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A New Domino Tiling Sequence

Roberto Tauraso
Dipartimento di Matematica
Università di Roma "Tor Vergata"
via della Ricerca Scientifica
00133 Roma, Italy
tauraso@mat.uniroma2.it

Abstract

In this short note, we prove that the sequence A061646 from Neil Sloane's Online Encyclopedia of Integer Sequences is connected with the number of domino tilings of a holey square.

1 Introduction

J. Bao discusses in [1] the following conjecture due to Edward Early (see also [2]): the number of tilings of a *holey square*, that is a $2n \times 2n$ square with a hole of size $2m \times 2m$ removed from the center, is $2^{n-m} \cdot k^2$ where k is an odd number which depends on n and m. The conjecture has been proven for $m = 1, 2, \ldots, 6$ and numerical experiments indicate that it holds for several other values of m. Here we are going to show that the conjecture is verified also for $m = n - 2$ and, perhaps what is more surprising, that the corresponding odd numbers k_n give a sequence already contained in Sloane's Encyclopedia [3], namely A061646.

Before proving this result we supply some definitions and notations. The Fibonacci numbers A000045 are defined as the sequence of integers

$$f_0 = 0, f_1 = 1, \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

We will need Cassini's formula, one of the most famous Fibonacci identities:

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n \text{ for all } n \geq 1. \tag{1}$$

The terms of the sequence A061646 satisfy the recurrence relation given by

$$a_0 = 1, a_1 = 1, a_2 = 3, \text{ and } a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3} \text{ for all } n \geq 3,$$

and the corresponding generating function is

$$A(x) = \frac{1 - x - x^2}{(1 + x)(1 - 3x + x^2)}.$$

The number a_n can be expressed in terms of the Fibonacci numbers, as follows:

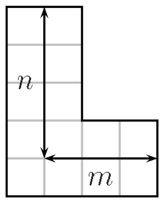
$$a_n = f_{n+1}f_{n-1} + f_n^2 = 2f_n^2 + (-1)^n \text{ for all } n \geq 1.$$

The second formula is obtained from the first by using (1) and it says that a_n is always an odd number. The first 20 values of the sequence a_n compared with the Fibonacci numbers are as follows:

n	f_n	a_n
0	0	1
1	1	1
2	1	3
3	2	7
4	3	19
5	5	49
6	8	129
7	13	337
8	21	883
9	34	2311
10	55	6051
11	89	15841
12	144	41473
13	233	108577
14	377	284259
15	610	744199
16	987	1948339
17	1597	5100817
18	2584	13354113
19	4181	34961521

2 The Result

Lemma 2.1. *For all $n, m \geq 1$ the number of domino tilings of the L-grid*



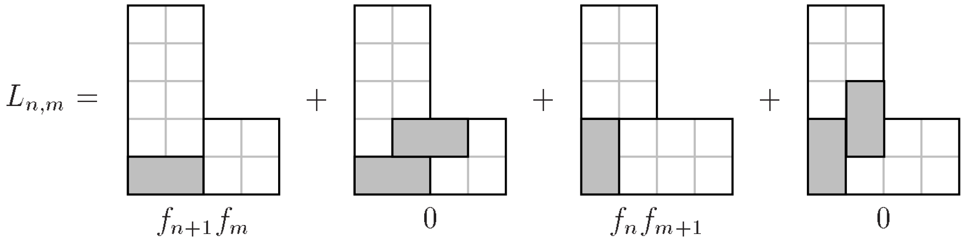
is

$$L_{n,m} = f_{n+1} f_m + f_n f_{m+1}. \tag{2}$$

Moreover, $L_{n,n-1} = a_n$ *for all* $n \geq 2$ *and*

$$L^2_{n,n-1} = L_{n,n} L_{n-1,n-1} + 1. \tag{3}$$

Proof. In order to prove (2), we start by covering the lower left corner. We can put there either a horizontal domino or a vertical one and then we can either cut along its smaller side or not. Then we get the following four different configurations:



where the number of domino tilings of each configuration has been calculated by recalling that the number of ways to tile a $2 \times n$ rectangle is equal to f_{n+1}.

By (2), the identity (3) is equivalent to

$$(f_{n+1} f_{n-1} + f_n^2)^2 = (2 f_{n+1} f_n)(2 f_{n-1} f_n) + 1 = 4 f_{n+1} f_{n-1} f_n^2 + 1,$$

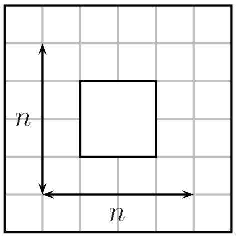
that is

$$(f_{n+1} f_{n-1} - f_n^2)^2 = 1$$

which holds by Cassini's formula (1). □

Finally we show that, as expected by the previous conjecture, the number of domino tilings of the considered holey square is the product of 2^2 by the square of an odd number. This odd number belongs to the sequence A061646.

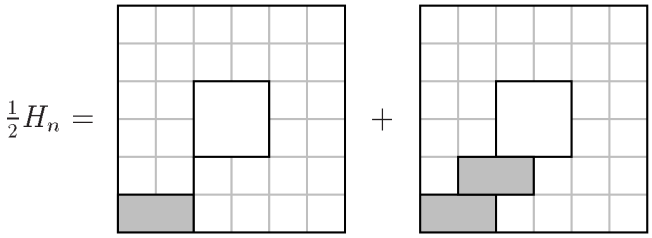
Theorem 2.2. *For all* $n \geq 3$ *the number of domino tilings of the holey square*



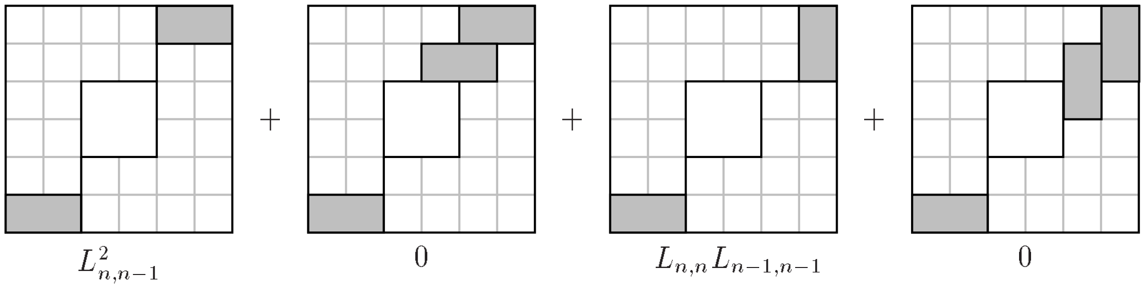
is

$$H_n = 4 L^2_{n,n-1} = 4 a_n^2. \tag{4}$$

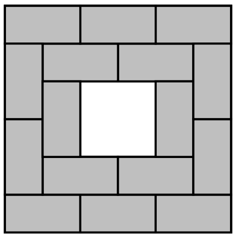
Proof. By symmetry, the tilings which have a horizontal domino in the lower left corner are one half of the total number H_n:



The first tiling can be continued by covering the upper right corner in these four ways:



On the other hand the second tiling can be completed in only one way:



Therefore

$$\frac{1}{2}H_n = L^2_{n,n-1} + L_{n,n}L_{n-1,n-1} + 1$$

and by (3) we obtain

$$\frac{1}{2}H_n = 2L^2_{n,n-1} = 2a_n^2$$

which gives us (4). $\square$

References

[1] J. Bao, On the number of domino tilings of the rectangular grid, the holey square, and related problems, Final Report, Research Summer Institute at MIT, (1997).

[2] L. Pachter, Combinatorial approaches and conjectures for 2-divisibility problems concerning domino tilings of polyominoes, *Electron. J. Comb.*, **4** (1997), 401–410.

[3] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Published electronically at http://www.research.att.com/~njas/sequences/.

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(Concerned with sequences A000045 and A061646.)

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