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Two Game-Set Inequalities

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Abstract

Two players compete in a contest where the first player to win a specified number of points wins the game, and the first player to win a specified number of games wins the set. This paper proves two generalized inequalities, each independent of the probability of winning a point, concerning the better player's chances of winning. Counterexamples are given for two additional conjectured inequalities. A sequence of integers which plays a significant role in this paper can be found in A033820 of the On-line Encyclopedia of Integer Sequences.

1 Introduction

A and B play a set of games. The winner of each game is the first player to win k points, and the winner of the set is the first player to win n games. The probabilities that A and B win each point are p and q, respectively, with $p + q = 1$. Let $P(n, k, p)$ and $Q(n, k, p)$ be the probabilities that A wins and loses the set, respectively . Let $P(n, p) = P(n, 1, p) = P(1, n, p)$ and $Q(n, p) = Q(n, 1, p) = Q(1, n, p)$ be the probabilities that A wins and loses an n point game, respectively. (Note that $P(n, k, p) = P(n, P(k, p))$ and $Q(n, k, p) = Q(n, P(k, p))$. This paper proves the following two theorems:

Theorem 1. *If $n \geq 2$, $k \geq 2$, and $.5 < p < 1$, then $P(n, k, p) > P(nk, p)$.*

Theorem 2. *If $2 \leq k < n$ and $.5 < p < 1$, then $P(n, k, p) > P(k, n, p)$.*

2 Proofs

Lemma 1. $P(k,p) = \frac{1}{2} + \frac{1}{2}(p-q)\sum_{i=0}^{k-1}\binom{2i}{i}(pq)^i.$

Proof. We show first that

$$P(k+1,p) - P(k,p) = \frac{1}{2}(p-q)\binom{2k}{k}(pq)^k,\ k \geq 1. \tag{1}$$

The only $(k+1)$-point games in which the winner might be different than the winner of the k-point game are those which are tied at k points each.

The increase in A's chance of winning a $k+1$-point game over his chance of winning a k-point game is the excess of (a) the probability that there is a k-point tie, B having won the k-point game, and A wins the next point, over (b) the probability that there is a k-point tie, A having won the k-point game, and B wins the next point. Note that in half of those tied games, B won the k-point game, and in the other half, A won the k-point game. Since $\binom{2k}{k}(pq)^k$ is the probability of a k-point tie, (1) holds.

Lemma 1 is true since it is true for $k = 1$, and its right hand side simply sums the differences in (1).

□

Lemma 2. *Let* $a_{k,i} := \frac{1}{2}\binom{2k}{k}\binom{2i}{i}\frac{k}{k+i}$*. Then* $P(k,p)Q(k,p) = (pq)^k\sum_{i=0}^{k-1} a_{k,i}(pq)^i$*,*

(Note that $a_{k,i}$ is equal to $a_{k+i-1,k-1}$ in A033820 of the On-line Encyclopedia of Integer Sequences.)

Proof. We see from Lemma 1, noting that $-(p-q)^2 = 4pq-1$, that

$$P(k,p)Q(k,p) = \frac{1}{4} + \frac{1}{4}(4pq-1)\left(\sum_{i=0}^{k-1}\binom{2i}{i}(pq)^i\right)^2. \tag{2}$$

Since

$$\left(\sum_{i=0}^{\infty}\binom{2i}{i}(pq)^i\right)^2 = \left(\frac{1}{\sqrt{1-4pq}}\right)^2 = \sum_{i=0}^{\infty}4^i(pq)^i,$$

we see that if $1 \leq t \leq k-1$, the coefficient of $(pq)^t$ in (2) equals $\frac{1}{4}\cdot 4\cdot 4^{t-1} - \frac{1}{4}\cdot 4^t = 0$, and equals 0 for $t = 0$ as well.

Hence, we can define $a_{k,i}$ such that

$$P(k,p)Q(k,p) = (pq)^k\sum_{i=0}^{k-1} a_{k,i}(pq)^i. \tag{3}$$

Thus, $a_{k,t-k}$ is the coefficient of $(pq)^t$ in (2).

We have, then

$$a_{k,t-k} = \sum_{j=t-k}^{k-1} \binom{2j}{j}\binom{2t-2j-2}{t-j-1} - \frac{1}{4}\sum_{j=t-k+1}^{k-1}\binom{2j}{j}\binom{2t-2j}{t-j}$$

$$= \binom{2t-2k}{t-k}\binom{2k-2}{k-1} + \sum_{j=t-k+1}^{k-1}\binom{2j}{j}\left(\binom{2t-2j-2}{t-j-1} - \frac{1}{4}\binom{2t-2j}{t-j}\right)$$

$$= \binom{2t-2k}{t-k}\binom{2k-2}{k-1} + \frac{1}{2}\sum_{j=t-k+1}^{k-1}\binom{2j}{j}\binom{2t-2j-2}{t-j-1}\frac{1}{t-j}. \tag{4}$$

$$\text{Let } g(t,j) = \frac{j(2t-2j-1)\binom{2j}{j}\binom{2t-2j-2}{t-j-1}}{(t-j)t}.$$

We show that

$$\binom{2j}{j}\binom{2t-2j-2}{t-j-1}\frac{1}{t-j} = g(t,j+1) - g(t,j) \tag{5}$$

by dividing both sides of the equation by $\binom{2j}{j}\binom{2t-2j-2}{t-j-1}$ to obtain

$$\frac{1}{t-j} = \frac{1+2j}{t} - \frac{j(2t-2j-1)}{(t-j)t} = \frac{1}{t-j}.$$

Let $S(t,k)$ equal the sum in (4). By summing both sides of (5) from $j = t-k+1$ to $k-1$, we see that $S(t,k)$ is equal to $g(t,k) - g(t,t-k+1)$, and we have

$$S(t,k) = \frac{k(2t-2k-1)\binom{2k}{k}\binom{2t-2k-2}{t-k-1}}{(t-k)t} - \frac{(t-k+1)(2k-3)\binom{2t-2k+2}{t-k+1}\binom{2k-4}{k-2}}{(k-1)t}.$$

We see from (4) that

$$a_{k,t-k} = \binom{2t-2k}{t-k}\binom{2k-2}{k-1} + \frac{1}{2}S(t,k).$$

Dividing both sides by $\binom{2k}{k}\binom{2t-2k}{t-k}$, we have

$$\frac{a_{k,t-k}}{\binom{2k}{k}\binom{2t-2k}{t-k}} = \frac{k}{4k-2} + \frac{1}{2}\left(\frac{k}{2t} - \frac{k(2t-2k+1)}{2(2k-1)t}\right) = \frac{k}{2t}.$$

Replacing t by $k+i$ gives

$$a_{k,i} = \frac{1}{2}\binom{2k}{k}\binom{2i}{i}\frac{k}{k+i}.$$

$\square$

Lemma 3.

$$P'(k,p) = \frac{1}{2}k\binom{2k}{k}(pq)^{k-1}, \textit{ where the derivative is taken with respect to } p.$$

Proof. The probability that A wins a k-point game on the $(k+i)^{th}$ point played is

$$p^k\binom{k-1+i}{i}(1-p)^i.$$

Hence,

$$P(k,p) = p^k\sum_{i=0}^{k-1}\binom{k-1+i}{i}(1-p)^i = p^k\sum_{j=0}^{k-1}(-1)^j\left(\sum_{i=j}^{k-1}\binom{i}{j}\binom{k-1+i}{i}\right)p^j.$$

We have

$$\sum_{i=j}^{k-1}\binom{i}{j}\binom{k-1+i}{i} = \sum_{i=j}^{k-1}\binom{k-1+i}{i-j}\binom{k-1+j}{j} = \binom{k-1+j}{j}\binom{2k-1}{k+j}$$

$$= \frac{k}{k+j}\binom{k+j}{j}\binom{2k-1}{k+j} = \frac{1}{2}\binom{2k}{k}\binom{k-1}{j}\frac{k}{k+j}.$$

Hence,

$$P(k,p) = \frac{1}{2}k\binom{2k}{k}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{k+j}p^{k+j}.$$

Taking the derivative with respect to p, we have

$$P'(k,p) = \frac{1}{2}k\binom{2k}{k}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}p^{k+j-1} = \frac{1}{2}k\binom{2k}{k}(pq)^{k-1}.$$

$\square$

Lemma 4. *The function*

$$r_{n,k}(p) = \frac{P'(n,k,p)}{P'(nk,p)}$$

is a decreasing function of p, for $.5 \le p < 1$.

Proof.

$$r_{n,k}(p) = \frac{P'(n,k,p)}{P'(nk,p)} = \frac{P'(n,P(k,p))}{P'(nk,p)}$$

$$= \frac{\frac{1}{2}n\binom{2n}{n}\left(P(k,p)Q(k,p)\right)^{n-1}\frac{1}{2}k\binom{2k}{k}(pq)^{k-1}}{\frac{1}{2}nk\binom{2nk}{nk}(pq)^{nk-1}}$$

$$= \frac{1}{2}\frac{\binom{2n}{n}\binom{2k}{k}\left(P(k,p)Q(k,p)\right)^{n-1}}{\binom{2nk}{nk}(pq)^{k(n-1)}}, \qquad .5 \leq p < 1. \qquad (6)$$

Substituting from Lemma 2, we have

$$r_{n,k}(p) = \frac{1}{2}\frac{\binom{2n}{n}\binom{2k}{k}}{\binom{2nk}{nk}}\left(\sum_{i=0}^{k-1} a_{k,i}(pq)^i\right)^{n-1},$$

where $a_{k,i} = \frac{1}{2}\binom{2k}{k}\binom{2i}{i}\frac{k}{k+i}$.

Since $a_{k,i} > 0$, and pq is a decreasing function of p, it follows that $r_{n,k}(p)$ is a decreasing function of p.

□

Lemma 5. *If* $2 \leq t \leq m$, *then*

$$4^t\frac{\binom{2m-2t}{m-t}}{\binom{2m}{m}}\frac{2m-t}{m} > 2 .$$

Proof. For $t = 2$, the left hand side of Lemma 5 reduces to

$$\frac{8(m-1)^2}{(2m-1)(2m-3)} = 2 + \frac{2}{(2m-1)(2m-3)}$$

which is clearly greater than 2 when $m \geq 2$. We show by induction that Lemma 5 holds for all t. Suppose it is true for some t, and consider the left hand side of Lemma 5 with t replaced by $t+1$. We have,

$$4^{t+1}\frac{\binom{2m-2t-2}{m-t-1}}{\binom{2m}{m}}\frac{2m-t-1}{m} = 4\left(4^t\frac{\binom{2m-2t}{m-t}}{\binom{2m}{m}}\frac{2m-t}{m}\right)\frac{(m-t)^2}{(2m-2t-1)(2m-2t)}\frac{2m-t-1}{2m-t}$$

$$> \frac{4(m-t)(2m-t-1)}{(2m-2t-1)(2m-t)}.$$

The right hand side of the inequality is greater than 2 when $m > t$ since

$$\frac{4(m-t)(2m-t-1)}{(2m-2t-1)(2m-t)} = 2 + \frac{2t}{(2m-2t-1)(2m-t)}.$$

When $m = t$, and $t \geq 2$, it is shown easily by induction that $\dfrac{4^t}{\binom{2t}{t}} > 2$.

□

Lemma 6. *If $n \geq 2$ and $k \geq 2$, then $r_{n,k}(.5) > 1$.*

Proof. Noting that $P(k, .5) = Q(k, .5) = .5$, we have from (6),

$$r_{n,k}(.5) = \frac{1}{2}\frac{\binom{2n}{n}\binom{2k}{k}}{\binom{2nk}{nk}}4^{(k-1)(n-1)}.$$

Lemma 6 is true for $k = 2$, since we have

$$r_{n,2}(.5) = 3\ 4^{n-1}\frac{\binom{2n}{n}}{\binom{4n}{2n}} > 1,$$

and we apply Lemma 5 with $m = 2n$ and $t = n$.

We show by induction that Lemma 6 holds for all k. Suppose it is true for some k, and consider Lemma 6 with k replaced by $k + 1$.

$$r_{n,k+1}(.5) = \frac{1}{2}\frac{\binom{2n}{n}\binom{2k+2}{k+1}}{\binom{2nk+2n}{nk+n}}4^{k(n-1)}$$

$$= 4^{n-1}\left(\frac{1}{2}\frac{\binom{2n}{n}\binom{2k}{k}}{\binom{2nk}{nk}}4^{(k-1)(n-1)}\right)\frac{(2k+2)(2k+1)}{(k+1)^2}\frac{\binom{2nk}{nk}}{\binom{2nk+2n}{nk+n}}$$

$$> \frac{1}{2}4^n\frac{\binom{2nk}{nk}}{\binom{2nk+2n}{nk+n}}\frac{2k+1}{k+1} > 1.$$

The final inequality is obtained by applying Lemma 5 with $m = nk + n$ and $t = n$.

□

Theorem 1. *If $n \geq 2$, $k \geq 2$, and $.5 < p < 1$, then $P(n, k, p) > P(nk, p)$.*

Proof. We have $P(n, k, .5) = P(nk, .5) = .5$, and $P(n, k, 1) = P(nk, 1) = 1$.

If there existed a point p_1, $.5 < p_1 < 1$, such that $P(n, k, p_1) = P(nk, p_1)$, then $r_{n,k}(p) = \dfrac{P'(n,k,p)}{P'(nk,p)}$ would need to be 1 at least twice, once on the interval $.5 < p < p_1$ and once on the interval $p_1 < p < 1$ (since $r_{n,k}(p)$ cannot be greater than 1 (or less than 1) over the full extent of either interval). But $r_{n,k}(p)$ cannot be 1 at least twice because we know from Lemma 4 that $r_{n,k}(p)$ is a decreasing function of p. Hence, either $P(n,k,p) > P(nk,p)$, or $P(nk,p) > P(n,k,p)$, on the interval $.5 < p < 1$. Since we know from Lemma 6 that $r_{n,k}(.5) > 1$, we must have $P(n,k,p) > P(nk,p)$ over that same range.

□

Lemma 7. *Let*

$$f(n,x) = \sum_{i=0}^{n-1} a_{n,i} x^i.$$

If the function

$$h(n,k,x) = (n-1) f(n,x) f'(k,x) - (k-1) f(k,x) f'(n,x)$$

has at most one positive zero, then the functions $P(n,k,p)$ and $P(k,n,p)$ cannot intersect on the interval $.5 < p < 1$.

Proof. Let

$$R_{n,k}(p) = \frac{P'(n,k,p)}{P'(k,n,p)}.$$

Looking at the proof of Lemma 4 we see that

$$R_{n,k}(p) = \frac{\frac{1}{2} n \binom{2n}{n} \left(P(k,p)Q(k,p)\right)^{n-1} \frac{1}{2} k \binom{2k}{k} (pq)^{k-1}}{\frac{1}{2} k \binom{2k}{k} \left(P(n,p)Q(n,p)\right)^{k-1} \frac{1}{2} n \binom{2n}{n} (pq)^{n-1}}.$$

Substituting from Lemma 2, and simplifying, we have

$$R_{n,k}(p) = \frac{f(k,pq)^{n-1}}{f(n,pq)^{k-1}}, \qquad .5 \le p < 1.$$

Following the simple argument made in the the proof of Theorem 1, if $P(n,k,p)$ and $P(k,n,p)$ intersected on $.5 < p < 1$, $R_{n,k}(p)$ would equal 1 at least twice on that interval. Furthermore, since $f(n,.25) = 4^{n-1}$ and $f(k,.25) = 4^{k-1}$ (see Lemma 2 with $p = .5$ and $pq = .25$), $R_{n,k}(.25) = 1$. Hence, there would be at least three different values of pq for which $R_{n,k}(p) = 1$.

Therefore, the logarithm of $R_{n,k}(p)$ would equal 0 at least three times, and the derivative of that logarithm with respect to pq would equal 0 at least twice. The derivative of the logarithm of $R_{n,k}(p)$ with respect to pq is equal to

$$\frac{h(n,k,pq)}{f(n,pq)f(k,pq)},$$

proving the Lemma.

□

Lemma 8. *Let $\{a_i\}_1^n$ be a sequence such that $a_1 < 0$, $\sum_{i=1}^{n} a_i \le 0$ and such that either*

Case 1: $a_i \le 0$ *if* $2 \le i \le n$, *or*
Case 2: there exists a $t \le n$ *such that* $a_i \le 0$ *if* $2 \le i < t$, *and* $a_i > 0$ *if* $i \ge t$.
holds.

Let $\{r_i\}_1^n$ *be a sequence such that for all* i, $r_i > 0$ *and* $r_{i+1} < r_i$.

Then $\displaystyle\sum_{i=1}^n r_i a_i < 0$.

Proof. The Lemma is obvious for Case 1.

For Case 2, $\displaystyle\sum_{i=1}^n r_i a_i - \sum_{i=1}^n r_t a_i < 0$, since $(r_1 - r_t)a_1$ is negative, and if $i > 1$, $r_i - r_t$ is positive when a_i is negative or zero, and zero or negative when a_i is positive. Hence,

$$\sum_{i=1}^n r_i a_i < \sum_{i=1}^n r_t a_i \le 0.$$

□

Lemma 9. *Each of the ratios* $\dfrac{a_{n,t}}{a_{n,t-1}}$ *and* $\dfrac{a_{n+1,t}}{a_{n,t-1}}$ *decreases as* t *decreases.*

Proof.

$$\frac{a_{n,t}}{a_{n,t-1}} - \frac{a_{n,t-1}}{a_{n,t-2}} = \frac{2\left(1 + n^2 + 2n(t-1) + t(3t-5)\right)}{t(t-1)(n+t-1)(n+t)},$$

which is positive for $t \ge 2$ and $n \ge 3$.

$$\frac{a_{n+1,t}}{a_{n,t-1}} - \frac{a_{n+1,t-1}}{a_{n,t-2}} = \frac{4(2n+1)\left(n(n-1) + 2nt + t(5t-7)\right)}{(t-1)tn(n+t)(n+1+t)},$$

which is positive for $t \ge 2$ and $n \ge 3$.

□

We note that, as defined, the quantity $R_{n,k}(1)$ is indeterminate. We take $R_{n,k}(1)$ to mean $\lim_{p\to 1} R_{n,k}(p)$.

Lemma 10. *If* $1 < k < n$ *then* $R_{n,k}(1) < 1$.

Proof. Noting that $pq = 0$ when $p = 1$,

$$R_{n,k}(1) = \frac{f(k,0)^{n-1}}{f(n,0)^{k-1}} = \frac{a_{k,0}^{n-1}}{a_{n,0}^{k-1}} = \frac{\left(\frac{1}{2}\binom{2k}{k}\right)^{n-1}}{\left(\frac{1}{2}\binom{2n}{n}\right)^{k-1}},$$

Let $u_{n,k} = \dfrac{R_{n+1,k}(1)}{R_{n,k}(1)} = \left(\dfrac{n+1}{4n+2}\right)^{k-1} \dfrac{1}{2}\dbinom{2k}{k}$, and

Let $v_{n,k} = \dfrac{u_{n,k+1}}{u_{n,k}} = \dfrac{n+1}{4n+2} \Big/ \dfrac{k+1}{4k+2}$

It is assumed in the following that $1 < k < n$.

$v_{n,k} < 1$, since $n > k$, and the function $\frac{r+1}{4r+2}$ is a decreasing function of r ;
Since $v_{n,k} < 1$, $u_{n,k}$ is a decreasing function of k, and since $u_{n,1} = 1$, $u_{n,k} < 1$;
Since $u_{n,k} < 1$, $R_{n,k}(1)$ is a decreasing function of n, and since $R_{k,k}(1) = 1$, $R_{n,k}(1) < 1$. □

Theorem 2. *If $2 \le k \le n$ and $.5 < p < 1$, then $P(n,k,p) > P(k,n,p)$.*

Proof. The $h(n,k,x)$ of Lemma 7 can be written as

$$h(n,k,x) = (n-1)\sum_{i=0}^{n-1} a_{n,i}x^i \sum_{i=0}^{k-1} i a_{k,i}x^{i-1} - (k-1)\sum_{i=0}^{k-1} a_{k,i}x^i \sum_{i=0}^{n-1} i a_{n,i}x^{i-1} = \sum_{r=0}^{n+k-3} c_{n,k,r}x^r.$$

We show that $h(n,k,x)$ has at most one positive zero by showing that the sequence $\{c_{n,k,r}\}_{r=0}^{n+k-3}$ has exactly one change in sign, and applying Descartes' Rule of Signs.

We do this by considering four cases,

Case 1: $0 \le r \le k-2$. We show that $c_{n,k,r} > 0$.
Case 2: $k-1 \le r \le n-2$. We show that if $c_{n,k,r} \le 0$, then $c_{n,k,r+1} < 0$.
Case 3: $n-1 \le r \le n+k-4$. We show that $c_{n,k,r} < 0$.
Case 4: $r = n+k-3$. We show that $c_{n,k,r} = 0$.

Case 1: $0 \le r \le k-2$.

$$c_{n,k,r} = \sum_{j=0}^{r+1} \left((n-1)j - (k-1)(r+1-j)\right) a_{k,j}a_{n,r+1-j}.$$

We see that $c_{n,k,r}$ is an increasing function of n, since the bracketed term in $c_{n,k,r}$ is an increasing funtion of n, and

$$\frac{a_{n+1,t}}{a_{n,t}} = 1 + \frac{3n^2 + n + 2t + 3nt}{n(n+t+1)} > 1.$$

Since $c_{k,k,r} = 0$, $c_{n,k,r} > 0$.

Case 2: $k-1 \le r \le n-2$.

$$c_{n,k,r} = \sum_{j=0}^{k-1} \left((n-1)j - (k-1)(r+1-j)\right) a_{k,j}a_{n,r+1-j}$$

Let

$$b(n,k,r,j) = \left((n-1)j - (k-1)(r+2-j)\right) a_{k,j}a_{n,r+1-j},$$

so that

$$c_{n,k,r+1} = \sum_{j=0}^{k-1} b(n,k,r,j)\frac{a_{n,r+2-j}}{a_{n,r+1-j}}.$$

We see that $b(n,k,r,0) < 0$, the bracketed term in $b(n,k,r,j)$ increases as j increases, and if $c_{n,k,r} \leq 0$, then $\sum_{j=0}^{k-1} b(n,k,r,j) < 0$(since the bracketed term in $b(n,k,r,j)$ is less than the bracketed term in $c_{n,k,r}$). Furthermore, we know from Lemma 9 that the sequence $\left\{\frac{a_{n,r+2-j}}{a_{n,r+1-j}}\right\}$ is decreasing as j increases.

Hence the conditions set forth in Lemma 8 are met, and we have $c_{n,k,r+1} < 0$.

Case 3: $n-1 \leq r \leq n+k-4$.

Let $r = n-1+t,\ \ 0 \leq t \leq k-3$.

$$c_{n,k,n-1+t} = \sum_{j=t+1}^{k-1} \left((n-1)j - (k-1)(n+t-j)\right) a_{k,j}a_{n,n+t-j}.$$

Let $b(n,k,t,j) = (nj - (k-1)(n+1+t-j))\, a_{k,j}a_{n,n+t-j}$, so that

$$c_{n+1,k,n+t} = \sum_{j=t+1}^{k-1} b(n,k,t,j)\frac{a_{n+1,n+1+t-j}}{a_{n,n+t-j}}.$$

We see that $b(n,k,t,t+1) = (n(t-(k-2)))\, a_{k,j}a_{n,n-1} < 0$, the bracketed term in $b(n,k,t,j)$ increases as j increases, and if $c_{n,k,n-1+t} < 0$, then $\sum_{j=t+1}^{k-1} b(n,k,t,j) < 0$ (since the bracketed term in $b(n,k,t,j)$ is less than or equal to the bracketed term in $c_{n,k,n-1+t}$). Furthermore, from Lemma 9 we know that the sequence $\left\{\frac{a_{n+1,n+1+t-j}}{a_{n,n+t-j}}\right\}$ is decreasing as j increases.

Hence the conditions set forth in Lemma 8 are met, and we have $c_{n+1,k,n+t} < 0$. Since $c_{k,k,k-1+t} = 0$, we have for all $n > k$, $c_{n,k,n-1+t} < 0$.

Case 4: $r = n+k-3$.

$c_{n,k,r} = ((n-1)(k-1) - (k-1)(n-1))\, a_{n,n-1}a_{k,k-1} = 0$.

We have proved that $h(n,k,x)$ has at most one positive zero. Hence we know from Lemma 7 that on the interval $.5 < p < 1$, $P(n,k,p)$ and $P(k,n,p)$

cannot intersect. From Lemma 10, we know that $\lim_{p\to 1} \frac{P'(n,k,p)}{P'(k,n,p)} < 1$. Hence, we must have $P(n,k,p) > P(k,n,p)$ on $.5 < p < 1$.

$\square$

3 Counterexamples for two conjectured inequalities

Define $\mathrm{Maxpoints}(m,n) := (2m-1)(2n-1)$, the maximum number of points possible where the winner is the first player to win m n-point games.

Initial examination of numerical values of $P(n,k,p)$ for a wide range of values of n, k and p suggested that the following conjectured inequalities might be universally true and provable ($.5 < p < 1$):

Conjecture 1. *If* $\mathrm{Maxpoints}(a,b) > \mathrm{Maxpoints}(c,d)$, *then* $P(a,b,p) > P(c,d,p)$.

Conjecture 2. *If* $ab = cd$ *and* $\min(a,b) > \min(c,d)$, *then* $P(a,b) > P(c,d)$.

The following computations show that neither of these conjectures is universally true:

$$\begin{array}{ll} \mathrm{Maxpoints}(3,2) = 15 & P(3,2,.6) = .7617\cdots \\ \mathrm{Maxpoints}(1,7) = 13 & P(7,.6) = .7711\cdots \end{array}$$

$$\begin{array}{ll} \min(4,3) = 3 & P(4,3,.99) = .99999999999999999670\cdots \\ \min(6,2) = 2 & P(6,2,.99) = .99999999999999999676\cdots \end{array}$$

4 Remark

The integers $a_{k,i}$, which play a significant role in this paper, are the same as the integers $a_{k+i-1,k-1}$ in A033820 of the On-line Encyclopedia of Integer Sequences. They appear in quite different contexts in [1, 2].

References

[1] A. Burstein, Enumeration of words with forbidden patterns, Ph. D. Thesis, U. of Pennsylvania, 1998.

[2] I. Gessel, Super ballot numbers, *J. Symbolic Computation* **14** (1992), 179–194.

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