Journal of Integer Sequences, Vol. 8 (2005),
Article 05.3.3

Prime and Composite Terms in Sloane's Sequence A056542

Tom Müller
Institute for Cusanus-Research
University and Theological Faculty of Trier
Domfreihof 3
54290 Trier
Germany
mue14503@uni-trier.de

Abstract

We give the complete factorization of the first fifty terms of the sequence $a_n := na_{n-1} + 1$ with $a_1 := 0$. We searched the terms a_n for primes up to $n = 1019$ with the result that only the indexes 4, 8, 18, 20 and 70 provide primes. A final section deals with some conjectures on prime terms in this sequence.

1 Introduction

Consider the sequence of integers defined by $a_n := na_{n-1} + 1$ with $a_1 := 0$. This sequence is A056542 in Neil Sloane's Online Encyclopedia of Integer Sequences [4]. In this paper we give the complete factorization of the first fifty terms and deal with the question what terms are prime numbers for $n < 1020$. To do this we need two equivalent expressions for the terms a_n.

Using a simple induction argument we can show that

$$a_n = \sum_{\nu=2}^{n} \frac{n!}{\nu!}. \tag{1}$$

The study of the convergence of the exponential series (which is, among other things, used to prove the irrationality of e) allows us moreover to write

$$a_n = \lfloor n!(e-2) \rfloor = \lfloor n!e \rfloor - 2n!. \tag{2}$$

Using the first expression it is easy to prove a criterion for the compositeness of certain terms.

2 Composite terms

From the definition of the sequence it follows immediately that for odd $n > 1$ the term a_n is even and greater than 3 and hence composite. So only even indexes can provide potential prime numbers. Expression (1) implies the following result.

Theorem 2.1 *Let p be a prime number and n be a natural number with $p \leq n$. If p is a divisor of a_n then p divides a_{n+p} as well.*

Proof: Writing

$$a_{n+p} = \sum_{\nu=2}^{p-1} \frac{(n+p)!}{\nu!} + \sum_{\nu=p}^{n+p} \frac{(n+p)!}{\nu!}, \tag{3}$$

we see that the first sum on the right side contains the factor p. Concerning the second sum, we obtain (e.g., by induction on n)

$$\sum_{\nu=p}^{n+p} \frac{(n+p)!}{\nu!} = \left(\sum_{\nu=0}^{n} \frac{n!}{\nu!} \right) + pk \tag{4}$$

where k is a natural number. This is equivalent to

$$a_{n+p} = a_n + 2n! + pK$$

with another natural number K absorbing k and the respective factor of the first sum in (3). Hence a_{n+p} is a multiple of p if a_n is. □

One consequence of the theorem is the fact that for a given prime p it is enough to search among the terms $a_p, a_{p+1}, \ldots, a_{2p-1}$ for multiples of p. Numerical computations lead to the results presented in Table 1.

Furthermore, we can now use the theorem to sift the prime candidates and retain more than half of them as composites. The a_n's that could not be identified being the product of two nontrivial factors with our sieve method were tested for compositeness by trying to find a small factor with a standard factorization algorithm, which again eliminated about half the terms. Each remaining term was subjected to the Fermat test

$$2^{a_n - 1} \equiv 1 \bmod a_n. \tag{5}$$

It turned out that in all the cases, exept for $n \in \{4, 8, 18, 20, 70\}$, the congruence failed, and hence the respective a_n was composite. We studied the indexes $n < 1020$. Testing one term took an average CPU time of about one hour.

p	n with $p \mid a_n$	p	n with $p \mid a_n$
2	3	113	163
5	7, 9	127	135
13	17, 23, 25	131	153, 164
19	25, 33	137	144, 153
23	37, 41	149	218
29	52	163	168
31	45, 50	167	242
37	53, 71, 73	173	336
41	78	179	351
43	62	181	276
59	100	193	250, 255
71	81, 120	197	228, 297
83	157	211	269, 357, 391, 415
97	123	223	363
103	109	227	411, 436
107	158	239	418
109	196, 213	251	279

Table 1: Prime numbers p dividing the term a_n

3 Prime terms

In the previous section we pointed out that only the terms

$$\begin{aligned} a_4 &= 17 \\ a_8 &= 28961 \\ a_{18} &= 4598708691828421 \\ a_{20} &= 17475093028948000001 \\ a_{70} &= 8603990361433692835766763032506384134769654780784715495311087517908153547994512075361554378508046501 \end{aligned}$$

remain potential prime numbers. For $n = 4, 8, 18, 20$ it is simple to prove their primality, for example by trial division up to the respective square roots. For a_{70}, which is a 100-digit number, primality can be established using two theorems of Brillhart, Lehmer and Selfridge [1].

Theorem 3.1 *Let $N - 1 = \prod p_i^{\alpha_i}$ be the complete factorization of the integer $N - 1$. If for each prime factor p_i there exists an a_i such that N fulfills $a_i^{N-1} \equiv 1 \pmod{N}$, but $a_i^{(N-1)/p_i} \not\equiv 1 \pmod{N}$, then N is prime.*

Theorem 3.2 *Let $N + 1 = \prod p_i^{\alpha_i}$ be the complete factorization of the integer $N + 1$. Let $\mathcal{U}$ be the set of Lucas sequences $\{U_k^{(i)}\}$ with the given discriminant D for which the Jacobi symbol $(D/N) = -1$. If for each p_i there exists a Lucas sequence in $\mathcal{U}$ such that N is a divisor of $U_{N+1}^{(i)}$, but not a divisor of $U_{(N+1)/p_i}^{(i)}$, then N is prime.*

Remark: Let P and Q be integers with the discriminant $D := p^2 - 4Q \neq 0$. Then the Lucas sequences are defined recursively by

$$U_{k+2} = PU_{k+1} - QU_k, \qquad k \geq 0, \quad U_0 = 0, \quad U_1 = 1, \tag{6}$$

and

$$V_{k+2} = PV_{k+1} - QV_k, \qquad k \geq 0, \quad V_0 = 2, \quad V_1 = P. \tag{7}$$

A simple method to compute large terms of a given Lucas sequence is presented in the paper of Brillhart et al. ([1, pp. 627–628]) as well.

With this we can settle the primality of a_{70}.

Theorem 3.3 *The number a_{70} is prime.*

Proof: We have

$$a_{70} + 1 = 2 \cdot 3 \cdot 11 \cdot 471193 \cdot p_{93},$$

where

$$p_{93} = 276666865434658243552094076373979681579672293479713404939810982616341330249301822966628239979$$

is a 93-digit probable prime. If we assume that p_{93} is prime, the claim follows with theorem 3.2 by setting $D = -3$. Then the divisibility criteria hold for the factor 2 with $P = Q = 1$, for the factor 3 with $P = Q = 3$ and for the other three factors with $P = 5$ and $Q = 7$.

So, we need the primality of p_{93}. To reach this goal, we consider the factorization

$$p_{93} - 1 = 2 \cdot 3 \cdot 167 \cdot 263 \cdot 457 \cdot 8377 \cdot p_{81}$$

with a 81-digit probable prime

$$p_{81} = 274238840573141405175902476234412618955309935656050257207045603051300234586527927.$$

Again, if we assume that p_{81} is prime the primality of p_{93} follows with $a_i = 3$ for all $i = 1, \ldots, 7$ in theorem 3.1.

The next step consists in proving the primality of p_{81}. For this, we apply theorem 3.1 again with the base $a_i = 3$ to

$$p_{81} - 1 = 2 \cdot q_{81}$$

where

$$q_{81} = 137119420286570702587951238117206309477654967828025128603522801525650117293263963$$

is another 81-digit probable prime. A final step shows the primality of q_{81}. This follows with theorem 3.2 and

$$\begin{aligned} q_{81} + 1 &= 2^2 \cdot 3 \cdot 13 \cdot 29 \cdot 4793 \cdot 15227 \cdot 20422008121 \cdot \\ &\cdot 6882703159854339959 \cdot 28276698587131486301 \cdot \\ &\cdot 104488479363890191610 9. \end{aligned}$$

These final factors are small enough to be easily proved prime. Here again a discriminant $D = -3$ gives the wanted results with $P = 5$ and $Q = 7$ for all ten prime factors. $\square$

The results of this section are summarized in Sloane's OEIS A070213 [4].

Remark: The sequence can be generalized to $b_n := nb_{n-1} + 1$ by defining another initial term $b_1 := b \in \mathbb{N}$. Then we can establish the relation

$$b_n = bn! + a_n.$$

Therefore the sieve method will work for the generalized case too. How many primes are there in the respective sequences?

4 Complete factorization of the first fifty terms

The following factorization results were obtained using Lenstra's elliptic curve method [3]. The factors found were furthermore tested for their primality again using trial division up to the square root. All computations in this sections were realized in almost 100 hours of CPU time on two Pentium-I computers. The results can be found in Tables 3 and 4.

5 Conclusions and conjectures

Using a heuristic argument similar to those used, e.g., by Hardy and Wright ([2, p. 15]) for Fermat numbers, or by Wagstaff [5] for Mersenne numbers, we can formulate some conjectures about the asymptotic behavior of prime numbers in sequence A056542.

In order to do this, we extended Table 1 up to the first 10,000 primes by steps of 1000 primes, noting only the number $N(p)$ of n in the interval $1 < n < p$, resp. $p \leq n < 2p - 1$, for which p divides a_n and computed the averages

$$\sigma(x) = \frac{\sum_{p \leq x} N(p)}{2\pi(x)}$$

with these data. Here $\pi(x) = \#\{p \in \mathbb{P} : p \leq x\}$ denotes the prime number counting function.

These results, which are given in Table 2, suggest that the average number of such n taken over all primes p less than a given x is approximately equal to 1, which might indicate that $\sigma(x) \to 1$ for $x \to \infty$.

$\pi(x) =$	1000	2000	3000	4000	5000
$1 < n < p$	1.019	1.024	1.006	1.005	1.000
$p \leq n < 2p$	0.973	1.001	1.005	1.003	1.010
$\sigma(x) =$	0.996	1.013	1.006	1.004	1.005
$\pi(x) =$	6000	7000	8000	9000	10000
$1 < n < p$	0.998	0.998	0.990	0.994	0.986
$p \leq n < 2p$	1.004	1.004	1.008	1.005	0.999
$\sigma(x) =$	1.001	1.001	0.999	1.000	0.993

Table 2: The averages of terms a_n that are divides by primes p (rounded)

If we assume that this is the case, this would say that, on average, a prime p divides a_n (for large n) with probability $\frac{1}{p}$. But this would mean that the terms of our sequence behave just like "typical" numbers of the same size, and hence are just as likely to be prime. So, according to the famous Prime Number Theorem, the probability for any given a_n to be prime, would be $\frac{1}{\ln(a_n)}$. Applying the well known Stirling formula to expression (2), we get

$$\ln(a_n) \approx \ln(n!) \approx A \cdot n \ln(n)$$

with a constant $A < 1$. If this all is true, there are two consequences worth mentioning. On the one hand, the number $P(x)$ of prime terms a_n with indices $n \leq x$ would be

$$P(x) \approx A^{-1} \sum_{n=2}^{\lfloor x \rfloor} \frac{1}{n \ln(n)}. \tag{8}$$

Hence, as the series in (8) diverges for $x \to \infty$, there would be infinitely many prime terms in sequence A056542.

On the other hand, having a glance at the known prime terms suggests that $A^{-1} \approx 2.3$. Using this parameter, solving the equation $P(x) = 6$ would predict a sixth prime term for an index $n \approx 780$. A seventh prime term is expected to accure with the index $n \approx 33700$. So, there should be a sixth prime in the interval $1020 \leq n < 33700$ (and more likely it should be found short after the index 1020), if we guessed right. Nevertheless, prime terms a_n look like being very rare after all.

Acknowledgements

The author is most grateful to the friendly referee who pointed out a mistake in the manuscript and made various most valuable suggestions to improve the paper.

References

[1] J. Brillhart, D. H. Lehmer and J. L. Selfridge, New primality criteria and factorizations of $2^m \pm 1$, *Math. Comp.* **29** (1975), 620–647.

[2] G. H. Hardy and E. L. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1979

[3] H. W. Lenstra, Jr., Factoring integers with elliptic curves, *Ann. Math.* **126** (1987), 649–673.

[4] Online Encyclopedia of Integer Sequences (OEIS), electronically published at: http://www.research.att.com/~njas/sequences/

[5] S.S. Wagstaff, Divisors of Mersenne numbers, *Math. Comp.* **40** (1983), 385–397.

2000 *Mathematics Subject Classification*: Primary 11Y05; Secondary 11Y55.
Keywords: factorization, integer sequence, prime number

(Concerned with sequences A056542 and A070213.)

Received November 2 2004; revised version received June 24 2005. Published in *Journal of Integer Sequences*, July 6 2005.

Return to Journal of Integer Sequences home page.

n	factor(a_n)
2	1
3	2^2
4	17
5	$2 \cdot 43$
6	$11 \cdot 47$
7	$2^2 \cdot 5 \cdot 181$
8	28961
9	$2 \cdot 5^2 \cdot 13 \cdot 401$
10	$67 \cdot 38903$
11	$2^3 \cdot 3583939$
12	$5 \cdot 68811629$
13	$2 \cdot 2236377943$
14	$5 \cdot 661 \cdot 2243 \cdot 8447$
15	$2^2 \cdot 234819684019$
16	$349 \cdot 8627 \cdot 4991479$
17	$2 \cdot 5 \cdot 13 \cdot 281 \cdot 6993808273$
18	4598708691828421
19	$2^5 \cdot 5^4 \cdot 1579 \cdot 12409 \cdot 222967$
20	1747509302894800001
21	$2 \cdot 107 \cdot 347 \cdot 461 \cdot 1071999585919$
22	$5 \cdot 48259 \cdot 3345901481329483$
23	$2^2 \cdot 13 \cdot 910619 \cdot 392147324904187$
24	$5 \cdot 1481 \cdot 41813 \cdot 287851 \cdot 5000304083$
25	$2 \cdot 13 \cdot 19 \cdot 103 \cdot 257 \cdot 852007193230945949$
26	$183644977 \cdot 1577374632467830901$
27	$2^3 \cdot 5 \cdot 1955319264674583248614731 37$
28	$119551 \cdot 373670483 \cdot 4902230134153477$
29	$2 \cdot 5 \cdot 635087697166304639150064748979$
30	$13 \cdot 127 \cdot 13511681593 \cdot 8540798558040523807$

Table 3: Factorization of a_n for $n \leq 30$

n	factor(a_n)
31	$2^2 \cdot 6793 \cdot 6623384304659 \cdot 32818224968844709$
32	$5^2 \cdot 157 \cdot 329993 \cdot 14592249210358062651130875 7$
33	$2 \cdot 19 \cdot 37 \cdot 1549 \cdot 2863807538994480264037164291179$
34	$5 \cdot 330679 \cdot 1341707 \cdot 2303179 \cdot 41504719765549572671$
35	$2^4 \cdot 1572242101 \cdot 2950449079340654083764855207 11$
36	$13 \cdot 179 \cdot 3389 \cdot 75793 \cdot 2334397613 \cdot 1914953129484444196 31$
37	$2 \cdot 5 \cdot 23 \cdot 1407281 \cdot 30543801770391751614358741298340689$
38	$13 \cdot 1307 \cdot 221103903242508396230663171400290389 49371$
39	$2^2 \cdot 5 \cdot 47 \cdot 373 \cdot 1913 \cdot 73181 \cdot 46498981 \cdot$ $\cdot 6419283046588931936893783$
40	$283 \cdot 2207 \cdot 452087 \cdot 27443371 \cdot 10961524829 \cdot$ $\cdot 6899549447021827237$
41	$2 \cdot 23 \cdot 67273 \cdot 855946423 \cdot 9984786337 \cdot$ $\cdot 9085311459561757173511 49$
42	$5 \cdot 51787 \cdot 59539 \cdot 1306501657 \cdot$ $\cdot 501038197445525902445707090742 33$
43	$2^3 \cdot 13 \cdot 674672980073 \cdot 167351313672951 7 \cdot$ $\cdot 3695606886962813288444 39$
44	$5^2 \cdot 19 \cdot 487 \cdot 833893 \cdot$ $\cdot 98983163056214222454453077616272192106892 01$
45	$2 \cdot 31 \cdot 503621 \cdot 5349857 \cdot$ $\cdot 5143622015274473746178862922882692448103 09$
46	$613981 \cdot 10339597 \cdot$ $\cdot 622595573454047871500475953343208441017551021$
47	$2^2 \cdot 5 \cdot 1740373 \cdot 25196940506279 \cdot$ $\cdot 211807964674353752193208733585193464069$
48	$262066528371751897 \cdot 22100157339031818463$ $\cdot 1539560451909744956702071$
49	$2 \cdot 5 \cdot 13 \cdot 36948815219 \cdot 5938235313533 \cdot$ $\cdot 153178733685070253638169472754447884 47$
50	$31 \cdot 293 \cdot 206506391500393 \cdot 31126232036943157 \cdot$ $\cdot 374179843710832651624027922347$

Table 4: Factorization of a_n for $30 < n \leq 50$