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The Number of Labelled k-Arch Graphs

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Abstract

In this note, we deal with k-arch graphs, a generalization of trees, which contain k-trees as a subclass. We show that the number of vertex-labelled k-arch graphs with n vertices, for a fixed integer $k \geq 1$, is $\binom{n}{k}^{n-k-1}$. As far as we know, this is a new integer sequence. We establish this result with a one-to-one correspondence relating k-arch graphs and words whose letters are k-subsets of the vertex set. This bijection generalises the well-known Prüfer code for trees. We also recover Cayley's formula n^{n-2} that counts the number of labelled trees.

1 Introduction

We recursively define the class of *k-arch graphs*, for $k \geq 1$, as the smallest class of simple graphs such that:

1. a $(k-1)$-simplex (i.e., a complete graph on k vertices) is a k-arch graph;
2. if a simple graph G has a vertex v of degree k such that the graph $G - \{v\}$ obtained from G by removing v and its incident edges is a k-arch graph, then G is a k-arch graph.

Figure 1 shows an unlabelled 2-arch graph (we simply say *arch graph* in this case) and a vertex-labelled 2-arch graph, each one built over 12 vertices. Note that, when $k = 1$, 1-arch graphs coincide with (Cayley) trees. In a constructive way, to build a k-arch graph of $n+1$ vertices from one on n vertices, we have to choose k vertices and join the new vertex to these selected vertices. The term *arch* evokes attaching the new vertex over the k chosen vertices.

There are few papers about k-arch graphs, apart from [18], where these graphs seem to appear, and [11]. However, there is an abundant literature about a subclass of k-arch

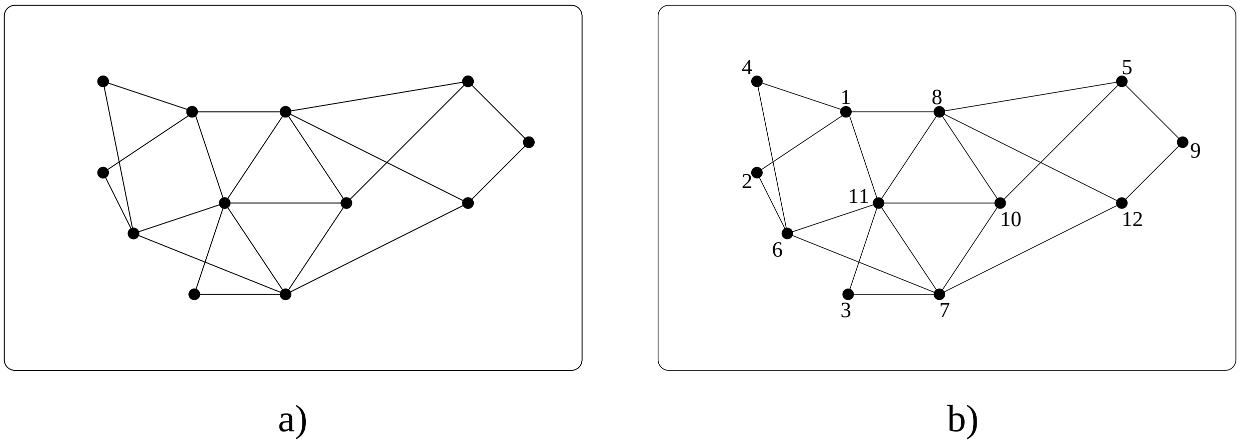

Figure 1: Arch graphs on 12 vertices a) unlabelled, b) vertex-labelled.

graphs, called k-trees, studied since the later 1960's. The essential difference between k-trees and k-arch graphs lie in the fact that for k-trees, we assume that the vertex v is attached to k mutually adjacent vertices (that is, it forms a complete graph on $k+1$ vertices). For instance, see [4, 13] for the labelled enumeration of k-trees, and [3, 14] for the particular case $k = 2$. Harary and Palmer [7] treat the unlabelled enumeration as well as Fowler et al. [6], who provide in addition asymptotic formulas. We also mention [8, 5] for the enumeration of generalizations of 2-trees and [10], where various specializations of 2-trees are enumerated. Finally, Labelle et al. [9] propose a classification of outerplanar 2-trees according to their symmetries. Although k-trees have been extensively studied, unlabelled enumeration of these structures is still an open problem. Apart from enumerative aspects of 2-trees, Leclerc and Makarenkov [12] give a correspondence between tree functions and 2-trees in the framework of tree metrics and tree dissimilarities (see [1, 2]).

In [18, 11], k-arch graphs are defined as maximal kd-acyclic graphs, where a graph is said kd-acyclic, if it contains no kd-cycle (see [11], definition 3.1). Todd shows that this definition is equivalent to the recursive one given above. Leclerc uses valuated arch graphs (that is, arch graph with valuated edges) to encode tree distances or tree functions (see [1, 2]).

We recall that the number of labelled k-trees over n vertices is given by ([4, 13])

$$a_n^k = \binom{n}{k}\left(k(n-k)+1\right)^{n-k-2}. \tag{1}$$

When $k = 1$, we recover the well-known Cayley formula $a_n = n^{n-2}$ counting labelled (Cayley) trees.

The aim of this note is to obtain the number of labelled k-arch graphs. We show this number is

$$\binom{n}{k}^{n-k-1}.$$

To achieve our goal, we propose in Section 2 a one-to-one correspondence between k-arch graphs on n vertices and words of length $n-k-1$ whose letters are k-subsets of the set of vertex labels. This correspondence generalizes the Prüfer code for trees ([15]).

2 The number of labelled k-arch graphs

In this section, we establish a formula giving the number of labelled k-arch graphs, for $k \geq 1$. We prove this formula using a bijective argument based on a generalization of the Prüfer code for vertex-labelled Cayley trees ([15]) which has been generalized for k-trees ([16]). Note that formula (2) first appear (without proof) in the conclusion of [10].

We call *leaf* of a k-arch graph, a vertex of degree k. For instance, in Figure 1 b), there are four leaves, respectively labelled 2,3,4,9. This definition of leaf is legitimate since, for the special case $k = 1$, a vertex of degree one in a Cayley tree is a leaf, in the common sense of graph theory.

Proposition 1 *Let $k \geq 1$ be a fixed integer. Then, the number $\mathcal{G}_n^k$ of k-arch graphs over n labelled vertices, for $n > k$, is given by*

$$\mathcal{G}_n^k = \binom{n}{k}^{n-k-1} \quad \text{and} \quad \mathcal{G}_k^k = 1. \tag{2}$$

Proof. We construct a one-to-one correspondence between k-arch graphs on n labelled vertices and words $w = w_1 w_2 \ldots w_{n-k-1}$ of length $n-k-1$ where each w_i is a (valid) k-letter of the following form:

$$\begin{pmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_k} \end{pmatrix} \tag{3}$$

such that

1. for all $1 \leq j \leq k$, $1 \leq i_j \leq n$;
2. for all $1 \leq j \leq n$, v_j is a vertex of the k-arch graph;
3. $i_1 < i_2 < \cdots < i_k$.

Such words are called *valid*. We implicitly assume that an order is given on the labels, i.e., $v_1 < v_2 < \ldots < v_n$. The one-to-one correspondence works by successive leaf deletions. Let g be a k-arch graph on n vertices with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. At the first step, we choose the leaf with the smallest label in g, we remove it from g as well as its incident edges and form a k-letter with its adjacent vertices by ordering them in increasing order. The fact that the degree of a leaf is k and the vertex ordering ensure we can always form a unique valid k-letter. After the first leaf deletion, we obtain a k-arch graph with $n-1$ vertices and we repeat the first step. Repeating this step $n-k-1$ times, we get a valid word of length $n-k-1$. Observe that after the last step, the k-arch graph g becomes a single $k-1$-simplex, that is, a complete graph on k vertices. For instance, if we apply this construction to the arch graph of Figure 1 b), we obtain the following valid word of length 9:

$$\begin{pmatrix} 1 & 7 & 1 & 8 & 7 & 5 & 8 & 8 & 10 \\ 6 & 11 & 6 & 11 & 11 & 12 & 10 & 7 & 11 \end{pmatrix}. \tag{4}$$

Conversely, given a valid word $w = w_1 w_2 \dots w_{n-k-1}$ of length $n-k-1$, we want to construct a k-arch graph with n labelled vertices. Together with w, we use a dynamic subset L of the vertex set V, that is, a subset with evolving entries and size. At the beginning, we fill L with all vertices not appearing in w and we extend w by appending a copy of the last k-letter, w_{n-k-1}, at the end of w. We also denote w this extended word

$$w := w || w_{n-k-1} = w_1 w_2 \dots w_{n-k-1} w_{n-k-1}.$$

For instance, according to the labelled k-arch graph of Figure 1 b), we obtain $L = \{2, 3, 4, 9\}$ and w becomes

$$\begin{pmatrix} 1 & 7 & 1 & 8 & 7 & 5 & 8 & 8 & 10 & 10 \\ 6 & 11 & 6 & 11 & 11 & 12 & 10 & 7 & 11 & 11 \end{pmatrix}.$$

We remove the smallest element of L and join it to every entry of the first k-letter (w_1) of w. We then delete w_1 from w and still call w this reduced word. We update L by adding every vertex $v_i \in w_1$ not appearing in the remaining letters of w. We repeat the above recursive step with the word $w = w_2 \dots w_{n-k-1} w_{n-k-1}$, and so on until we reach the empty word, keeping at each step the connected component created. To end the converse map, we have to connect together the k vertices of the last k-letter w_{n-k-1} of w.

We have to verify that we obtain a connected graph which is a k-arch graph corresponding to the word w when the leaf deletion algorithm is applied. It is quite straightforward using a recursive argument, as the reader can check. ■

Figure 2 shows a complete example of the reverse map for a labelled arch graph of six vertices.

Remark 1 *It is interesting to note that, for $k = 1$, formula (2) remains valid. Indeed, this formula becomes Cayley formula ($a_n = n^{n-2}$) enumerating trees with n labelled vertices.*

Remark 2 *The converse map constructed in the proof of Proposition 1 induces easily an algorithm of random generation of labelled k-arch graphs by means of valid words.*

3 Conclusion

The following table gives the first few values of the number of k-arch graphs over n labelled vertices, for k from 1 up to 7, and for n from k up to 10. Only the first line of this table (corresponding to Cayley formula $a_n = n^{n-2}$) is listed in the on-line Encyclopedia of Integer Sequences [17] (sequence A000272).

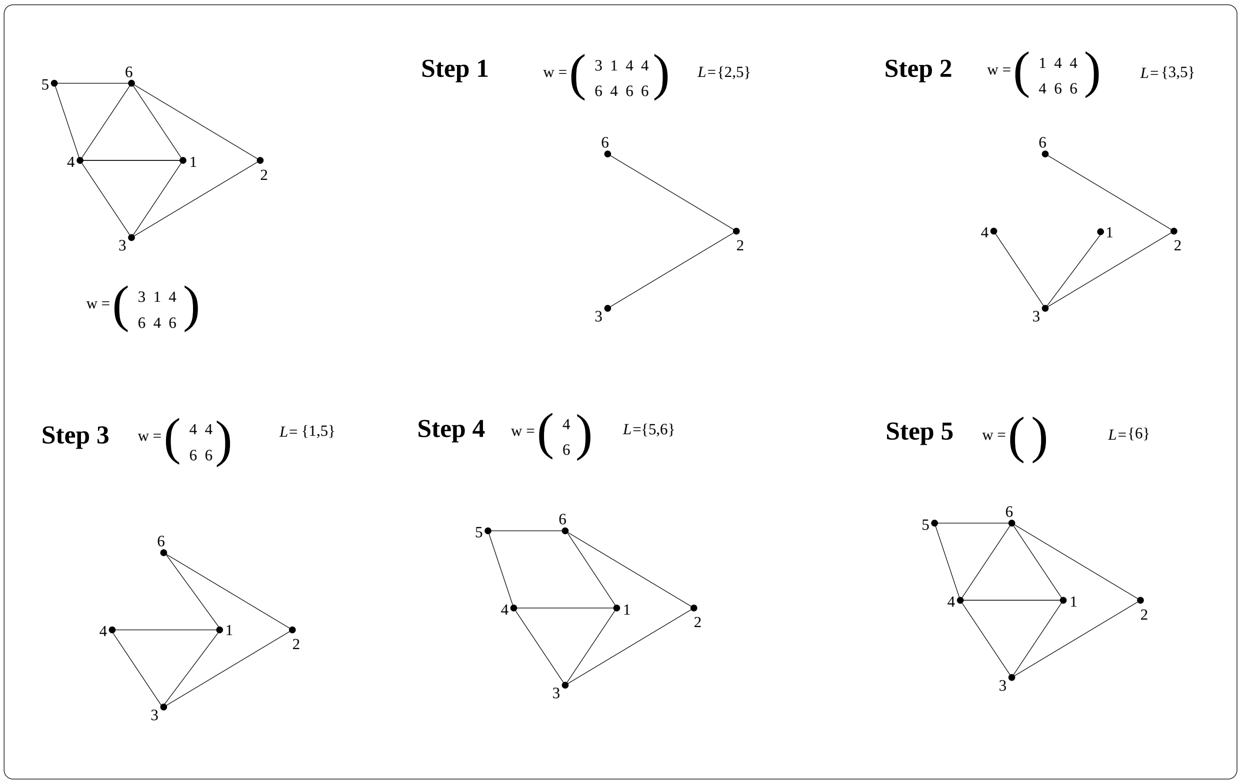

Figure 2: Illustration of the reverse map.

In this note, we showed by a bijective way that the number of labelled k-arch graphs is given by a new integer sequence. The class of k-arch graphs are a natural multidimensional generalization of trees, encompassing k-trees. To go further in the study of k-arch graphs, we have to wonder about the unlabelled enumeration of these graphs. Unfortunately, no result is known about the unlabelled enumeration of both k-arch graphs and k-trees.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	1	3	16	125	1296	16807	262144	4782969	100000000
2	-	1	1	6	100	3375	194481	17210368	2176782336	373669453125
3	-	-	1	1	10	400	42875	9834496	4182119424	2985984000000
4	-	-	-	1	1	15	1225	343000	252047376	408410100000
5	-	-	-	-	1	1	21	3136	2000376	4032758016
6	-	-	-	-	-	1	1	28	7056	9261000
7	-	-	-	-	-	-	1	1	36	14400

Table 1: Values of $\mathcal{G}_n^k$, for $1 \leq k \leq 7$ and $k \leq n \leq 10$.

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