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How to Differentiate a Number

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Abstract. We define the derivative of an integer to be the map sending every prime to 1 and satisfying the Leibnitz rule. The aim of the article is to consider the basic properties of this map and to show how to generalize the notion to the case of rational and arbitrary real numbers. We make some conjectures and find some connections with Goldbach's Conjecture and the Twin Prime Conjecture. Finally, we solve the easiest associated differential equations and calculate the generating function.

1 A derivative of a natural number

Let n be a positive integer. We would like to define a derivative n' such that $(n, n') = 1$ if and only if n is square-free (as is the case for polynomials). It would be nice to preserve some natural properties, for example $(n^k)' = kn^{k-1}n'$. Because $1^2 = 1$ we should have $1' = 0$ and $n' = (1 + 1 \cdots + 1)' = 0$, if we want to preserve linearity. But if we ignore linearity and use the Leibnitz rule only, we will find that it is sufficient to define p' for primes p. Let us try to define n' by using two natural rules:

- $p' = 1$ for any prime p,

- $(ab)' = a'b + ab'$ for any $a, b \in \mathbf{N}$ (Leibnitz rule).

For instance,

$$6' = (2 \cdot 3)' = 2' \cdot 3 + 2 \cdot 3' = 1 \cdot 3 + 2 \cdot 1 = 5.$$

Here is a list of the first 18 positive integers and their first, second and third derivatives:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
n'	0	1	1	4	1	5	1	12	6	7	1	16	1	9	8	32	1	21
n''	0	0	0	4	0	1	0	16	5	1	0	32	0	6	12	80	0	10
n'''	0	0	0	4	0	0	0	32	1	0	0	80	0	5	16	176	0	7

It looks quite unusual but first of all we need to check that our definition makes sense and is well-defined.

Theorem 1 *The derivative n' can be well-defined as follows: if $n = \prod_{i=1}^{k} p_i^{n_i}$ is a factorization in prime powers, then*

$$n' = n \sum_{i=1}^{k} \frac{n_i}{p_i}. \tag{1}$$

It is the only way to define n' that satisfies desired properties.

Proof. Because $1' = (1 \cdot 1)' = 1' \cdot 1 + 1 \cdot 1' = 2 \cdot 1'$, we have only one choice for $1'$: it should be zero. Induction and Leibnitz rule show that if the derivative is well-defined, it is uniquely determined. It remains to check that the equation (1) is consistent with our conditions. It is evident for primes and clear that (1) can be used even when some n_i are equal to zero. Let $a = \prod_{i=1}^{k} p_i^{a_i}$ and $b = \prod_{i=1}^{k} p_i^{b_i}$. Then according to (1) the Leibnitz rule looks as

$$ab \sum_{i=1}^{k} \frac{a_i + b_i}{p_i} = \left(a \sum_{i=1}^{k} \frac{a_i}{p_i} \right) b + a \left(b \sum_{i=1}^{k} \frac{b_i}{p_i} \right)$$

and the consistency is clear. ∎

For example

$$(60)' = (2^2 \cdot 3 \cdot 5)' = 60 \cdot \left(\frac{2}{2} + \frac{1}{3} + \frac{1}{5} \right) = 60 + 20 + 12 = 92.$$

We can extend our definition to $0' = 0$, and it is easy to check that this does not contradict the Leibnitz rule.

Note that linearity does not hold in general; for many a, b we have $(a+b)' \neq a' + b'$. Furthermore $(ab)'' \neq a'' + 2a'b' + b''$ because we need linearity to prove this. It would be interesting to describe all the pairs (a, b) that solve the differential equation $(a+b)' = a' + b'$. We can find one of the solutions, $(4, 8)$ in our table above. This solution can be obtained from the solution $(1, 2)$ by using the following result.

Theorem 2 *If* $(a+b)' = a' + b'$*, then for any natural* k*, we have*

$$(ka+kb)' = (ka)' + (kb)'.$$

The same holds for the inequalities

$$(a+b)' \geq a' + b' \Rightarrow (ka+kb)' \geq (ka)' + (kb)',$$

$$(a+b)' \leq a' + b' \Rightarrow (ka+kb)' \leq (ka)' + (kb)'.$$

Moreover, all these can be extended for linear combinations, for example:

$$(\sum \gamma_i a_i)' = \sum \gamma_i (a_i)' \Rightarrow (k \sum \gamma_i a_i)' = \sum \gamma_i (ka_i)'.$$

Proof. The proof is the same for all the cases, so it is sufficient to consider only one of them, for example the case $\geq$ with two summands:

$$(ka+kb)' = (k(a+b))' = k'(a+b) + k(a+b)' =$$

$$k'a + k'b + k(a+b)' \geq k'a + k'b + ka' + kb' = (ka)' + (kb)'.$$

■

Corollary 1

$$(3k)' = k' + (2k)'; (2k)' \geq 2k'; (5k)' \leq (2k)' + (3k)'; (5k)' = (2k)' + 3(k)'.$$

Proof.

$$3' = 1' + 2'; 2' \geq 1' + 1'; 5' \leq 2' + 3'; 5' = 2' + 3 \cdot 1'.$$

■

Here is the list of all (a,b) with $a < b \leq 100, \gcd(a,b) = 1$, for which $(a+b)' = a' + b'$:

$$(1,2), (4,35), (4,91), (8,85), (11,14), (18,67), (26,29),$$

$$(27,55), (35,81), (38,47), (38,83), (50,79), (62,83), (95,99).$$

A similar result is

Theorem 3 *For any natural* $k > 1$,

$$n' \geq n \Rightarrow (kn)' > kn.$$

Proof.

$$(kn)' = k'n + kn' > kn' \geq kn.$$

■

The following theorem shows that every $n > 4$ that is divisible by 4 satisfies the condition $n' > n$.

Theorem 4 *If $n = p^p \cdot m$ for some prime p and natural $m > 1$, then $n' = p^p(m + m')$ and* $\lim_{k\to\infty} n^{(k)} = \infty$.

Proof. According to the Leibnitz rule and (1), $n' = (p^p)' \cdot m + p^p \cdot m' = p^p(m + m') > n$ and by induction $n^{(k)} \geq n + k$. ∎

The situation changes when the exponent of p is less than p.

Theorem 5 *Let p^k be the highest power of prime p that divides the natural number n. If $0 < k < p$, then p^{k-1} is the highest power of p that divides n'. In particular, all the numbers $n, n', n'', \ldots, n^{(k)}$ are distinct.*

Proof. Let $n = p^k m$. Then $n' = kp^{k-1}m + p^k m' = p^{k-1}(km + pm')$, and the expression inside parentheses is not divisible by p. ∎

Corollary 2 *A positive integer n is square-free if and only if $(n, n') = 1$.*

Proof. If $p^2|n$, then $p|n'$ and $(n, n') > 1$. On the other hand, if $p|n$ and $p|n'$ then $p^2|n$. ∎

2 The equation $n' = n$

Let us solve some differential equations (using our definition of derivative) in positive integers.

Theorem 6 *The equation $n' = n$ holds if and only $n = p^p$, where p is any prime number. In particular, it has infinitely many solutions in natural numbers.*

Proof. If prime p divides n, then according to Theorem 5, at least p^p should divide n or else $n' \neq n$. But Theorem 4 implies that in this case $n = p^p$, which according to (1) is evidently equal to n'. ∎

Thus, considering the map $n \longrightarrow n'$ as a dynamical system, we have a quite interesting object. Namely, we have infinitely many fixed points, 0 is a natural attractor, because all the primes after two differentiations become zero. Now it is time to formulate the first open problem.

Conjecture 1 *There exist infinitely many composite numbers n such that $n^{(k)} = 0$ for sufficiently large natural k.*

As we will see later, the Twin Prime Conjecture would fail if this conjecture is false. Preliminary numerical experiments show that for non-fixed points either the derivatives $n^{(k)}$ tend to infinity or become zero; however, we do not know how to prove this.

Conjecture 2 *Exactly one of the following could happen: either $n^{(k)} = 0$ for sufficiently large k, or $\lim_{k\to\infty} n^{(k)} = \infty$, or $n = p^p$ for some prime p.*

According to Theorem 4, it is sufficient to prove that, for some k, the derivative $n^{(k)}$ is divisible by p^p (for example by 4). In particularly we do not expect periodic point except fixed points p^p.

Conjecture 3 *The differential equation $n^{(k)} = n$ has only trivial solutions p^p for primes p.*

Theorem 5 gives some restrictions for possible nontrivial periods: if p^k divides n the period must be at least $k+1$.

Conjecture 3 is not trivial even in special cases. Suppose, for example, that n has period 2, i.e. $m = n' \neq n$ and $m' = n$. According to Theorem 4 and Theorem 5, both n and m should be the product of distinct primes: $n = \prod_{i=1}^{k} p_i$, $m = \prod_{j=1}^{l} q_j$, where all primes p_i are distinct from all q_j. Therefore, our conjecture in this case is equivalent to the following:

Conjecture 4 *For any positive integers k, l, the equation*

$$\left(\sum_{i=1}^{k} \frac{1}{p_i}\right)\left(\sum_{j=1}^{l} \frac{1}{q_j}\right) = 1$$

has no solutions in distinct primes.

3 The equation $n' = a$

We start with two easy equations.

Theorem 7 *The differential equation $n' = 0$ has only one positive integer solution $n = 1$.*

Proof. Follows immediately from (1). ■

Theorem 8 *The differential equation $n' = 1$ in natural numbers has only primes as solutions.*

Proof. If the number is composite then according to Leibnitz rule and the previous theorem, the derivative can be written as the sum of two positive integers and is greater than 1. ■

All other equations $n' = a$ have only finitely many solutions, if any.

Theorem 9 *([1]) For any positive integer n*

$$n' \leq \frac{n \log_2 n}{2}. \tag{2}$$

If n is not a prime, then

$$n' \geq 2\sqrt{n}. \tag{3}$$

More generally, if n is a product of k factors larger than 1, then

$$n' \geq k n^{\frac{k-1}{k}}. \tag{4}$$

Proof. If $n = \prod_{i=1}^{k} p_i^{n_i}$, then

$$n \geq \prod_{i=1}^{k} 2^{n_i} \Rightarrow \log_2 n \geq \sum_{i=1}^{k} n_i.$$

According to (1) we now have

$$n' = n \sum_{i=1}^{k} \frac{n_i}{p_i} \leq \frac{n \sum_{i=1}^{k} n_i}{2} \leq \frac{n \log_2 n}{2}.$$

If $n = n_1 n_2 n_3 \cdots n_k$ then, according to the Leibnitz rule,

$$n' = n_1' n_2 n_3 \cdots n_k + n_1 n_2' n_3 \cdots n_k + n_1 n_2 n_3' \cdots n_k + \ldots + n_1 n_2 n_3 \cdots n_k' \geq$$

$$n_2 n_3 n_4 \cdots n_k + n_1 n_3 n_4 \cdots n_k + n_1 n_2 n_4 \cdots n_k + \ldots + n_1 n_2 \cdots n_{k-1} =$$

$$n\left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}\right) \geq n \cdot k \left(\frac{1}{n_1} \cdot \frac{1}{n_2} \cdots \frac{1}{n_k}\right)^{\frac{1}{k}} = k \cdot n \cdot n^{\frac{-1}{k}} = k \cdot n^{\frac{k-1}{k}}.$$

Here we have replaced the arithmetic mean by the geometric mean. ∎

Note that bounds (2) and (4) are exact for $n = 2^k$.

Corollary 3 *If the differential equation $n' = a$ has any solution in natural numbers, then it has only finitely many solutions if $a > 1$.*

Proof. The number n cannot be a prime. According to (3) the solutions must be no greater than $\frac{a^2}{4}$. ∎

What about the existence of solutions? We start with the even numbers.

Conjecture 5 *The differential equation $n' = 2b$ has a positive integer solution for any natural number $b > 1$.*

A motivation for this is the famous

Conjecture 6 (Goldbach Conjecture) *Every even number larger than 3 is a sum of two primes.*

So, if $2b = p + q$, then $n = pq$ is a solution that we need. Inequality (3) helps us easy to prove that the equation $n' = 2$ has no solutions. What about odd numbers larger than 1? It is easy to check with the help of (3) that the equation $n' = 3$ has no solutions. For $a = 5$ we have one solution and more general have a theorem:

Theorem 10 *Let p be a prime and $a = p + 2$. Then $2p$ is a solution for the equation $n' = a$.*

Proof. $(2p)' = 2'p + 2p' = p + 2$. ∎

Some other primes also can be obtained as a derivative of a natural number (e.g. 7), but it is more interesting which of numbers cannot. Here is a list of all $a \leq 1000$ for which the equation $n' = a$ has no solutions (obtained using Maple and (3)):

$$2, 3, 11, 17, 23, 29, 35, 37, 47, 53, 57, 65, 67, 79, 83, 89, 93, 97, 107, 117, 125, 127,$$

$$137, 145, 149, 157, 163, 173, 177, 179, 189, 197, 205, 207, 209, 217, 219, 223, 233,$$

$$237, 245, 257, 261, 277, 289, 303, 305, 307, 317, 323, 325, 337, 345, 353, 367, 373,$$

$$377, 379, 387, 389, 393, 397, 409, 413, 415, 427, 429, 443, 449, 453, 457, 473, 477,$$

$$485, 497, 499, 509, 513, 515, 517, 529, 531, 533, 537, 547, 553, 561, 569, 577, 593,$$

$$597, 605, 613, 625, 629, 639, 657, 659, 665, 673, 677, 681, 683, 697, 699, 709, 713,$$

$$715, 733, 747, 749, 757, 765, 769, 777, 781, 783, 785, 787, 793, 797, 805, 809, 817,$$

$$819, 827, 833, 835, 845, 847, 849, 853, 857, 869, 873, 877, 881, 891, 895, 897, 907,$$

$$917, 925, 933, 937, 947, 953, 963, 965, 967, 981, 989, 997.$$

Note that a large portion of them (69 from 153) are primes, one of them ($529 = 23^2$) is a square, and some of them (e.g. $765 = 3^2 \cdot 5 \cdot 17$) have at least 4 prime factors. In general it is interesting to investigate the behavior of the "integrating" function $I(a)$ which calculates for every a the set of solutions of the equation $n' = a$ and its weaker variant $i(a)$ that calculates the number of such solutions. As we have seen above $I(0) = \{0, 1\}$, $I(1)$ consist of all primes and $i(2) = i(3) = i(11) = \cdots = i(997) = 0$. Here is a list of the those numbers $a \leq 100$ that have more than one "integral" (i.e. $i(a) \geq 2$). For example 10 has two "integrals" (namely $I(10) = \{21, 25\}$) and 100 has six ($I(100) = \{291, 979, 1411, 2059, 2419, 2491\}$).

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[10, 2], [12, 2], [14, 2], [16, 3], [18, 2], [20, 2],
[21, 2], [22, 3], [24, 4], [26, 3], [28, 2], [30, 3],
[31, 2], [32, 4], [34, 4], [36, 4], [38, 2], [39, 2],
[40, 3], [42, 4], [44, 4], [45, 2], [46, 4], [48, 6],
[50, 4], [52, 3], [54, 5], [55, 2], [56, 4], [58, 4],
[60, 7], [61, 2], [62, 3], [64, 5], [66, 6], [68, 3],
[70, 5], [71, 2], [72, 7], [74, 5], [75, 3], [76, 5],
[78, 7], [80, 6], [81, 2], [82, 5], [84, 8], [86, 5],
[87, 2], [88, 4], [90, 9], [91, 3], [92, 6], [94, 5],
[96, 8], [98, 3], [100, 6].
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Note that only three of them are primes. To complete the picture it remains to list the set of those $a <= 100$ for which $i(a) = 1$.

$$4, 5, 6, 7, 8, 9, 13, 15, 19, 25, 27, 33, 41,$$

$$43, 49, 51, 59, 63, 69, 73, 77, 85, 95, 99.$$

Theorem 11 *The function $i(n)$ is unbounded for $n > 1$.*

Proof. Suppose that $i(n) < C$ for all $n > 1$ for some constant C. Then

$$\sum_{k=2}^{2n} i(k) < 2Cn$$

for any n. But for any two primes p, q the product pq belongs to $I(p+q)$ thus

$$\sum_{k=2}^{2n} i(k) > \sum_{p \le q \le n}{}' 1 = \frac{\pi(n)(\pi(n)+1)}{2} > \frac{\pi(n)^2}{2},$$

where $\sum'$ means that the sum runs over the primes, and $\pi(n)$ is the number of primes not exceeding n. This leads to the inequality

$$2Cn > \frac{\pi(n)^2}{2} \Rightarrow \pi(n) < 2\sqrt{Cn},$$

which contradicts the known asymptotic behavior $\pi(n) \approx \frac{n}{\ln n}$. ■

It would be interesting to prove a stronger result.

Conjecture 7 *For any nonnegative m there exists infinitely many a such that $i(a) = m$.*

Another related conjecture is the following:

Conjecture 8 *There exists an infinite sequence a_n of different natural numbers such that $a_1 = 1, (a_n)' = a_{n-1}$ for $n = 2, 3 \ldots$*

Here is an example of possible beginning of such a sequence:

$$1 \leftarrow 7 \leftarrow 10 \leftarrow 25 \leftarrow 46 \leftarrow 129 \leftarrow 170 \leftarrow 501 \leftarrow 414 \leftarrow 2045.$$

The following table shows the maximum of $i(n)$ depending of the number m of (not necessary different) prime factors in the factorization of n for $n \le 1000$.

m	1	2	3	4	5	6	7	8	9
$i(n)$	8	22	35	46	52	52	40	47	32

The next more detailed picture shows the distribution of $i(n)$ depending of the number m for $i(n) < 33$. Note that maximum possible $i(n)$ is equal 52, so we have only part of a possible table. We leave to the reader the pleasure of making some natural conjectures.

$i(n)\backslash m$	1	2	3	4	5	6	7	8	9
0	69	49	28	6	1	0	0	0	0
1	46	89	35	8	3	1	0	0	0
2	25	44	18	7	1	0	0	0	0
3	13	16	17	7	0	0	0	0	0
4	9	12	8	5	2	0	1	0	0
5	2	6	3	4	0	1	0	0	0
6	1	7	8	1	2	0	0	0	0
7	1	10	4	3	2	1	0	0	0
8	2	3	8	3	2	2	0	1	0
9	0	8	6	7	4	0	0	0	0
10	0	3	7	5	1	1	0	0	0
11	0	8	13	2	1	2	0	0	0
12	0	4	4	5	2	0	1	0	1
13	0	3	10	5	2	2	1	0	0
14	0	7	7	5	4	1	1	0	0
15	0	8	8	3	3	1	0	0	0
16	0	1	15	6	5	1	0	0	0
17	0	10	4	8	2	0	0	0	0
18	0	3	4	5	2	1	1	0	0
19	0	4	5	9	4	2	1	1	0
20	0	3	7	1	0	1	0	1	0
21	0	0	5	2	4	3	0	1	0
22	0	1	2	5	1	0	1	0	0
23	0	0	4	1	1	1	2	0	0
24	0	0	1	6	3	1	0	0	0
25	0	0	3	2	1	1	0	0	0
26	0	0	1	2	4	1	0	1	0
27	0	0	2	1	2	1	0	0	0
28	0	0	1	1	0	1	1	0	0
29	0	0	2	2	1	0	1	0	0
30	0	0	1	1	1	0	0	0	0
31	0	0	1	4	3	0	0	0	0
32	0	0	0	6	1	1	1	0	1

4 The equation $n'' = 1$

The main conjecture for the second-order equations is the following:

Conjecture 9 *The differential equation $n'' = 1$ has infinitely many solutions in natural numbers.*

Theorem 10 shows that $2p$ is a solution if $p, p+2$ are primes. So the following famous conjecture would be sufficient to prove.

Conjecture 10 **(prime twins)** *There exists infinitely many pairs $p, p+2$ of prime numbers.*

The following problem is another alternative which would be sufficient:

Conjecture 11 **(prime triples)** *There exists infinitely many triples p, q, r of prime numbers such that $P = pq + pr + qr$ is a prime.*

Such a triple gives a solution $n = pqr$ to our equation, because $n' = P$. In reality all the solutions can be described as follows.

Theorem 12 *A number n is a solution of the differential equation $n'' = 1$ if and only if the three following conditions are valid:*

1. *The number n is a product of different primes: $n = \prod_{i=1}^{k} p_i$.*
2. *$\sum_{i=1}^{k} 1/p_i = \frac{p}{n}$, where p is a prime.*
3. *If k is even, then the smallest prime of p_i should be equal to 2.*

Proof. If $n = p^2 m$ for some prime p then $n' = p(2m + pm')$ is not prime and according to Theorem 8 the number n cannot be a solution. So, it is a product of different primes. Then the second condition means that n' is a prime and by Theorem 8 it is necessary and sufficient to be a solution. As to the number k of factors it cannot be even if all primes p_i are odd, because n' in this case is (as the sum of k odd numbers) even and larger than two. ■

5 Derivative for integers

It is time to extend our definition to integers.

Theorem 13 *A derivative is uniquely defined over the integers by the rule*

$$(-x)' = -x'.$$

Proof. Because $(-1)^2 = 1$ we should have (according to the Leibnitz rule) $2(-1)' = 0$ and $(-1)' = 0$ is the only choice. After that $(-x)' = ((-1) \cdot x)' = 0 \cdot x' + (-1) \cdot x' = -x'$ is the only choice for negative $-x$ and as a result is true for positive integers also. It remains to check that the Leibnitz rule is still valid. It is sufficient to check that it is valid for $-a$ and b if it was valid for a and b. It follows directly:

$$((-a) \cdot b)' = -(a \cdot b)' = -(a' \cdot b + a \cdot b') = -a' \cdot b + (-a) \cdot b' = (-a)' \cdot b + (-a) \cdot b'.$$

■

6 Derivative for rational numbers

The next step is to differentiate a rational number. We start from the positive rationals. The shortest way is to use (1). Namely, if $x = \prod_{i=1}^{k} p_i^{x_i}$ is a a factorization of a rational number x in prime powers, (where some x_i may be negative) then we put

$$x' = x \sum_{i=1}^{k} \frac{x_i}{p_i} \tag{5}$$

and the same proof as in Theorem 1 shows that this definition is still consistent with the Leibniz rule.

Here is a table of derivatives of i/j for small i, j.

i/j	1	2	3	4	5	6	7	8	9	10
1	0	$\frac{-1}{4}$	$\frac{-1}{9}$	$\frac{-1}{4}$	$\frac{-1}{25}$	$\frac{-5}{36}$	$\frac{-1}{49}$	$\frac{-3}{16}$	$\frac{-2}{27}$	$\frac{-7}{100}$
2	1	0	$\frac{1}{9}$	$\frac{-1}{4}$	$\frac{3}{25}$	$\frac{-1}{9}$	$\frac{5}{49}$	$\frac{-1}{4}$	$\frac{-1}{27}$	$\frac{-1}{25}$
3	1	$\frac{-1}{4}$	0	$\frac{-1}{2}$	$\frac{2}{25}$	$\frac{-1}{4}$	$\frac{4}{49}$	$\frac{-7}{16}$	$\frac{-1}{9}$	$\frac{-11}{100}$
4	4	1	$\frac{8}{9}$	0	$\frac{16}{25}$	$\frac{1}{9}$	$\frac{24}{49}$	$\frac{-1}{4}$	$\frac{4}{27}$	$\frac{3}{25}$
5	1	$\frac{-3}{4}$	$\frac{-2}{9}$	-1	0	$\frac{-19}{36}$	$\frac{2}{49}$	$\frac{-13}{16}$	$\frac{-7}{27}$	$\frac{-1}{4}$
6	5	1	1	$\frac{-1}{4}$	$\frac{19}{25}$	0	$\frac{29}{49}$	$\frac{-1}{2}$	$\frac{1}{9}$	$\frac{2}{25}$
7	1	$\frac{-5}{4}$	$\frac{-4}{9}$	$\frac{-3}{2}$	$\frac{-2}{25}$	$\frac{-29}{36}$	0	$\frac{-19}{16}$	$\frac{-11}{27}$	$\frac{-39}{100}$
8	12	4	$\frac{28}{9}$	1	$\frac{52}{25}$	$\frac{8}{9}$	$\frac{76}{49}$	0	$\frac{20}{27}$	$\frac{16}{25}$
9	6	$\frac{3}{4}$	1	$\frac{-3}{4}$	$\frac{21}{25}$	$\frac{-1}{4}$	$\frac{33}{49}$	$\frac{-15}{16}$	0	$\frac{-3}{100}$
10	7	1	$\frac{11}{9}$	$\frac{-3}{4}$	1	$\frac{-2}{9}$	$\frac{39}{49}$	-1	$\frac{1}{27}$	0

A natural property is the following:

Theorem 14 *For any two rationals a, b we have*

$$\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2}.$$

A derivative can be well defined for rational numbers using this formula and this is the only way to define a derivative over rationals that preserves the Leibniz rule.

Proof. If $a = \prod_{i=1}^{k} p_i^{a_i}, b = \prod_{i=1}^{k} p_i^{a_i}$ then we have

$$\left(\frac{a}{b}\right)' = (\prod_{i=1}^{k} p_i^{a_i-b_i})' = (\prod_{i=1}^{k} p_i^{a_i-b_i}) \sum_{i=1}^{k} \frac{a_i - b_i}{p_i} =$$

$$\left(\frac{a}{b}\right) \sum_{i=1}^{k} \frac{a_i}{p_i} - \left(\frac{ab}{b^2}\right) \sum_{i=1}^{k} \frac{b_i}{p_i} = \frac{a'}{b} - \frac{ab'}{b^2} = \frac{a'b - ab'}{b^2}.$$

Let us check uniqueness. If n is an integer then $n \cdot \frac{1}{n} = 1$ and the Leibnitz rule demands

$$n' \cdot \frac{1}{n} + n \left(\frac{1}{n}\right)' = 0 \Rightarrow \left(\frac{1}{n}\right)' = -\frac{n'}{n^2}.$$

After that

$$\left(\frac{a}{b}\right)' = \left(a \cdot \frac{1}{b}\right)' = a' \cdot \frac{1}{b} + a \cdot \left(\frac{1}{b}\right)' = \frac{a'}{b} - a \cdot \left(\frac{b'}{b^2}\right) = \frac{a'b - ab'}{b^2}$$

is the only choice that satisfies the Leibnitz rule. This proves uniqueness. To prove that such a definition is well-defined, it is sufficient to see that

$$\left(\frac{ac}{bc}\right)' = \frac{(ac)'(bc) - (ac)(bc)'}{(bc)^2} = \frac{(a'c + ac')(bc) - (ac)(b'c + bc')}{b^2c^2} =$$

$$\frac{(a'bc^2 + abc'c) - (ab'c^2 + abcc')}{b^2c^2} = \frac{a'b - ab'}{b^2}$$

has the same value. ■

For negative rationals we can proceed as above and put $(-x)' = -x'$.

7 Rational solutions of the equation $x' = a$.

Unexpectedly the equation $x' = 0$ has nontrivial rational solutions, for instance $x = 4/27$. We can describe all of them.

Theorem 15 *Let k be some natural number, $\{p_i, i = 1, \ldots k\}$ be a set of different prime numbers and $\{\alpha_i, i = 1, \ldots k\}$ be a set of integers such that $\sum_{i=1}^{k} \alpha_i = 0$. Then*

$$x = \pm \prod_{i=1}^{k} p_i^{\alpha_i p_i}$$

are solutions of the differential equation $x' = 0$ and any other nonzero solution can be obtained in this manner.

Proof. Because $(-x)' = -x'$ it is sufficient to consider positive solutions only. Let $x = \prod_{i=1}^{k} p_i^{a_i}$ Then from (5)

$$\sum_{i=1}^{k} \frac{a_i}{p_i} = 0 \Rightarrow \sum_{i=1}^{k} a_i \cdot Q_i = 0,$$

where $Q_i = \left(\prod_{j=1}^{k} p_j\right)/p_i$ is not divisible by p_i. Thus a_i should be divisible by p_i and $\alpha_i = \frac{a_i}{p_i}$. ∎

Other equations are more difficult.

Conjecture 12 *The equation $x' = 1$ has only primes as positive rational solutions.*

Note that there exists a negative solution, namely $x = -\frac{5}{4}$. One possible solution of this equation would be $x = \frac{n}{p^p}$ for some natural n and prime p. Because $x' = \frac{n'-n}{p^p}$ in this case we can reformulate the conjecture as

Conjecture 13 *Let p be a prime. The equation $n' = n + p^p$ has no natural solutions except $n = qp^p$, where q is a prime.*

Note, that according to Theorem 5 if a solution n is divisible by p it should be divisible by p^p. Therefore $n = mp^p$ and $p^p(m' + m) = p^p(m + 1)$ by Theorem 4 and m should be a prime. Thus it is sufficient to prove that any solution is divisible by p.

We do not expect that it is possible to integrate every rational number, though we do not know a counterexample.

Conjecture 14 *There exists a such that the equation $x' = a$ has no rational solutions.*

The first natural candidates do not verify the conjecture:

$$(-\frac{21}{16})' = 2; (-\frac{13}{4})' = 3; (-\frac{22}{27})' = \frac{1}{3}.$$

8 Logarithmic derivative

One thing that is still absent in our picture is the analogue of the logarithm – the primitive of $\frac{1}{n}$. Because our derivative is not linear we cannot expect that the logarithm of the product is equal to the sum of logarithms. Instead this is true for its derivative. So let us define a logarithmic derivative $\mathrm{ld}(x)$ as follows. If $x = \prod_{i=1}^{k} p_i^{x_i}$ for different primes p_i and some integers x_i, then

$$\mathrm{ld}(x) = \sum_{i=1}^{k} \frac{x_i}{p_i},\ \mathrm{ld}(-x) = \mathrm{ld}(x),\ \mathrm{ld}(0) = \infty.$$

In other words

$$\mathrm{ld}(x) = \frac{x'}{x}.$$

Theorem 16 *For any rational numbers*

$$ld(xy) = ld(x) + ld(y).$$

Proof.
$$\mathrm{ld}(xy) = \frac{(xy)'}{xy} = \frac{x'y + xy'}{xy} = \frac{x'}{x} + \frac{y'}{y} = \mathrm{ld}(x) + \mathrm{ld}(y).$$
■

It is useful to divide every integer number into large and small parts. Let $\mathrm{sign}(x)x = |x| = \prod_{i=1}^{k} p_i^{x_i}$ and $x_i = a_i p_i + r_i$, where $0 \le r_i < p_i$. We define

$$P(x) = \mathrm{sign}(x) \prod_{i=1}^{k} p_i^{a_i p_i}, R(x) = \prod_{i=1}^{k} p_i^{r_i}, A(x) = \sum_{i=1}^{k} a_i.$$

Theorem 17 *The following properties hold*

- $ld(x) = A(x) + ld(R(x))$.
- $x' = A(x)x + P(x)(R(x))' = x(A(x) + ld(R(x)))$.
- *If x is a nonzero integer, then*
$$x|x' \Leftrightarrow ld(x) \in \mathbf{Z} \Leftrightarrow R(x) = 1.$$
- *if $\left(\frac{a}{b}\right)'$ is an integer, and* $\gcd(a, b) = 1$ *then* $R(b) = 1$.

Proof. First we have

$$\mathrm{ld}(x) = \mathrm{ld}(P(x)R(x)) = \mathrm{ld}(P(x)) + \mathrm{ld}(R(x)) = A(x) + \mathrm{ld}(R(x)).$$

Using this we get

$$x' = x\mathrm{ld}(x) = x(A(x) + \mathrm{ld}(R(x))) = xA(x) + x\mathrm{ld}(R(x)) =$$
$$xA(x) + P(x)R(x)\mathrm{ld}(R(x)) = A(x)x + P(x)(R(x))'.$$

If $R(x) \neq 1$ then the sum
$$\mathrm{ld}(R(x)) = \sum_{i=1}^{k} \frac{r_i}{p_i}$$
cannot be an integer. Otherwise
$$\mathrm{ld}(R(x)) \prod_{i=1}^{k} p_i = \sum_{i=1}^{k} r_i Q_i,$$
and if $0 < r_j < p_j$ then an integer on the left hand side is divisible by p_j, but on the right hand side is not because $Q_j = \frac{\prod_{i=1}^{k} p_i}{p_j}$ and all primes p_i are different. The last statement follows from Theorem 14. ■

Now we are able to solve the equation $x' = \alpha x$ with rational α in the rationals. We have already solved this equation in the case $\alpha = 0$, so let $\alpha \neq 0$.

Theorem 18 *Let $\alpha = \frac{a}{b}$ be a rational number with $\gcd(a, b) = 1, b > 0$. Then*

- *The equation*
$$x' = \alpha x \tag{6}$$
has nonzero rational solutions if and only if b is a product of different primes or $b = 1$.

- *If x_0 is a nonzero particular solution (6) and y is any rational solution of the equation $y' = 0$ then $x = x_0 y$ is also a solution of (6) and any solution of (6) can be obtained in this manner.*

- *To obtain a particular solution of the equation (6) it is sufficient to decompose α into the elementary fractions:*
$$\alpha = \frac{a}{b} = \lfloor \alpha \rfloor + \sum_{i=1}^{k} \frac{c_i}{p_i},$$
where $b = \prod_{i=1}^{k} p_i, 1 \leq |c_i| < p_i$. Then
$$x_0 = 4^{\lfloor \alpha \rfloor} \prod_{i=1}^{k} p_i^{c_i}$$
is a particular solution. (Of course the number 4 can be replaced by p^p for any prime p).

Proof. The equation (6) is equivalent to the equation
$$\mathrm{ld}(x) = \alpha \Leftrightarrow A(x) + \mathrm{ld}(R(x)) = \alpha = \frac{a}{b}.$$
Because $A(x)$ is an integer and $\mathrm{ld}(R(x)) = \sum_{i=1}^{k} \frac{r_i}{p_i}$, the natural number b should be equal to the product of the different primes or should be equal to 1. Suppose that b is of this type. Then
$$\mathrm{ld}\left(4^{\lfloor \alpha \rfloor} \prod_{i=1}^{k} p_i^{c_i}\right) = \lfloor \alpha \rfloor + \sum_{i=1}^{k} \frac{c_i}{p_i} = \alpha$$
and we obtain a desired particular solution. If $y' = 0$ and x_0 any particular solution then
$$(x_0 y)' = x_0' y + x_0 y' = \alpha x_0 y,$$
also satisfies (6). Finally, if $x' = \alpha x$ and $y = \frac{x}{x_0}$ then
$$\mathrm{ld}(y) = \mathrm{ld}(x) - \mathrm{ld}(x_0) = 0$$
means that y is a solution of the equation $y' = 0$. ∎

For instance the equation $x' = \frac{x}{4}$ has no solutions, $x_0 = \frac{2}{3}$ is a partial solution of the equation $x' = \frac{x}{6}$ and to obtain all nonzero solutions we need to multiply x_0 with any y such that $R(y) = 1, A(y) = 0$.

9 How to differentiate irrational numbers

The next step is to try to generalize our definition to irrational numbers. The equation (1) can still be used in the more general situation. But first we need to think about the correctness of the definition.

Lemma 1 *Let $\{p_1, \ldots, p_k\}$ be a set of different primes and $\{x_1, \ldots, x_k\}$ a set of rationals. Then*

$$P = \prod_{i=1}^{k} p_i^{x_i} = 1 \Leftrightarrow x_1 = x_2 = \cdots = x_k = 0.$$

Proof. It is evident if all x_i are integers, because the primes are different. If they are rational, let us choose a natural m such that all $y_i = mx_i$ are integer. Then $P^m = 1$ too and we get $y_i = 0 \Rightarrow x_i = 0$. ∎

Now we can extend our definition to any real number x that can be written as a product $x = \prod_{i=1}^{k} p_i^{x_i}$ for different primes p_i and some nonzero rationals x_i. The previous lemma shows that this form is unique and as above we can define

$$x' = x \sum_{i=1}^{k} \frac{x_i}{p_i}.$$

The proof for the Leibnitz rule is still valid too and we skip it. For example we have

$$(\sqrt{3})' = (3^{1/2})' = 3^{1/2}\frac{1/2}{3} = \frac{\sqrt{3}}{6}.$$

More generally we have the following convenient formula:

Theorem 19 *Let x, y be rationals and x be positive. Then*

$$(x^y)' = yx^{y-1}x' = \frac{yx'}{x}x^y = yx^y ld(x). \tag{7}$$

Proof. If $x = \prod_{i=1}^{k} p_i^{x_i}$, then

$$(x^y)' = (\prod_{i=1}^{k} p_i^{yx_i})' = x^y \sum_{i=1}^{k} \frac{yx_i}{p_i} = yx^{y-1}x \sum_{i=1}^{k} \frac{x_i}{p_i} = yx^{y-1}x'.$$

∎

An interesting corollary is

Corollary 4 *Let a, b, c, d be rationals such that $a^b = c^d$ (a, c being positive). Then*

$$b \cdot ld(a) = d \cdot ld(c)$$

and

$$a'bc = c'ad.$$

In particular, for the case $a = b, c = d$, we have

$$a^a = c^c \Rightarrow a' = c'.$$

■

As an example we can check directly that $x^y = y^x$ has the solutions

$$x = \left(\frac{m+1}{m}\right)^m ; y = \left(\frac{m+1}{m}\right)^{m+1},$$

thus

$$\frac{x'}{x^2} = \frac{y'}{y^2},$$

so the equation $x' = \frac{x^2}{4}$ has at least two solutions obtained from $m = 1$.

Another example is

$$(1/2)^{1/2} = (1/4)^{1/4} \Rightarrow \left(\frac{1}{2}\right)' = \left(\frac{1}{4}\right)' = -\left(\frac{1}{4}\right).$$

In general it is not difficult to prove that all rational solutions of the equation $x^x = y^y$ have the form

$$x = \left(\frac{m}{m+1}\right)^m, y = \left(\frac{m}{m+1}\right)^{m+1}$$

for some natural m. Direct calculations give the same result as above:

$$x' = m\left(\frac{m}{m+1}\right)^{m-1}\left(\frac{m}{m+1}\right)' = (m+1)\left(\frac{m}{m+1}\right)^m\left(\frac{m}{m+1}\right)' = y'$$

and shows that this works even for rational m.

It would be natural to extend our definition to infinite products: if $x = \prod_{i=1}^{\infty} p_i^{x_i}$ is convergent then it is easy to show that the sum $x\sum_{i=1}^{\infty}\frac{x_i}{p_i}$ is also convergent. However, the problem is that the sum is not necessary convergent to zero, when $x = 1$. This is a reason why such a natural generalization of the derivative is not well-defined. Maybe a more natural approach is to restrict possible products, but we still do not know a nice solution of the problems that arise. But there is another way, which we consider in Section 11.

10 Arithmetic Derivative for UFD

The definition of the derivative and most of the proofs are based only on the fact that that every natural number has a unique factorization into primes. So it is not difficult to transfer it to an arbitrary UFD (unique factorization domain) R using the same definition: $p' = 1$ for every "canonical" prime (irreducible) element, the Leibnitz rule and additionally $u' = 0$ for all units (invertible elements) in R. For example we can do it for a polynomial ring $K[x]$ or for the Gaussian numbers $a + bi$. In the first case the canonical irreducible polynomials are monic, in the second the canonical primes are "positive" primes [3]. This leads to a well-defined derivative for the field of fractions. Note also, that even the condition UFD is not necessary – we only need to have a well-defined derivative, i.e. independent of factorization. We do not plan to develop the theory in this more abstract direction and restrict ourselves by the following trivial (but interesting) result.

Theorem 20 *Let K be a field of characteristic zero and with the derivative f' in $K[x]$ is defined as above. Let $\frac{d}{dx}$ be a usual derivative. Then $f'(x) = \frac{df(x)}{dx}$ if and only if the polynomial $f(x)$ is a product of linear factors.*

Proof. Because both derivatives are equal to zero on constants they coincide on linear polynomials. If $f(x)$ has no linear irreducible factors then $f'(x)$ has smaller degree then $\frac{df(x)}{dx}$. Otherwise $f(x) = l(x)g(x)$ for some linear polynomial $l(x)$ and

$$f'(x) - \frac{df(x)}{dx} = l'(x)g(x) + l(x)g'(x) - \frac{dl(x)}{dx}g(x) - l(x)\frac{dg(x)}{dx} =$$

$$l(x)\left(g'(x) - \frac{dg(x)}{dx}\right)$$

and we can use induction. ■

So, for the complex polynomials both definitions coincide. On the other hand $(x^2 + x + 1)' = \frac{d}{dx}(x^2 + x + 1) = 1$ in $\mathbf{Z_2}[x]$, though $(x^2 + x + 1)$ is irreducible, thus characteristic restrictions are essential.

Let us now look at the Gaussian numbers. We leave to the reader the pleasure of creating similar conjectures as for integers, for example the analogs of Goldbach and prime twins conjectures (twins seem to be pairs with distance $\sqrt{2}$ between two elements; more history and variants can be found in "The Gaussian zoo" [5]). We go into another direction.

Note, that because $2+i$ and $2-i$ are "positive" primes and $5 = (2+i)(2-i)$, we should have $5' = (2+i) + (2-i) = 4$, but this does not coincide with the earlier definition. So it may be is time to change our point of view radically.

11 Generalized derivatives

Our definition is based on two key points – the Leibnitz rule and $p' = 1$ for primes. If we skip the second one and use the Leibnitz rule only we get a more general definition of $D(x)$. Now, if $x = \prod_{i=1}^{k} p_i^{x_i}$, then

$$D(x) = x\sum_{i=1}^{k}\frac{x_i D(p_i)}{p_i},$$

and we can again repeat most of the proofs above. But it is much more natural to use another approach.

Theorem 21 *Let R be a commutative ring without zero divisors and let $L : R^* \longrightarrow R^+$ be a homomorphism of its multiplicative semigroup to the additive group. Then a map*

$$D : R \longrightarrow R, D(x) = xL(x), D(0) = 0$$

satisfies the Leibnitz rule. Conversely, if $D(xy) = D(x)y + xD(y)$ then $L(x) = \frac{D(x)}{x}$ is a homomorphism. If R is a field then L is a group homomorphism and

$$D\left(\frac{x}{y}\right) = \frac{D(x)y - xD(y)}{y^2}.$$

Proof.

$$D(xy) - D(x)y - xD(y) = xyL(xy) - xL(x)y - xyL(y) = xy(L(xy) - L(x) - L(y))$$

and we see that the Leibnitz rule is equivalent to the homomorphism condition. If R is a field then the semigroup homomorphism is automatically the group homomorphism and $L(1/x) = -L(x)$ which is sufficient to get

$$D\left(\frac{1}{y}\right) = \left(\frac{1}{y}\right)(-L(y)) = \frac{-D(y)}{y^2}.$$

Then it remains to repeat the proof of Theorem 14. ■

Corollary 5 *There exist infinitely many possibilities to extend the derivative x', constructed in Section 9 on* $\mathbf{Q}$ *to all real numbers preserving the Leibnitz rule.*

Proof. We start from the positive numbers. It is sufficient to extend $ld(x)$. Note that the multiplicative group of positive real numbers is isomorphic to the additive group and both of them are vector spaces over rationals. In Section 9 a map $ld(x)$ is defined over a subspace and there are infinitely many possibilities to extend a linear map from a subspace to the whole space. Obviously it would be a group homomorphism and this gives a derivative for positive numbers. For the negative numbers we proceed as in Section 5. ■

Note that the Axiom of Choice is being used here. It would be nice to find some "natural" extension, which preserves condition (7), but note that no such extension can be continuous. To show this let us consider a sequence

$$x_n = \frac{2^{a_n}}{3^n}, a_n = \lfloor n \log_2 3 \rfloor.$$

It is bounded and has a convergent subsequence (even convergent to 1.) But

$$\lim_{n\to\infty} (x_n)' = \lim_{n\to\infty} x_n \left(\frac{a_n}{2} - \frac{n}{3}\right) = \infty.$$

An example of continuous generalized derivative gives us $D(x) = x \ln x$. It is easy to construct a surjective generalized derivative in the set of integers, and is impossible to make it injective (because $D(1) = D(-1) = D(0) = 0$). But probably even the following conjecture is true.

Conjecture 15 *There is no generalized derivative $D(x)$ which is bijection between the set of natural numbers and the set of nonnegative integers.*

We can even hope for a stronger variant:

Conjecture 16 *For any generalized derivative $D(x)$ on the set of integers there exist two different positive integers which have the same derivative.*

Returning to the generalized derivatives in $\mathbf{Q}$ or $\mathbf{R}$ let us investigate their structure as a set.

Theorem 22 *If D_1, D_2 are two generalized derivatives and a, b are some real numbers then $aD_1 + bD_2$ and $[D_1, D_2] = D_1D_2 - D_2D_1$ are generalized derivatives too. Nevertheless the set of all generalized derivatives is* not *a Lie algebra.*

Proof. We have

$$(aD_1 + bD_2)(xy) = aD_1(xy) + bD_2(xy) = aD_1(x)y + axD_1(y) + bD_2(x)y + bxD_2(y) =$$

$$aD_1(x)y + bD_2(x)y + xaD_1(y) + xbD_2(y) = (aD_1 + bD_2)(x)y + x(aD_1 + bD_2)(y).$$

In the same way:

$$[D_1, D_2](xy) = (D_1D2 - D_2D_1)(xy) = D_1D_2(xy) - D_2D_1(xy) =$$

$$D_1(D_2(x)y + xD_2(y)) - D_2(D_1(x)y + xD_1(y)) =$$
$$D_1(D_2(x))y + D_2(x)D_1(y) + D_1(x)D_2(y) + xD_1(D_2(y)) -$$
$$(D_2(D_1(x))y + D_1(x)D_2(y) + D_2(x)D_1(y) + xD_2(D_1(y))) =$$
$$D_1(D_2(x))y + xD_1(D_2(y)) - D_2(D_1(x))y - xD_2(D_1(y)) =$$
$$[D_1, D_2](x)y + x[D_1, D_2](y).$$

But the commutator is not bilinear: in general

$$[aD_1 + bD_2, D_3] \neq a[D_1, D_3] + b[D_2, D_3]$$

so we have no Lie algebra structure. ■

Let us define $D_{(p_i)}$ as a derivative which maps a prime p_i to 1 and other primes p_j to zero. Then $[D_{(p_i)}, D_{(p_j)}] = 0$, but already $[3D_{(2)}, D_{(3)}] = -D_{(2)}$. Nevertheless every generalized derivative D can be uniquely written as

$$D = \sum_{i=1}^{\infty} D(p_i)D_{(p_i)}.$$

12 The generating function

Let $D(x)$ be a generalized derivative over the reals and $L(x) = \frac{D(x)}{x}$ be corresponding logarithmic derivative. Let

$$H_D(t) = \sum_{n=0}^{\infty} D(n)t^n, H_L(t) = \sum_{n=1}^{\infty} L(n)t^n$$

be their generating functions.

Theorem 23 *The generating functions $H_D(t), H_L(t)$ can be be calculated as follows:*

$$H_D(t) = t\frac{d}{dt}\left(H_L(t)\right).$$

$$H_L(t) = {\sum_p}' L(p) \sum_{j=1}^{\infty} \frac{t^p}{1-t^{p^j}},$$

where the first sum runs over all primes.

Proof. The first formula is equivalent to the condition $D(n) = n \cdot L(n)$. As to the second formula it is sufficient to prove it for the special case when $L(p) = 1$ for some prime p and $L(q) = 0$ for all other primes. Then we need to prove that

$$\sum_{n=0}^{\infty} L(n)t^n = \sum_{j=1}^{\infty} \frac{t^p}{1-t^{p^j}}.$$

If $n = p^k m$ and $\gcd(p, m) = 1$ then t^n appears exactly in k sums

$$\frac{t^p}{1-t^{p^j}} = \sum_{i=1}^{\infty} t^{ip^j}$$

for $j = 1, 2, \ldots, k$. It only remains to note that $L(n) = k$. ∎

Corollary 6

$$L(n!) = \sum_{i=1}^{n} L(i) = {\sum_{p\le n}}' L(p) \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

Proof. If we replace every $\frac{t^p}{1-t^{p^j}}$ by $\sum_{i=1}^{\lfloor \frac{n}{p^j} \rfloor} t^{ip^j}$ we do not change the coefficients in t^k for $k \le n$ and make them equal to zero for $k > n$. So it is sufficient to put $t = 1$ to get the desired $\lfloor \frac{n}{p^j} \rfloor$ in every summand. ∎

If we use the same $L(x)$ that we used in the proof of the Theorem 23, we get the classical Legendre theorem that calculates the maximal power of a prime p in $n!$.

On the other hand if we use $L(x) = \mathrm{ld}(x)$ we will be able, following Barbeau [1], to estimate $\sum_{i=1}^{n} \mathrm{ld}(i)$. Let $m = \lfloor \log_2 n \rfloor$. Then we can change infinity in our sums to m. Using standard estimates

$${\sum_{p\le n}}' \frac{1}{p} = O(\ln m),$$

$${\sum_{p>n}}' \frac{n}{p(p-1)} < \sum_{k>n} \frac{n}{k(k-1)} = \sum_{k>n} n\left(\frac{1}{k} - \frac{1}{k-1}\right) \le 1,$$

$${\sum_{p\le n}}' \frac{n}{p^{m+1}(p-1)} < {\sum_{p\le n}}' \frac{2n}{2^{m+1}p(p-1)} < \sum_{k\le n} \frac{2n}{nk(k-1)} \le 2,$$

we get

$$\sum_{i=1}^{n} \mathrm{ld}(i) = {\sum_{p\le n}}' \frac{1}{p} \sum_{j=1}^{m} \left\lfloor \frac{n}{p^j} \right\rfloor = {\sum_{p\le n}}' \frac{1}{p} \left(\sum_{j=1}^{m} \frac{n}{p^j} + O(m) \right) =$$

$${\sum_{p\le n}}' \frac{n}{p^{m+1}} \left(\frac{p^m - 1}{p-1} \right) + O(\ln m) O(m) = {\sum_{p}}' \frac{n}{p(p-1)} -$$

$$-{\sum_{p>n}}' \frac{n}{p(p-1)} - {\sum_{p\le m}}' \frac{n}{p^{m+1}(p-1)} + O(\ln m) O(m) =$$

$${\sum_{p}}' \frac{n}{p(p-1)} + O(m \ln m).$$

Theorem 24 *[1] Let*

$$C = {\sum_{p}}' \frac{1}{p(p-1)} = 0.749\ldots$$

Then

$$ld(n!) = \sum_{i=1}^{n} ld(i) = Cn + O\left((\ln n)(\ln \ln n)\right)$$

$$\sum_{k=1}^{n} k' = \frac{C}{2} n^2 + O(n^{1+\delta})$$

for any $\delta > 0$.

Proof. The first formula is already proved. As to the second we have

$$\sum_{k=1}^{n} k' = \sum_{k=1}^{n} k \cdot \mathrm{ld}(k) = \sum_{k=1}^{n} \sum_{i=k}^{n} \mathrm{ld}(i) =$$

$$\sum_{k=1}^{n} (\mathrm{ld}(n!) - \mathrm{ld}((k-1)!)) = n \mathrm{ld}(n!) - \sum_{k=1}^{n-1} \mathrm{ld}(k!) =$$

$$n(Cn + O(n^\delta)) - \sum_{k=1}^{n-1} (Ck + O(n^\delta)) =$$

$$Cn^2 - C\frac{n(n-1)}{2} + O(n^{1+\delta}) = \frac{C}{2} n^2 + O(n^{1+\delta}).$$

■

We leave to the reader the pleasure to play with $\zeta_D(s) = \sum \frac{n'}{n^s}$.

13 Logical dependence of the conjectures

Here we would like to exhibit some of the logical dependence between the different conjectures we have mentioned above. As we see the Conjectures 8 and 9 seem to be the key problems.

Theorem 25 *The following picture describes the logical dependence between the different conjectures.*

$$(2) \Rightarrow (3) \Rightarrow (4),$$

$$(12) \Rightarrow (13),$$

$$(5) \Leftarrow (6, \textit{Goldbach}),$$

$$(15) \Leftarrow (16),$$

$$\begin{array}{ccccc} & & (11, \textit{Triples}) & & (8) \\ & & \Downarrow & & \Downarrow \\ (10, \textit{Twins}) & \Rightarrow & (9) & \Rightarrow & (1) \end{array}.$$

Additionally if Conjecture 1 is valid then either Conjecture 8 or Conjecture 9 is valid (or both).

Proof. The only nontrivial dependence is the last one. Suppose that Conjecture 9 is wrong, but Conjecture 1 is true. We need to show that Conjecture 8 is valid. Let Γ be the tree having vertices 1 (the root), the primes p with $i(p) > 0$ and all composite n with $n^{(k)} = 0$ for some $k \geq 1$. Further, let Γ have edges from n to n'. By Conjecture 1, Γ is infinite. By Theorem 8 and Corollary 3 the degree at each vertex different from 1 is finite. Also the vertex 1 has finite degree since 9 is false. By Köning infinity lemma Γ contains an infinite chain, ending in 1, which is Conjecture 8. ■

14 Concluding remarks

This article is our expression of the pleasure being a mathematician. We have written it because we found the subject to be very attractive and wanted to share our joy with others. To our surprise we did not find many references. In the article of A. Buium [2] and other articles of this author (which are highly recommended) we at least have found that there exists authors who can imagine a derivative without the linearity property. But the article of E. J. Barbeau [1] was the only article that has direct connection to our topic. Most of the material from this article we have repeated here (not always citing). We omitted only the description of the numbers with derivatives that are divisible by 4 and his conjecture that for every n there exists a prime p such that all derivatives $n^{(k)}$ are divisible by p for sufficiently large k. In fact according to Theorems 4, 5 it is equivalent to be divisible by p^p for sufficiently large k. Thus this conjecture is a bit stronger then Conjecture 2.

The definition of the arithmetic derivative itself and its elementary properties was already in the Putnam Prize competition (it was Problem 5 of the morning session in March 25, 1950, [4]) and probably was known in folklore even earlier. What we have done is mainly to generalize this definition in different directions, to solve some differential equations, to

calculate the generating function and to invite the reader to continue work in this area. We are grateful to our colleagues for useful discussion, especially to G. Almkvist, A. Chapovalov, S. Dunbar, G. Galperin, S. Shimorin and the referee, who helped to improve the text. We are especially grateful to J. Backelin, who helped us to reduce the number of conjectures by suggesting ideas that translated them into theorems.

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