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# On Families of Nonlinear Recurrences Related to Digits

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**Abstract**

Consider the sequence of positive integers $(u_n)_{n\geq 1}$ defined by $u_1 = 1$ and $u_{n+1} = \lfloor \sqrt{2}\left(u_n + \frac{1}{2}\right)\rfloor$. Graham and Pollak discovered the unexpected fact that $u_{2n+1} - 2u_{2n-1}$ is just the $n$-th digit in the binary expansion of $\sqrt{2}$. Fix $w \in \mathbb{R}_{>0}$. In this note, we first give two infinite families of similar nonlinear recurrences such that $u_{2n+1} - 2u_{2n-1}$ indicates the $n$-th binary digit of $w$. Moreover, for all integral $g \geq 2$, we establish a recurrence such that $u_{2n+1} - gu_{2n-1}$ denotes the $n$-th digit of $w$ in the $g$-ary digital expansion.

## 1 Introduction

In 1969, Hwang and Lin [6] studied Ford and Johnson's algorithm for sorting partially-sorted sets (see also [7]). In doing so, they came across the sequence of integers

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 27,\ 38,\ 54,\ 77,\ 109\ldots$$

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defined by the nonlinear recurrence

$$u_1 = 1, \qquad u_{n+1} = \left\lfloor \sqrt{2u_n(u_n+1)} \right\rfloor, \quad n \geq 1. \tag{1}$$

Since there is no integral square between $2u_n^2 + 2u_n$ and $2u_n^2 + 2u_n + \frac{1}{2} = 2(u_n + \frac{1}{2})^2$ we can rewrite the recurrence in a more striking form, i.e.,

$$u_1 = 1, \qquad u_{n+1} = \left\lfloor \sqrt{2}(u_n + 1/2) \right\rfloor, \quad n \geq 1. \tag{2}$$

While investigating closed-form expressions for $u_n$ in (2), Graham and Pollak [4] discovered the following amazing fact:

**Fact 1 (Graham/Pollak).** *We have that*

$$d_n = u_{2n+1} - 2u_{2n-1}$$

*is the $n$-th digit in the binary expansion of* $\sqrt{2} = (1.011010100\ldots)_2$.

Since then, sequences arising from the recurrence relation given in (2) are referred to as Graham-Pollak sequences [9, 10]. Sloane [9] gives three special sequences depending on the initial term $u_1$, i.e., sequence A001521 for $u_1 = 1$, A091522 for $u_1 = 5$ and sequence A091523 for $u_1 = 8$.

The curiosity of Fact 1 has drawn the attention of several mathematicians and has been cited a few times, see Ex. 30 in Guy [5], Ex. 3.46 in Graham/Knuth/Patashnik [3] and in Borwein/Bailey [1, pp. 62–63]. A generalization to numbers other than $\sqrt{2}$ is, however, not straightforward from Graham and Pollak's proof. Nevertheless, Erdős and Graham [2, p. 96] suspected that similar results would also hold "for $\sqrt{m}$ and other algebraic numbers", but they concluded that "we have no idea what they are."

By applying a computational guessing approach, Rabinowitz and Gilbert [8] could give an answer in the binary case:

**Theorem 1.1 (Rabinowitz/Gilbert).** *Let $w \in \mathbb{R}_{>0}$ and $t = w/2^m$, where $m = \lfloor \log_2 w \rfloor$. Furthermore, set*

$$a = 2\left(1 - \frac{1}{t+2}\right), \qquad b = \frac{2}{a}.$$

*Define a sequence $(u_n)_{n\geq 1}$ by the recurrence*

$$u_1 = 1$$
$$u_{n+1} = \begin{cases} \lfloor a(u_n + 1/2) \rfloor, & \text{if } n \text{ is odd;} \\ \lfloor b(u_n + 1/2) \rfloor, & \text{if } n \text{ is even.} \end{cases}$$

*Then $u_{2n+1} - 2u_{2n-1}$ is the $n$-th digit in the binary expansion of $w$.*

Note that for $w = \sqrt{2}$ we get $a = b = \sqrt{2}$ and the statement of Fact 1 is obtained. However, the values of $a$ and $b$ in Theorem 1.1 are somehow wrapped in mystery. Rabinowitz and Gilbert first varied $a$ and $b$ in order that $u_{2n+1} - 2u_{2n-1} \in \{0, 1\}$. They found that

$ab = 2$ and discovered that the represented $w$ indeed equals $2(a-1)/(2-a)$ provided that $1 < a < 3/2$.

It is a natural question to ask, whether there exist other values of $a$ and $b$ such that the binary expansion of $w$ is obtained. Here we prove

**Theorem 1.2.** *Let $w \in \mathbb{R}_{>0}$ and $t = w/2^m$, where $m = \lfloor \log_2 w \rfloor$. Furthermore, let $j \in \mathbb{Z}_{>0}$ and set the values of $a$ and $b$ according to one of the following cases:*

CASE I:

$$a = 2\left(j - \frac{1}{t+2}\right), \qquad b = \frac{2}{a}.$$

CASE II:

$$a = 2j - \frac{t}{t+2}, \qquad b = \frac{2}{a}.$$

*Define a sequence $(u_n)_{n\geq 1}$ by the recurrence*

$$u_1 = 1$$
$$u_{n+1} = \begin{cases} \lfloor a(u_n + 1/2) \rfloor, & \text{if } n \text{ is odd;} \\ \lfloor b(u_n + \varepsilon) \rfloor, & \text{if } n \text{ is even,} \end{cases}$$

*where $1/3 \leq \varepsilon < 2/3$ in* CASE I *and $\varepsilon = 1/2$ in* CASE II*, respectively. Then $u_{2n+1} - 2u_{2n-1}$ is the $n$-th digit in the binary expansion of $w$.*

In the closing paragraph of [8], the authors finally posed the question, whether there exists an analogous statement for ternary digits. Here we prove

**Theorem 1.3.** *Let $w \in \mathbb{R}_{>0}$ and $g \geq 2$ be an integer. Furthermore, set $t = w/g^m$, where $m = \lfloor \log_g w \rfloor$ and*

$$a = \frac{g}{(g-1)(t+g)}, \qquad b = \frac{g}{a}.$$

*Define a sequence $(u_n)_{n\geq 1}$ by the recurrence*

$$u_1 = 1$$
$$u_{n+1} = \begin{cases} \lfloor a(u_n + \varepsilon) \rfloor, & \text{if } n \text{ is odd;} \\ \lfloor b(u_n + 1/(g-1)) \rfloor, & \text{if } n \text{ is even,} \end{cases}$$

*where $-1/g \leq \varepsilon < (g+1)(g-2)/g$. Then $u_{2n+1} - gu_{2n-1}$ is the $n$-th digit in the $g$-ary digital expansion of $w$.*

In view of Fact 1, we note two immediate consequences of Theorem 1.2 and Theorem 1.3. To begin with, we substitute $w = t = \sqrt{2}$ in CASE I and CASE II of Theorem 1.2. This implies $a = 2j - 2 + \sqrt{2}$ (CASE I) and $a = 2j + 1 - \sqrt{2}$ (CASE II) for $j \geq 1$. By ordering all such values into a single sequence, we obtain

**Corollary 1.1.** *Let* $a_j = j + (-1)^j\sqrt{2}$ *for* $j = 0, 2, 3\ldots$ *and* $b_j = 2/a_j$. *Define a sequence* $(u_n)_{n\geq 1}$ *by*

$$u_1 = 1$$
$$u_{n+1} = \begin{cases} \lfloor a_j(u_n + 1/2)\rfloor, & \textit{if } n \textit{ is odd;} \\ \lfloor b_j(u_n + 1/2)\rfloor, & \textit{if } n \textit{ is even.} \end{cases}$$

*Then* $u_{2n+1} - 2u_{2n-1}$ *is the* $n$*-th digit in the binary expansion of* $\sqrt{2} = (1.011010100\ldots)_2$.

Note that for $j = 1$ we have $a_1 = 1 - \sqrt{2} < 0$ and $u_5 - 2u_3 = 7 - 2 \cdot 2 = 3$, which is not a binary digit.

On the other hand, if we take $g = 3$, $w = t = \sqrt{2}$ and $\varepsilon = 1/2$ in Theorem 1.3, we get

**Corollary 1.2.** *Define a sequence* $(u_n)_{n\geq 1}$ *by*

$$u_1 = 1$$
$$u_{n+1} = \begin{cases} \lfloor a(u_n + 1/2)\rfloor, & \textit{if } n \textit{ is odd;} \\ \lfloor b(u_n + 1/2)\rfloor, & \textit{if } n \textit{ is even,} \end{cases}$$

*where* $a = (9 - 3\sqrt{2})/14$ *and* $b = 6 + 2\sqrt{2}$. *Then* $u_{2n+1} - 3u_{2n-1}$ *is the* $n$*-th digit in the ternary expansion of* $\sqrt{2} = (1.102011221\ldots)_3$.

# 2 Proofs

For later reference we state an easy, but useful proposition.

**Proposition 2.** *Let* $g \geq 2$ *be an integer and* $w = (d_1d_2d_3\ldots)_g$ *be the* $g$*-ary digital expansion of* $w$ *with* $d_1 \neq 0$ *and* $0 \leq d_n < g$ *for* $n \geq 1$. *Suppose further that for* $n \geq 1$ *not all of* $d_n, d_{n+1}, \ldots$ *equal* $g - 1$. *Then*

- $t = (d_1.d_2d_3\ldots)_g$ *with* $1 \leq t < g$,
- $d_n = \lfloor tg^{n-1}\rfloor - g\lfloor tg^{n-2}\rfloor$.

*Proof.* Since $m = \lfloor \log_g w\rfloor$ it is immediate that $1 \leq w/g^m < g$. Moreover,

$$\lfloor tg^{n-1}\rfloor - g\lfloor tg^{n-2}\rfloor = (d_1d_2\ldots d_n)_g - (d_1d_2\ldots d_{n-1}0)_g = d_n.$$

□

## 2.1 Proof of Theorem 1.2

First, we prove that in Case I there hold

$$u_{2k} = 2^{k-1} + \lfloor t2^{k-1}\rfloor + (j-1)(2^k + 2\lfloor t2^{k-2}\rfloor + 1),$$
$$u_{2k+1} = 2^k + \lfloor t2^{k-1}\rfloor,$$

so that Proposition 2 gives $u_{2n+1} - 2u_{2n-1} = d_n$. To begin with, we have $u_1 = 2^0 + \lfloor t/2 \rfloor = 1$ because of $1 \le t < 2$. We are going to employ an induction argument. Suppose that the result holds true for $u_{2k-1}$. Then

$$\begin{aligned} u_{2k} &= \left\lfloor 2\left(j - \frac{1}{t+2}\right)\left(2^{k-1} + \lfloor t2^{k-2} \rfloor + \frac{1}{2}\right)\right\rfloor \\ &= (j-1)(2^k + 2\lfloor t2^{k-2}\rfloor + 1) + \left\lfloor \left(1 - \frac{1}{t+2}\right)\left(2^k + 2\left\lfloor t2^{k-2}\right\rfloor + 1\right)\right\rfloor . \end{aligned}$$

Thus, it suffices to show that

$$\left\lfloor \frac{t+1}{t+2} \cdot \left(2^k + 2\left\lfloor t2^{k-2}\right\rfloor + 1\right)\right\rfloor = 2^{k-1} + \left\lfloor t2^{k-1}\right\rfloor . \tag{3}$$

Since $2\left\lfloor t2^{k-2}\right\rfloor = \left\lfloor t2^{k-1}\right\rfloor - d_k$ by Proposition 2, we may rewrite (3) in the equivalent form

$$\begin{aligned} (t+2)\left(2^{k-1} + \left\lfloor t2^{k-1}\right\rfloor\right) &\le (t+1)\left(2^k + \left\lfloor t2^{k-1}\right\rfloor - d_k + 1\right) \\ &< (t+2)\left(2^{k-1} + \left\lfloor t2^{k-1}\right\rfloor + 1\right). \end{aligned}$$

Straightforward algebraic manipulation leads to

$$t2^{k-1} + \left\lfloor t2^{k-1}\right\rfloor \le t2^k + (1-d_k)(t+1) < \left(t2^{k-1} + \left\lfloor t2^{k-1}\right\rfloor + 1\right) + 1\cdot(t+1),$$

which is obviously true because of $\left\lfloor t2^{k-1}\right\rfloor \le t2^{k-1} < \left\lfloor t2^{k-1}\right\rfloor + 1$.
Now, assume that the result is true for $u_{2k}$. Thus, we have to show that

$$u_{2k+1} = \left\lfloor \frac{t+2}{j(t+2)-1}(u_{2k} + \varepsilon)\right\rfloor = 2^k + \lfloor t2^{k-1}\rfloor. \tag{4}$$

The equality of integer floors (4) can be rewritten in terms of two inequalities, i.e.,

$$\begin{aligned} (j(t+2)-1)(2^k + \left\lfloor t2^{k-1}\right\rfloor) &\le (t+2)\left(2^{k-1} + \left\lfloor t2^{k-1}\right\rfloor + \varepsilon + (j-1)(2^k + 2\left\lfloor t2^{k-2}\right\rfloor + 1)\right) \\ &< (j(t+2)-1)(2^k + \left\lfloor t2^{k-1}\right\rfloor + 1). \end{aligned}$$

Again, we use Proposition 2 and proper term cancelling such that (4) translates into

$$0 \le \left\lfloor t2^{k-1}\right\rfloor - t2^{k-1} + (t+2)\left(\varepsilon + (j-1)(1-d_k)\right) < j(t+2) - 1.$$

Since $-1 < \left\lfloor t2^{k-1}\right\rfloor - t2^{k-1} \le 0$ and $\varepsilon < 2/3$ we have

$$\left\lfloor t2^{k-1}\right\rfloor - t2^{k-1} + (t+2)\left(\varepsilon + (j-1)(1-d_k)\right) < (t+2)(2/3 + (j-1)) \le j(t+2) - 1.$$

On the other hand, $\varepsilon \ge 1/3$ implies

$$\left\lfloor t2^{k-1}\right\rfloor - t2^{k-1} + (t+2)\left(\varepsilon + (j-1)(1-d_k)\right) > -1 + (t+2)\varepsilon \ge 0.$$

This finishes the proof of Theorem 1.2 for Case I.

Let now $a$, $b$ and $\varepsilon$ be according to Case II. Again, by Proposition 2 it suffices to show that

$$\begin{aligned} u_{2k} &= 2^k + \lfloor t2^{k-2} \rfloor + (j-1)(2^k + 2\lfloor t2^{k-2}\rfloor + 1), \\ u_{2k+1} &= 2^k + \lfloor t2^{k-1} \rfloor. \end{aligned}$$

As before, we have $u_1 = 2^0 + \lfloor t/2 \rfloor = 1$. Assume that the closed-form expression holds true for $u_{2k-1}$. Then

$$\begin{aligned} u_{2k} &= \left\lfloor \left(2j - \frac{t}{t+2}\right)\left(2^{k-1} + \lfloor t2^{k-2}\rfloor + \frac{1}{2}\right)\right\rfloor \\ &= (j-1)(2^k + 2\lfloor t2^{k-2}\rfloor + 1) + \left\lfloor \left(1 - \frac{t}{2(t+2)}\right)\left(2^k + 2\left\lfloor t2^{k-2}\right\rfloor + 1\right)\right\rfloor. \end{aligned}$$

Hence, it is sufficient to prove that

$$2^k + \lfloor t2^{k-2} \rfloor \le \frac{t+4}{2(t+2)} \cdot (2^k + 2\lfloor t2^{k-2}\rfloor + 1) < 2^k + \lfloor t2^{k-2}\rfloor + 1. \tag{5}$$

By multiplying (5) with $2(t+2)$ and simply canceling out all terms with $\lfloor t2^{k-2} \rfloor$, (5) simplifies to

$$0 \le 4\left(\lfloor t2^{k-2}\rfloor - t2^{k-2}\right) + t + 4 < 2t + 4. \tag{6}$$

Relation (6) is obviously true, since $-1 < \lfloor t2^{k-2} \rfloor - t2^{k-2} \le 0$.
For the induction step from $u_{2k}$ to $u_{2k+1}$ we have to ensure that

$$u_{2k+1} = \left\lfloor \frac{2(t+2)}{2j(t+2)-t}\left(u_{2k} + \frac{1}{2}\right)\right\rfloor = 2^k + \lfloor t2^{k-1} \rfloor,$$

or equivalently, that

$$\begin{aligned} 2^k + \lfloor t2^{k-1}\rfloor &\le \frac{2(t+2)}{2j(t+2)-t}\left(2^k + \lfloor t2^{k-2}\rfloor + \frac{1}{2} + (j-1)(2^k + 2\lfloor t2^{k-2}\rfloor + 1)\right) \\ &< 2^k + \lfloor t2^{k-1} \rfloor + 1. \end{aligned}$$

We replace all $\lfloor t2^{k-2} \rfloor$ by $\left(\lfloor t2^{k-1}\rfloor - d_k\right)/2$ and after some term sorting we obtain

$$0 \le (t+2)(2j-1)(1-d_k) + t2^k - 2\lfloor t2^{k-1}\rfloor < 2j(t+2) - t. \tag{7}$$

Since $0 \le t2^k - 2\lfloor t2^{k-1} \rfloor = d_{k+1} + t2^k - \lfloor t2^k \rfloor = (d_{k+1}.d_{k+2}d_{k+3}\ldots)_2 < 2$, the inequalities given in (7) hold true for all $k \ge 1$. The proof of Theorem 1.2, Case II is done. It is not difficult to see that $\varepsilon = 1/2$ cannot be replaced by any other value.

## 2.2 Proof of Theorem 1.3

Here we prove

$$\begin{aligned} u_{2k} &= (g^{k-1} - 1)/(g-1), \\ u_{2k+1} &= g^k + \lfloor tg^{k-1} \rfloor. \end{aligned}$$

Similarly as before, the statement of Theorem 1.3 is then obtained from Proposition 2. Again, $u_1 = g^0 + \lfloor t/g \rfloor = 1$. Suppose first, the result holds for $u_{2k}$. Then

$$u_{2k+1} = \left\lfloor b\left(u_{2k} + \frac{1}{g-1}\right)\right\rfloor = \lfloor (t+g)(g^{k-1}-1) + (t+g)\rfloor = g^k + \lfloor tg^{k-1}\rfloor.$$

Vice versa, assume the result holds for $u_{2k+1}$. Let $\{x\}$ denote the fractional part of $x \in \mathbb{R}_{>0}$. Then

$$\begin{aligned} u_{2k+2} &= \lfloor a(u_{2k+1} + \varepsilon)\rfloor = \lfloor a \lfloor g^{k-1}(t+g)\rfloor + a\varepsilon\rfloor \\ &= \left\lfloor a\left\lfloor \frac{g^k}{a(g-1)}\right\rfloor + a\varepsilon\right\rfloor = \frac{g^k-1}{g-1} + \left\lfloor \frac{1}{g-1} - a\left\{\frac{g^k}{a(g-1)}\right\} + a\varepsilon\right\rfloor. \end{aligned}$$

Since $0 < a \le g/(g^2-1)$, we have $1/(g-1) - a \ge a/g$. Thus, for $\varepsilon \ge -1/g$ we get

$$\frac{1}{g-1} - a\left\{\frac{g^k}{a(g-1)}\right\} + a\varepsilon > \frac{1}{g-1} - a + a\varepsilon \ge \frac{a}{g} - \frac{a}{g} = 0.$$

On the other hand, if $\varepsilon < (g+1)(g-2)/g$ then

$$\frac{1}{g-1} - a\left\{\frac{g^k}{a(g-1)}\right\} + a\varepsilon \le \frac{1}{g-1} + a\varepsilon < \frac{1}{g-1} + \frac{g}{g^2-1}\cdot\frac{(g+1)(g-2)}{g} = 1.$$

This finishes the proof of Theorem 1.3.


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(Concerned with sequences A001521, A091522 and A091523.)

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