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# Animals and 2-Motzkin Paths

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**Abstract**

We consider an animal $S$ as a set of points in the coordinate plane that are reachable from the origin $(0,0)$ through points in $S$ by steps from $\{(1,0),(0,1),(1,1),(-1,-1)\}$. In this paper, we give a combinatorial bijection with 2-Motzkin paths, i.e., the Motzkin paths with two different horizontal steps.

## 1 Introduction

We start by dividing the plane into eight equal octants. In this paper we count animals $A_i$, $1 \leq i \leq 3$, in the first $i$ octants . The count of $A_1$ was first done by Gouyou-Beauchamps and Viennot [5] and the idea of classifying it by the number of points lying on the $x$-axis is due to Aigner [1]. Bousquet-Melou [3] includes the possibility of diagonal steps, which changes the count of $A_3$ from $3^n$ to $4^n$, and that is the case we will consider here. In Theorems 8, 13, and 18 we give a bijection between animals and 2-Motzkin paths. For definitions and references, see Stanley [7].

**Definition 1** An **animal** $S$ is a set of points in the $xy$-plane with integer coordinates that satisfy the following conditions:

1. $(0,0) \in S$,
2. if $(a,b) \in S$ and $b \neq -a$, let $C(a,b) := \{(a-1,b),(a,b-1),(a-1,b-1)\}$, then $C(a,b) \cap S \neq \emptyset$,
3. if $(0,0) \neq (-b,b) \in S$, then $(-(b-1),(b-1)) \in S$.

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For $A_1$ we require also that $0 \le b \le a$, i.e., the first octant. For $A_2$ we want $0 \le a, b$, i.e., the first quadrant or the first two octants. Then $A_3$ is defined by $0 \le b$ and $a + b \ge 0$, i.e., the first three octants.

**Definition 2** We start with partial Motzkin paths beginning at $(0,0)$ with steps from $\{U = (1,1),\ D = (1,-1),\ H = (1,0)\}$. Then bicoloring the horizontal steps we have partial 2-Motzkin paths with steps from $\{U = (1,1),\ D = (1,-1),\ H_r = (1,0)$ and $H_g = (1,0)\}$, where $H_r$ is a horizontal step colored red and where $H_g$ is a horizontal step colored green. Let $M(n) = M_3(n)$ be the set of all partial 2-Motzkin paths of $n$ steps, let $M_2(n) \subset M_3(n)$ be the set of paths that never go below the $x$-axis and let $M_1(n) \subset M_2(n)$ denote the set of paths that end on $x$-axis at $(n,0)$ and let $m_i(n) = |M_i(n)|$ and $m_i(n,k) = |M_i(n,k)|$, where $M_i(n,k)$ is the set of partial 2-Motzkin paths that end at $(n,k)$.

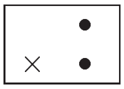
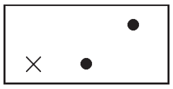
For $m, k \le 6$, the entries $(m_3(n,k))$ and $(m_2(n,k))$ are as follows:

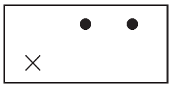
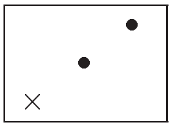
$$(m_3(n,k)) = \begin{bmatrix} n/k & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 \\ 4 & 0 & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 \\ 5 & 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix},$$

$$(m_2(n,k)) = \begin{bmatrix} n/k & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 3 & 14 & 14 & 6 & 1 & 0 & 0 \\ 4 & 42 & 48 & 27 & 8 & 1 & 0 \\ 5 & 132 & 165 & 110 & 44 & 10 & 1 \end{bmatrix}.$$

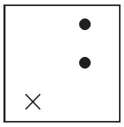
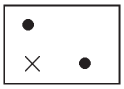
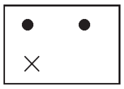
Let $A(n)$ be the set of all animals of size $n = |S - \{(0,0)\}|$, i.e., we do not count the origin $(0,0)$ for the size. Let $A_i(n)$ be the set of animals in the first $i$ octants of size $n$ and $a_i(n) = |A_i(n)|$ be the number of elements. We shall construct a bijection between $A_i(n)$ and $M_i(n)$.

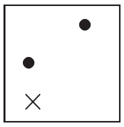
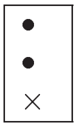
**Example 3** For $n = 2$, we illustrate the 5 elements in $A_1(2)$, and their counterparts in $M_1(2)$; $\times$ marks source points on the line $y = -x$. Note that the lowest source point is the origin, $(0,0)$.

 $\to H_r H_r$,  $\to UD$,  $\to H_r H_g$,

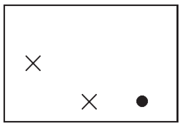
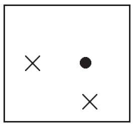
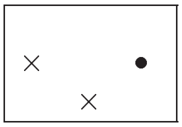
 $\to H_gH_r,$  $\to H_gH_g.$

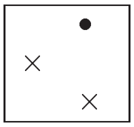
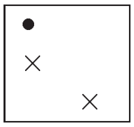
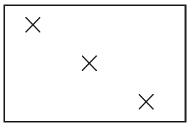
Both $A_2(2)$ and $M_2(2)$ have 10 elements. The following 5 elements are those not in $A_1(2)$:

 $\to H_gU,$  $\to H_rU,$  $\to UH_r$

 $\to UH_g,$  $\to UU.$

We have 16 elements in $A_3(2), M_3(2)$. The following 6 elements are those not in $A_2(2)$

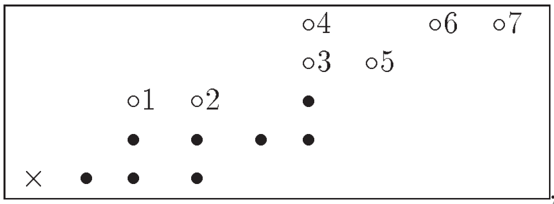
 $\to H_rD,$  $\to DH_r,$  $\to H_gD,$

 $\to DH_g,$  $\to DU,$  $\to DD.$

**Algorithm 4** We describe a decomposition method of an animal into two smaller parts, the top $T$ and the bottom $B$. The bottom part is an animal while the top part will be an animal after we apply Algorithm 5. Let $S$ be an animal and start the partition path at $P_S(a_0, a_0)$ at a point $(a_0, a_0)$ with the least $a_0 > 0$. If $(a_i, b_i) \in S$, go $E = (1, 0)$ one unit; otherwise go diagonally $D = (1, 1)$ one unit. Keep going until there are no more points in $S$ with larger first coordinate than this point. Let $T \subset S$ be the set of points on or above the path and let $B \subset S$ denote the set of points below the path.

**Algorithm 5** Let us define $T(i)$ inductively: $T(1) = T$, and $T(i+1)$ is constructed from $T(i)$ by replacing each $(a, b) \in T(i)$ with $(a-1, b-1)$ whenever $C(a, b) \cap T(i) = \emptyset$ and $a, b > 0$. Continue until $T(i+1) = T(i) = T'$.

**Example 6** For the following example $S$, we start with $(2, 2)$ and by Algorithm 4 the partition path of $S$ is $P = EEDEEDE$, i.e., $(2, 2) \to (3, 2) \to (4, 2) \to (5, 3) \to (6, 3) \to (7, 3) \to (8, 4) \to (9, 4)$,

$S =$ 

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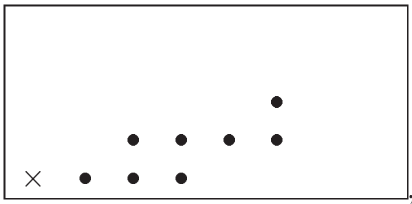
The partition path partitions $S$ into $B, T$ as follows:

$B =$ 
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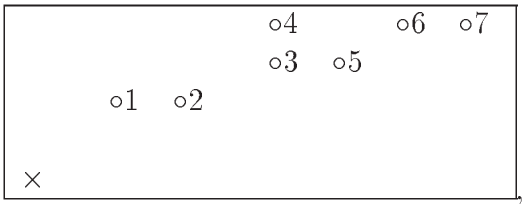
$T =$ 

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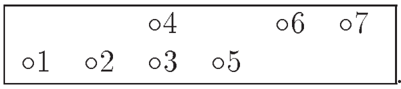
Applying Algorithm 5 on $T$, we have

$T^{'} =$ 

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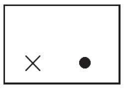
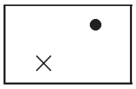
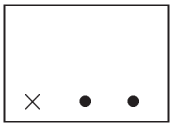
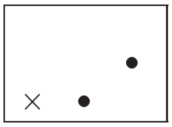
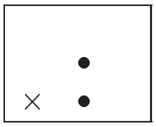
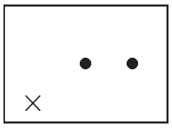
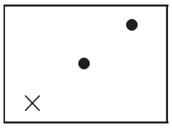
## 2 Animals $A_1(m)$ and 2-Motzkin Paths $M_1(m)$

Here we construct a bijection between animals and 2-Motzkin paths. For other bijections and partitions, see [2, 4].

**Example 7** For $m = 1, 2$. $A_1(m) \Leftrightarrow M_1(m)$

 $\rightarrow H_r$,  $\rightarrow H_g$ ,

 $\rightarrow H_r H_r$,  $\rightarrow H_r H_g$,  $\rightarrow UD$,

 $\rightarrow H_g H_r$,  $\rightarrow H_g H_g$.

**Theorem 8** *The number of the animals in the first octant of size $n$ is given by $c_{n+1}$, the $(n+1)^{th}$ Catalan number.*

**Proof**. By induction, assume that the theorem is true for size less than $n$. Let $S \in A_1(n)$. We apply Algorithm 4 by starting at the smallest $d > 0$ such that $(d,d) \in S$ to partition $S$ into $T, B$. We apply Algorithm 5 to obtain the $T'$. Let $B \ \to B' \in A_1(k-1)$, by removing the first point (the origin). If $T' = \emptyset$, then $B'$ is of size $k-1 = n-1$, the first step is $H_r$ and by induction $S \to H_r B^*$. If $B' = \emptyset$, then the first step is $H_g$ and by induction $S \to H_g T^*$. Otherwise, the first step is $U$ and the $k^{th}$ step is $D$, by induction fill in steps 2 to $(k-1)$ by $B' \to B^*$ and steps $(k+1)^{th}$ to the $n^{th}$ by $T' \to T^*$, i.e., $S \to P = U(B^*)D(T^*)$.

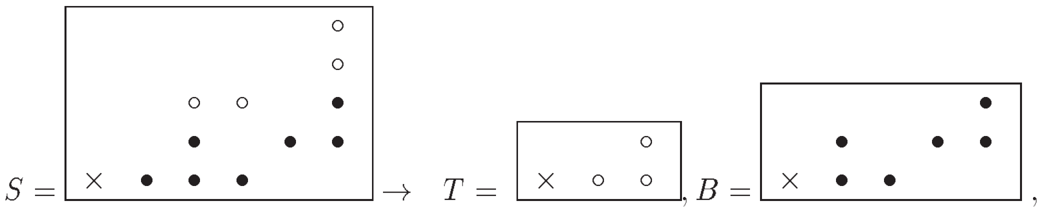
Conversely, by induction let $P = a_1a_2\cdots a_n \in M_1(n)$. If $a_1 = H_r$, then $P' = a_2a_3\cdots a_n$ is of size $m-1$, by induction $P' \to S^*$ and $S = (S^* + (1,0)) \cup \{(0,0)\} \in A_1(n)$ (shift $S^*$ to the right one unit). If $a_1 = H_g$, then $S = \{S^* + (1,1) \cup (0,0)\}$ (shift $S^*$ diagonally up one unit). If $a_1$ is $U$, then find the first $k$ such that $a_1a_2\cdots a_k \in M(k)$, $B = a_2a_3\cdots a_{k-1}$ and $T = a_{k+1}a_{k+2}\cdots a_n$. By induction $T \to T^*$, $B \to B^* \to B' = (B^* + (1,0)) \cup \{(0,0)\}$. Let $T' = T^* + (j+1, j+1)$, where $j = \max\{b : (a,b) \in B\}$, and then apply Algorithm 5 by starting with the union of $B'$ and $T'$. By induction the total count is

$$\begin{aligned} a_1(n) &= a_1(n-1) + a_1(n-2)a_1(0) + a_1(n-3)a_1(1) + a_1(n-4)a_1(2) + \cdots \\ &= c_nc_0 + c_{n-1}c_1 + c_{n-2}c_2 + \cdots + c_0c_{n-1} \\ &= \sum_{i=0}^{n} c_{n-i}c_i = c_{n+1}, \end{aligned}$$

where the first term represents the case that $T$ is empty and the second term represents the case that $T$ is one point. Similarly, the last term represents the case that $B$ is empty and next-to-last term represents the case that $B$ is one point.

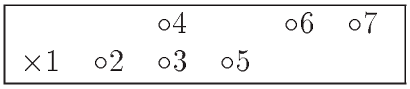
The generating function is $\sum a_1(n)x^n = 1 + 2x + 5x^2 + 14x^3 + 42x^4 + \cdots = C^2$.

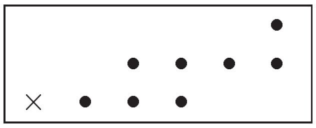
**Example 9** This example is for the first part of the proof in Theorem 8. The partition path is $(2,2) \to (3,2) \to (4,2) \to (5,3) \to (6,3)$, which partitions $S$ into two animals $T, B$. By induction we produce 2-Motzkin paths $T^*$ and $B^*$, and by Theorem 8 we produce a 2-Motzkin path $P$ for $S$.




$P = U(B^*)D(T^*) \to U(UH_rH_gUDD)D(H_rUD)$.

**Example 10** This example is for the converse of the bijection. We start with a 2-Motzkin path $P = U(UH_rDH_rH_rUD)D(H_rUH_rH_gH_rD)$, locate the first $D$ such that the path $P$ comes back to $x$-axis. The subpath $B = UH_rDH_rH_rUD$ is the section of $P$ between the first step($U$) and this $D$, the section after this $D$ is the subpath $T = H_rUH_rH_gH_rD$. By induction we produce subanimals $T^*$ and $B^*$, using Theorem 8 we produce the animal $S$ for $P$.

$T \to T^* =$ 

, $B \to B^* \to B' =$ 

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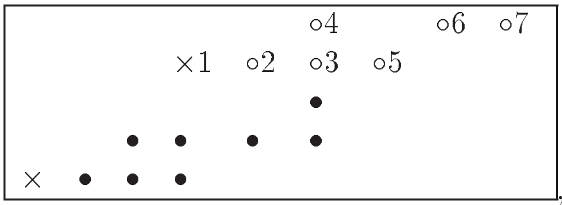
$B' \cup T' =$ 

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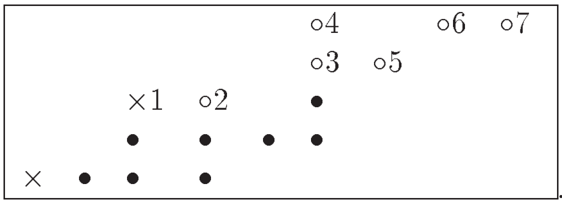
$P \to S =$ 

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**Remark 11** Let us partition $A_1(n)$ by the number points on the line $y = x$. Let $A_1(n,k) = \{S \in A_1(n) : |S \cap \{(x,x) : x > 0\}| = k\}$ and $a_1(n,k) = |A_1(n,k)|$. Then the following is the matrix $(a_1(n,k))$ for $n, k$ up to 5:

$$
\begin{array}{c|c}
n\backslash k & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 \\
5 & 5 & 3 & 1 & 0 & 0 \\
14 & 14 & 9 & 4 & 1 & 0 \\
42 & 42 & 28 & 14 & 5 & 1
\end{bmatrix}
\end{array}.
$$

We say that an infinite lower triangular matrix $L = (g, f)$ is a *Riordan matrix* if the generating function of the $k^{th}$ column is $gf^k$ for all $k$. Here $(a_1(n,k)) = (C, xC)$. For more about the Riordan matrix, see [6].

**Remark 12** By using the Lagrange Inversion Formula (Wilf [8]) with some index adjustment we derive the explicit formula $a_1(n,k) = \frac{k+1}{2n-k+1}\binom{2n-k+1}{n-k}$.

## 3 Animals $A_2(m)$ and 2-Motzkin Paths$M_2(m)$

Here we construct a bijection between animals and 2-Motzkin paths. For other bijections and partitions, see [2, 4].

**Theorem 13** *There is a bijection between $M_2(m)$ and $A_2(m)$. Moreover, for all $m \geq 0$, we have $a_2(m) = \binom{2m+1}{m}$.*

**Proof**. Let $S \in A_2(m)$, we apply Algorithm 4 by starting at $(0,1)$ to partition $S$ into $T$, $B$. Then $B \in A_1(k)$ and by applying Algorithm 5, $T \to T'$. If $T = \emptyset$, then $S \in A_1(m)$ and by Theorem 8 we are done. Otherwise, $T' \to T^* = T' - (0,1), B \to B^*$; and $S \to P = B^*UT^*$.

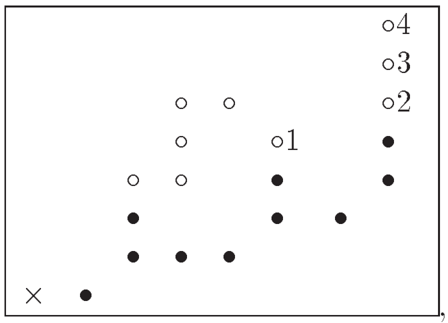
Conversely, by induction let $P = a_1a_2\cdots a_m \in M_2(m) - M_1(m)$. We look for the largest $k$ such that $(k,0) \in P$, i.e., the last time $P$ is on the $x$-axis. By induction $a_1a_2\cdots a_k \to B$ and $a_{k+2}\cdots a_m \to T$. $T \to T' = T + (0,1) \to T^* = T' + (j+1, j+1)$, where $j = \max\{b : (a,b) \in B\}$. Apply Algorithm 5 by starting with $B \cup T^*$; we have $P \to S = (B \cup T^*)'$.

The generating function of the counts of $A_2(m)$ is

$$C^2(1 + xC^2 + (xC^2)^2 + (xC^2)^3 + \cdots) = \frac{C^2}{1 - xC^2} = \frac{C^2}{C\sqrt{1-4x}} = \frac{C}{\sqrt{1-4x}}.$$

These numbers appear as the nonzero entries in column one in the Pascal's Triangle.

**Example 14** This example is for the first part of Theorem 13. We start an animal $S$ with the partition path $(2,3) \to (3,3) \to (4,3) \to (5,4) \to (6,4) \to (7,5) \to (8,5)$, which partitions $S$ into $B, T$. By induction $B \to B^*, T \to T^*$ we derive the path $P$.

$S =$ 

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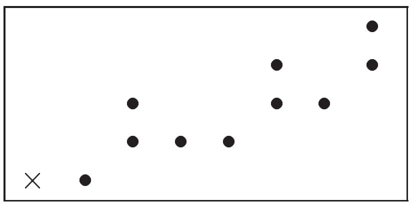
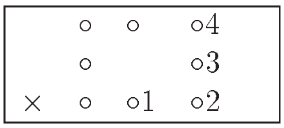
$B =$  $\to B^* = U(H_gH_rUH_gH_rH_gDH_r)D,$

$T =$ 

$\to T^* = (U(H_rUD)D)UH_rH_r \to$

$$P = (B^*)U(T^*) = (UH_gH_rUH_gH_rH_gDH_rD)U(UH_rUDDUH_rH_r).$$

**Remark 15** Let us partition $A_2(m)$ by

$$A_2(m,k) = \{S \in A_2(m) : k = \max\{b - a : (a,b) \in S\}\}$$

and $a_2(m,k) = |A_2(m,k)|$. By using the Lagrange Inversion Formula (Wilf [8]) and simple algebraic operations we obtain the explicit formula $a_2(m,k) = \frac{2k+2}{2m+2}\binom{2m+2}{m-k}$. Then by Theorem 13 the following is the matrix $(a_2(m,k)) = (C^2, xC^2)$ for $m,k$ up to 5:

$$\begin{bmatrix} m/k & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 01 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 3 & 14 & 14 & 6 & 1 & 0 & 0 \\ 4 & 42 & 48 & 27 & 8 & 1 & 0 \\ 5 & 132 & 165 & 110 & 44 & 10 & 1 \end{bmatrix}.$$

**Remark 16** Let us partition $A_2(m)$ by

$$A_2^*(m,k) = \{S \in A_2(m) : k = \max\{b \ : (0,b) \in S\}\}$$

and $a_2^*(m,k) = |A_2^*(m,k)|$ . The set $A_2^*(m,0)$ consists of two copies of $A_2(m-1)$; one copy consists of those with no points on $x$-axis except the origin and the other copy with such points. Hence the generating function is

$$1 + \frac{2xC}{\sqrt{1-4x}} = \frac{\sqrt{1-4x} + 2xC}{\sqrt{1-4x}} = \frac{\frac{2-C}{C} + 2xC}{\sqrt{1-4x}} = \frac{1}{\sqrt{1-4x}}.$$

Note that if $(0,1) \in S$, the partition is $T, B$ in Theorem 13 with $B$ containing no point on $y = x > 0$. By using the Lagrange Inversion Formula (Wilf [8]) and simple algebraic operations we can derive the explicit formula $a_2^*(m,k) = \binom{2m-k}{m-k}$.

The generating function for $B$ is $C$ and the following is the matrix $(a_2^*(m,k)) = (\frac{1}{\sqrt{1-4x}}, xC)$ for $m,k$ up to 5:

$$\begin{bmatrix} m/k & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 6 & 3 & 1 & 0 & 0 & 0 \\ 3 & 20 & 10 & 4 & 1 & 0 & 0 \\ 4 & 70 & 35 & 15 & 5 & 1 & 0 \\ 5 & 252 & 126 & 56 & 21 & 6 & 1 \end{bmatrix}.$$

**Remark 17** Let us go a step further. Let $D(m) = A_2(m, 0)$, the set of animals in the first quadrant containing no point on the $y$-axis except the origin and partition $D(m)$ by the number of points on the $x$-axis $D(m, k) = \{S \in D(m) : (k, 0) \in S, (k + 1, 0) \notin S\}$. Let $d(m, k) = |D(m, k)|$. Then by using the Lagrange Inversion Formula (Wilf [8]) and simple algebraic operations we can derive the explicit formula $d(m, k) = \binom{2m-k-1}{m-k}$ for $k > 0$ and $d(m, 0) = \binom{2m-1}{m-1}$.

The following is the matrix $(d(m, k))$ for $m, k$ up to 5:

$$\begin{bmatrix} m/k & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 10 & 6 & 3 & 1 & 0 & 0 \\ 4 & 35 & 20 & 10 & 4 & 1 & 0 \\ 5 & 126 & 70 & 35 & 15 & 5 & 1 \end{bmatrix} = \left( 1 + \frac{xC}{\sqrt{1-4x}}, xC(x) \right).$$

# 4 Animals $A_3(m)$ and 2-Motzkin Paths $M_3(m)$

Here we construct a bijection between animals and 2-Motzkin paths. For other bijections and partitions, please see [2, 4].

**Theorem 18** *There is a bijection between $M_3(m) = M(m)$ and $A_3(m) = A(m)$. Moreover, the number of animals in the first three octants of size $m$ is $4^m$, $m \geq 0$.*

**Proof**. Let $S \in A_3(m)$, apply Algorithm 4 by starting at $(-1, 1)$ to partition $S$ into $T$, $B$. Then $B \in A_1(k)$ and apply Algorithm 5 to $T \to T' \in A_3(m - k - 1)$. Then $T' \to T^* = T' + (1, -1)$. If $S \in A_2(m)$, then by Theorem 13 we are done. Otherwise, by induction $S \to (B^*)D(T^*) \in M_3(m)$.

Conversely, let $P = a_1 a_2 \cdots a_m \in M_3(m)$. If $P \in M_2(m)$, then by Theorem 13, we are done. Otherwise, we look for the first $k$ such that $(k, 0) \in P$ and $P$ goes under the $x$-axis after that. Then by induction $a_1 a_2 \cdots a_k \to B^*$ and $a_{k+2} \cdots a_m \to T$ , $T \to T' = T + (-1, 1)$. $T^* = E \cup ((T' - E) + (j + 1, j + 1))$, where $E$ is the set of points in $T'$ that are not in the first quadrant and $j = \max\{(b : (a, b) \in B\}$. Apply Algorithm 5 by starting with $B^* \cup T^*$. We have $P \to S$.

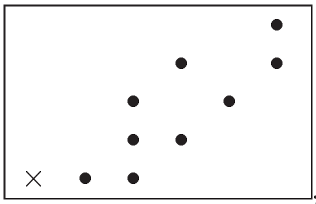
In terms of generating functions we have

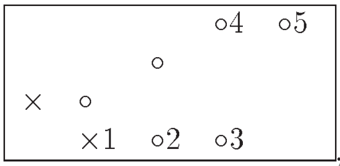
$$\frac{C}{\sqrt{1-4x}}((1+xC^2+(xC^2)^2+(xC^2)^3+\cdots) = \frac{C}{\sqrt{1-4x}}\frac{1}{1-xC^2} = \frac{C}{\sqrt{1-4x}}\frac{1}{C\sqrt{1-4x}} = \frac{1}{1-4x}.$$

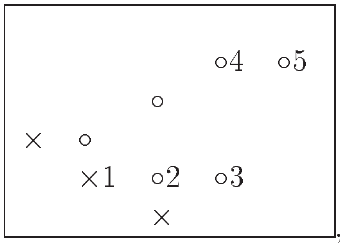
**Example 19** This example is the converse of the proof of Theorem 18. Let

$$P = (UUDH_rH_gH_gDH_gH_r)D(H_rH_rDH_rH_gH_gH_r)$$

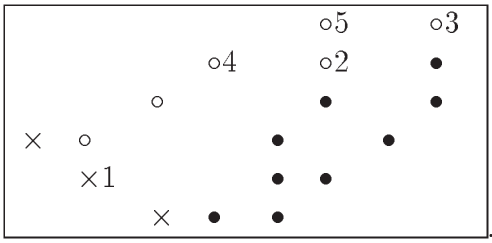
be a 2-Motzkin path and locate the first $D$ step, where the path goes below the $x$-axis. The section of $P$ before the $D$ is $B$ and the section after the $D$ is $T$. Then by induction we derive the animal $S$ for $P$.

$$B = U(UDH_rH_gH_g)D(H_gH_r) \to B^* =$$  ,

$$(H_rH_r)D(H_rH_gH_gH_r) \quad \to \quad T =$$  ,

$$T^{'} \quad =$$  ,

$$P = (UUDH_rH_gH_gDH_gH_r)D(H_rH_rDH_rH_gH_gH_r) = (B^*)D(T^*),$$

$$\to S =$$  .

**Remark 20** Let us partition $A(m)$ by the number of source points on the line $y = -x > 0$. Let $A(m,k) = \{S \in A(m) : (-k,k) \in S,\ (-(k+1), k+1) \notin S\}$ and $a(m,k) = |A(m,k)|$. By using the Lagrange Inversion Formula (Wilf [8]) and simple algebraic operations we obtain the explicit formula $a(m,k) = \binom{2m+1}{m-k}$.

The following is the matrix $(a(m,k))$ for $m,k$ up to 5:

$$(a(m,k)) = \begin{array}{cc} m/k & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 10 & 5 & 1 & 0 & 0 & 0 \\ 35 & 21 & 7 & 1 & 0 & 0 \\ 126 & 84 & 36 & 9 & 1 & 0 \\ 462 & 330 & 165 & 55 & 11 & 1 \end{bmatrix} \end{array} = \left(\frac{C}{\sqrt{1-4x}}, xC^2\right).$$

The $k^{th}$ column is the condensed version of the $(2k+1)^{th}$ column of Pascal's Triangle.

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