$$

Subsequently, we have

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\,H_{2k}=\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\,H_{2k}+$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}+\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)^2}=\frac{21}{16}\,\zeta(3)+\frac{1}{8}\,(\pi^2-8\ln 2). \tag{32}$$

From this and the known result [1]

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\,H_k=\frac{1}{4}\,(\pi^2-\pi^2\ln 2-8\ln 2+7\zeta(3)),$$

we finally get

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\,h_k=\frac{7}{16}\,\zeta(3)+\frac{3}{4}\,\zeta(2)\ln 2. \tag{33}$$

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